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NOTE ON THE MULTIDIMENSIONAL GEBELEIN INEQUALITY

Abstract. We generalize the Gebelein inequality for Gaussian random vectors in $\mathbb{R}^d$.

1. The Mehler kernel in $\mathbb{R}^d$. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and let

$$V = (X, Y) = (X_1, \ldots, X_d, Y_1, \ldots, Y_d)$$

be a Gaussian vector on $(\Omega, \mathcal{F}, P)$ such that

$$\widehat{R} = \operatorname{cov}(V) = \begin{bmatrix} I & R \\ R & I \end{bmatrix},$$

where I is the identity matrix and R is a square symmetric matrix, both of order d. By $N_d(0, I)$ we denote the family of all Gaussian vectors on $(\Omega, \mathcal{F}, P)$ with mean zero and the identity covariance matrix. It follows that $X = (X_1, \ldots, X_d)$, $Y = (Y_1, \ldots, Y_d) \in N_d(0, I)$. We denote by μ the normalized d-dimensional Gaussian measure, i.e.

$$d\mu(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}\|x\|^2\right) dx,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. In $L^2(\mu) = L^2(\mathbb{R}^d, \mu)$ we have the scalar product

$$(f, g)_\mu = \int_{\mathbb{R}^d} f(x) g(x)\, d\mu(x).$$

Throughout the paper we shall assume that $\|R\|_\infty < 1$, where $\|\cdot\|_\infty$ is a norm of the operator $R : l_\infty^d \to l_\infty^d$ (which we denote by the same letter as its matrix in the standard basis). Hence, for all $x \neq 0$ we have

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$((I-R^2)(x),x)_d > 0$, in particular $\det(I-R^2) > 0$, where $(\cdot,\cdot)_d$ is the standard inner product in $\mathbb{R}^d$.

Let $Z \in N_d(0,I)$ be a Gaussian vector such that Z, Y are independent. Introducing $U = RY + \sqrt{I-R^2}\,Z$, we see that the Gaussian vectors (X,Y) and (U,Y) have the same joint distribution.

We can introduce an *Ornstein–Uhlenbeck* type linear operator $P_R : L^2(\mu) \to L^2(\mu)$ by

$$\begin{aligned}(P_R)f(y) &= E[f(X)\,|\,Y=y] = E[f(U)\,|\,Y=y]\\ &= \int_{\mathbb{R}^d} f(Ry+\sqrt{I-R^2}\,z)\,d\mu(z), \qquad y\in\mathbb{R}^d.\end{aligned}$$

It is easily seen that P_R can be defined on $L^1(\mu)$ and from the Jensen inequality it follows that P_R is a contraction in $L^p(\mu)$ for $p \geq 1$. Moreover it turns out that the operator P_R has a kernel:

Proposition 1.1. *Under the above assumptions, we have*

$$(P_R f)(x) = \int_{\mathbb{R}^d} k_R(x,y)f(y)\,d\mu(y), \qquad x\in\mathbb{R}^d,\, f\in L^2(\mu),$$

where

$$k_R(x,y) = \frac{1}{\sqrt{\det(E)}}\exp\Big\{-\frac{1}{2}[-\|y\|^2+(E^{-1}(y-Rx),y-Rx)_d]\Big\}, \quad x,y\in\mathbb{R}^d,$$

and $E = I-R^2$.

Proof. It is known that the density f_V of the random vector $V=(X,Y)$ has the form

$$f_V(v) = \frac{1}{(2\pi)^d}\frac{1}{\sqrt{\det(\widehat{R})}}\exp\Big\{-\frac{1}{2}(\widehat{R}^{-1}v,v)_{2d}\Big\}, \qquad v\in\mathbb{R}^{2d}.$$

Using the formulas for the determinant and the inverse of a block matrix we obtain $\det(\widehat{R}) = \det(I-R^2)$ and

$$\widehat{R}^{-1} = \begin{bmatrix} I & R\\ R & I\end{bmatrix}^{-1} = \begin{bmatrix} I+RE^{-1}R & -RE^{-1}\\ -E^{-1}R & E^{-1}\end{bmatrix},$$

where $E = I-R^2$. Hence for $v=(x,y)$, $x,y\in\mathbb{R}^d$ we have

$$\begin{aligned}&f_V(v) = f_{(X,Y)}(x,y) = \frac{1}{(2\pi)^d}\frac{1}{\sqrt{\det(E)}}\\ &\times\exp\Big\{-\frac{1}{2}[\|x\|^2+(E^{-1}Rx,Rx)_d-(E^{-1}y,Rx)_d-(E^{-1}Rx,y)_d+(E^{-1}y,y)_d]\Big\}.\end{aligned}$$

By the definition of the operator P_R we have

$$k_R(x,y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)f_Y(y)},$$

where $f_Y(x) = f_X(x) = (2\pi)^{-d/2}\exp\left(-\frac{1}{2}\|x\|^2\right)$. Hence the conclusion follows. ∎

2. The Gebelein inequality. For $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $k = (k_1,\ldots,k_d) \in \mathbb{N}_0^d = (\mathbb{N}\cup\{0\})^d$, we denote as usual

$$|x| = \sum_{i=1}^d x_i, \qquad x^k = \prod_{i=1}^d x_i^{k_i}, \qquad |k| = \sum_{i=1}^d k_i, \qquad k! = \prod_{i=1}^d k_i!.$$

The set of all $d\times d$ matrices with elements from $\mathbb{R}$ (or $\mathbb{N}_0$) is denoted by $\mathcal{M}_d(\mathbb{R})$ (resp. $\mathcal{M}_d(\mathbb{N}_0)$). If $R \in \mathcal{M}_d(\mathbb{R})$, the jth column and ith row of R are denoted by R_j and R^i respectively. From time to time we shall use the shorthand notation $R = [R_j^i]$. As usual we identify rows and columns of R with vectors from $\mathbb{R}^d$. If $R \in \mathcal{M}_d(\mathbb{R})$ and $K \in \mathcal{M}_d(\mathbb{N}_0)$, we denote

$$|K| = (|K^1|,\ldots,|K^d|), \qquad |R| = (|R^1|,\ldots,|R^d|),$$

$$K! = K^1!\cdots K^d! = \prod_{i,j=1}^d K_j^i!, \qquad R^K = {R^1}^{K^1}\cdots {R^d}^{K^d} = \prod_{i,j=1}^d {R_j^i}^{K_j^i},$$

with the convention $0^0=1$. Given $R \in \mathcal{M}_d(\mathbb{R})$, a multiindex $n=(n_1,\ldots,n_d) \in \mathbb{N}_0^d$ and a vector $t = (t_1,\ldots,t_d) \in \mathbb{R}^d$, it is easy to check that

$$(Rt)^n = \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K|=n}} \frac{n!}{K!} R^K t^{|K^T|} \tag{1.1}$$

(here T stands for transposition) . Putting $t = (1,\ldots,1) \in \mathbb{R}^d$ in the above formula we obtain

$$|R|^n = \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K|=n}} \frac{n!}{K!} R^K. \tag{1.2}$$

Let H_n, $n \ge 0$, be the Hermite polynomial on $\mathbb{R}$ of degree n, i.e.

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}), \qquad x \in \mathbb{R}.$$

Hermite polynomials on $\mathbb{R}^d$ are defined as tensor products of Hermite polynomials on $\mathbb{R}$, namely for $n=(n_1,\ldots,n_d)\in\mathbb{N}_0^d$ and $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ we put

$$H_n(x) = \prod_{i=1}^d H_{n_i}(x_i) \quad\text{and}\quad h_n(x) = \prod_{i=1}^d h_{n_i}(x_i),$$

where $h_{n_i}(x_i) = \frac{1}{\sqrt{n_i!}} H_{n_i}(x_i)$. It is known that the collection $\{h_n\}_{n\in\mathbb{N}_0^d}$ forms an orthonormal basis in $L^2(\mu)$. The polynomials H_n divided by $n!$ are the coefficients of the expansion in powers of $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ of the function $w_t(x) = \exp(-\|t\|^2/2 + (t,x)_d)$. In fact, we have

$$w_t(x) = \sum_{n\in\mathbb{N}_0^d} \frac{t^n}{n!} H_n(x), \qquad t, x \in \mathbb{R}^d.$$

PROPOSITION 1.2. *Let $R \in \mathcal{M}_d(\mathbb{R})$ be a symmetric matrix such that $\|R\|_\infty < 1$. Then*

$$(P_R H_n)(x) = \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{|K^T|!}{K!} R^K H_{|K|}(x), \quad x \in \mathbb{R}^d. \tag{1.3}$$

Proof. By the definition of the operator P_R and of the generating function w_t of Hermite polynomials we have

$$\begin{aligned}(P_R w_t)(x) &= \int_{\mathbb{R}^d} \exp\left[-\frac{\|t\|^2}{2} + (t, Rx + \sqrt{I-R^2}\, y)_d\right] d\mu(y)\\
&= \exp\left[-\frac{\|t\|^2}{2} + (t, Rx)_d\right] \int_{\mathbb{R}^d} \exp[(\sqrt{I-R^2}\, t, y)_d]\, d\mu(y)\\
&= \exp\left[-\frac{\|t\|^2}{2} + (t, Rx)_d\right] \exp\left[\frac{1}{2}((I-R^2)t, t)_d\right]\\
&= \exp\left[(Rt, x)_d - \frac{1}{2}\|Rt\|^2\right] = \sum_{n\in\mathbb{N}_0^d} \frac{(Rt)^n}{n!} H_n(x).\end{aligned}$$

Hence and from (1.1) we conclude that

$$\begin{aligned}(P_R w_t)(x) &= \sum_{n\in\mathbb{N}_0^d} \sum_{\substack{K\in\mathcal{M}_d(N_0)\\ |K|=n}} \frac{R^K}{K!} t^{|K^T|} H_{|K|}(x)\\
&= \sum_{n\in\mathbb{N}_0^d} \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{R^K}{K!} t^{|K^T|} H_{|K|}(x) = \sum_{n\in\mathbb{N}_0^d} \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{R^K}{K!} t^n H_{|K|}(x)\\
&= \sum_{n\in\mathbb{N}_0^d} \frac{t^n}{n!} \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{|K^T|!}{K!} R^K H_{|K|}(x).\end{aligned}$$

On the other hand we have

$$(P_R w_t)(x) = \sum_{n\in\mathbb{N}_0^d} \frac{t^n}{n!}(P_R H_n)(x),$$

and finally

$$(P_R H_n)(x) = \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{|K^T|!}{K!} R^K H_{|K|}(x). \ \blacksquare$$

We observe that for normalized (in $L^2(\mu)$) Hermite polynomials h_n the formula (1.3) has the form

$$(1.4)\qquad (P_R h_n)(x) = \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} R^K h_{|K|}(x), \qquad x\in\mathbb{R}^d.$$

We can now formulate the following generalization of the classical Gebelein inequality (see [BC], [DK], [G])

THEOREM 1.1. *Let $R\in\mathcal{M}_d(\mathbb{R})$ be a symmetric matrix such that $\|R\|_\infty < 1$. Then for $f\in L^2(\mu)$ such that $\int_{\mathbb{R}^d} f\,d\mu = 0$ we have*

$$\|P_R f\|_{L^2(\mu)} \le \|R\|_\infty \|f\|_{L^2(\mu)}.$$

Proof. Fix $f\in L^2(\mu)$ with $\int_{\mathbb{R}^d} f\,d\mu = 0$. Expanding f with respect to the orthonormalized Hermite system $\{h_n\}_{n\in\mathbb{N}_0^d}$ and using (1.4) we obtain

$$\begin{aligned}
\|P_R f\|^2_{L^2(\mu)} &= \int_{\mathbb{R}^d} \Big| \sum_{n\in\mathbb{N}_0^d} (f,h_n)_\mu (P_R h_n)\Big|^2 d\mu \\
&= \int_{\mathbb{R}^d} \Big| \sum_{n\in\mathbb{N}_0^d} (f,h_n)_\mu \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K^T|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} R^K h_{|K|}\Big|^2 d\mu \\
&= \int_{\mathbb{R}^d} \Big| \sum_{n\in\mathbb{N}_0^d} \sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} R^K (f,h_{|K^T|})_\mu h_n\Big|^2 d\mu \\
&= \sum_{n\in\mathbb{N}_0^d} \Big(\sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} R^K (f,h_{|K^T|})_\mu \Big)^2 \\
&\le \sum_{n\in\mathbb{N}_0^d} \Big(\sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\ |K|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} \overline{R}^K |(f,h_{|K^T|})_\mu| \Big)^2,
\end{aligned}$$

where $\overline{R} = [|R^i_j|]$. Hence and by the Schwarz inequality, the observation that $R^K = R^{K^T}$, $K! = K^T!$, and (1.2), we conclude that

$$\|P_R f\|^2_{L^2(\mu)} \le \sum_{n\in\mathbb{N}_0^d}\Big(\sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\|K|=n}}\frac{n!}{K!}\overline{R}^K\Big)\Big(\sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\|K|=n}}\frac{|K^T|!}{K^T!}\overline{R}^{K^T}(f,h_{|K^T|})^2_\mu\Big)$$
$$\le \sum_{n\in\mathbb{N}_0^d}|\overline{R}|^n\sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\|K|=n}}\overline{R}^{K^T}(f,h_{|K^T|})^2_\mu$$
$$\le \|R\|_\infty\sum_{n\in\mathbb{N}_0^d}\sum_{\substack{K\in\mathcal{M}_d(\mathbb{N}_0)\\|K^T|=n}}\frac{n!}{K^T!}\overline{R}^{K^T}(f,h_n)^2_\mu$$
$$= \|R\|_\infty\sum_{n\in\mathbb{N}_0^d}|\overline{R}|^n(f,h_n)^2_\mu \le \|R\|^2_\infty\|f\|^2_{L^2(\mu)}. \ \blacksquare$$

3. Applications of Gebelein's inequality. Suppose the normalized Gaussian sequence $X = (X_i, i = 1,2,\ldots)$ of random vectors in $\mathbb{R}^d$ is given. In particular $X_i \in N(0,I)$ for each $i \ge 1$. It is assumed that the matrices $R_{i,j} = E(X_iX_j)$ are symmetric for $i,j = 1,2,\ldots$ and satisfy the following hypothesis:

$$\text{(1.5)}\qquad \|R_{i,j}\|_\infty < 1,\quad i,j = 1,2,\ldots,\qquad C = \sup_{i\ge1}\sum_{j\ge1}\|R_{i,j}\|_\infty < \infty.$$

By the Frobenius Theorem (see [HLP]) and Theorem 1.1 we get the estimate

$$\mathrm{Var}\Big(\sum_{i=1}^n f_i(X_i)\Big) \le C\sum_{i=1}^n \mathrm{Var}(f_i(X_i)),\qquad n = 1,2,\ldots,$$

where $f_i \in L^2(\mu)$, $i = 1,2,\ldots$ (see [B], [BC], [V]). Using this inequality and adopting the methods from [B] and [BC] we obtain the following two statements:

Lemma 1.1 (Borel–Cantelli Lemma). *Let $X = (X_n, n = 1,2,\ldots)$ be a Gaussian sequence with $X_i \in N(0,I)$ for $i \ge 1$ and suppose that X satisfies* (1.5). *Then for every sequence of Borel sets $(A_n,, n = 1,2,\ldots)$ in $\mathbb{R}^d$ such that*

$$\sum_{n=1}^\infty P\{X_n \in A_n\} = \infty$$

we have $P\{X_n \in A_n \text{ i.o.}\} = 1$. $\blacksquare$

Theorem 1.2 (Strong Law of Large Numbers). *Let $X = (X_i, i=1,2,\ldots)$ be a Gaussian sequence with $X_i \in N(0,I)$ for $i \ge 1$ and suppose that X*

satisfies (1.5). *Then for* $f \in L^1(\mu)$ *we have*

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \xrightarrow[n\to\infty]{} Ef(X_1) \qquad a.s. \ \blacksquare$$

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