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NOTE ON THE MULTIDIMENSIONAL GEBELEIN INEQUALITY

Abstract. We generalize the Gebelein inequality for Gaussian random vectors in \mathbb{R}^d .

1. The Mehler kernel in \mathbb{R}^d . Let (Ω, \mathcal{F}, P) be a fixed probability space and let

$$V = (X, Y) = (X_1, \dots, X_d, Y_1, \dots, Y_d)$$

be a Gaussian vector on (Ω, \mathcal{F}, P) such that

$$\widehat{R} = \text{cov}(V) = \begin{bmatrix} I & R \\ R & I \end{bmatrix},$$

where I is the identity matrix and R is a square symmetric matrix, both of order d . By $N_d(0, I)$ we denote the family of all Gaussian vectors on (Ω, \mathcal{F}, P) with mean zero and the identity covariance matrix. It follows that $X = (X_1, \dots, X_d), Y = (Y_1, \dots, Y_d) \in N_d(0, I)$. We denote by μ the normalized d -dimensional Gaussian measure, i.e.

$$d\mu(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}\|x\|^2\right) dx,$$

where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^d . In $L^2(\mu) = L^2(\mathbb{R}^d, \mu)$ we have the scalar product

$$(f, g)_\mu = \int_{\mathbb{R}^d} f(x)g(x) d\mu(x).$$

Throughout the paper we shall assume that $\|R\|_\infty < 1$, where $\|\cdot\|_\infty$ is a norm of the operator $R : l_\infty^d \rightarrow l_\infty^d$ (which we denote by the same letter as its matrix in the standard basis). Hence, for all $x \neq 0$ we have

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$((I - R^2)(x), x)_d > 0$, in particular $\det(I - R^2) > 0$, where $(\cdot, \cdot)_d$ is the standard inner product in \mathbb{R}^d .

Let $Z \in N_d(0, I)$ be a Gaussian vector such that Z, Y are independent. Introducing $U = RY + \sqrt{I - R^2}Z$, we see that the Gaussian vectors (X, Y) and (U, Y) have the same joint distribution.

We can introduce an *Ornstein-Uhlenbeck* type linear operator $P_R : L^2(\mu) \rightarrow L^2(\mu)$ by

$$\begin{aligned} (P_R)f(y) &= E[f(X) | Y = y] = E[f(U) | Y = y] \\ &= \int_{\mathbb{R}^d} f(Ry + \sqrt{I - R^2}z) d\mu(z), \quad y \in \mathbb{R}^d. \end{aligned}$$

It is easily seen that P_R can be defined on $L^1(\mu)$ and from the Jensen inequality it follows that P_R is a contraction in $L^p(\mu)$ for $p \geq 1$. Moreover it turns out that the operator P_R has a kernel:

PROPOSITION 1.1. *Under the above assumptions, we have*

$$(P_R f)(x) = \int_{\mathbb{R}^d} k_R(x, y) f(y) d\mu(y), \quad x \in \mathbb{R}^d, f \in L^2(\mu),$$

where

$$k_R(x, y) = \frac{1}{\sqrt{\det(E)}} \exp\left\{-\frac{1}{2}[-\|y\|^2 + (E^{-1}(y - Rx), y - Rx)_d]\right\}, \quad x, y \in \mathbb{R}^d,$$

and $E = I - R^2$.

Proof. It is known that the density f_V of the random vector $V = (X, Y)$ has the form

$$f_V(v) = \frac{1}{(2\pi)^d} \frac{1}{\sqrt{\det(\hat{R})}} \exp\left\{-\frac{1}{2}(\hat{R}^{-1}v, v)_{2d}\right\}, \quad v \in \mathbb{R}^{2d}.$$

Using the formulas for the determinant and the inverse of a block matrix we obtain $\det(\hat{R}) = \det(I - R^2)$ and

$$\hat{R}^{-1} = \begin{bmatrix} I & R \\ R & I \end{bmatrix}^{-1} = \begin{bmatrix} I + RE^{-1}R & -RE^{-1} \\ -E^{-1}R & E^{-1} \end{bmatrix},$$

where $E = I - R^2$. Hence for $v = (x, y)$, $x, y \in \mathbb{R}^d$ we have

$$\begin{aligned} f_V(v) &= f_{(X,Y)}(x, y) = \frac{1}{(2\pi)^d} \frac{1}{\sqrt{\det(E)}} \\ &\times \exp\left\{-\frac{1}{2}[\|x\|^2 + (E^{-1}Rx, Rx)_d - (E^{-1}y, Rx)_d - (E^{-1}Rx, y)_d + (E^{-1}y, y)_d]\right\}. \end{aligned}$$



By the definition of the operator P_R we have

$$k_R(x, y) = \frac{f_{(X,Y)}(x, y)}{f_X(x)f_Y(y)},$$

where $f_Y(x) = f_X(x) = (2\pi)^{-d/2} \exp(-\frac{1}{2}\|x\|^2)$. Hence the conclusion follows. ■

2. The Gebelein inequality. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d = (\mathbb{N} \cup \{0\})^d$, we denote as usual

$$|x| = \sum_{i=1}^d x_i, \quad x^k = \prod_{i=1}^d x_i^{k_i}, \quad |k| = \sum_{i=1}^d k_i, \quad k! = \prod_{i=1}^d k_i!$$

The set of all $d \times d$ matrices with elements from \mathbb{R} (or \mathbb{N}_0) is denoted by $\mathcal{M}_d(\mathbb{R})$ (resp. $\mathcal{M}_d(\mathbb{N}_0)$). If $R \in \mathcal{M}_d(\mathbb{R})$, the j th column and i th row of R are denoted by R_j and R^i respectively. From time to time we shall use the shorthand notation $R = [R_j^i]$. As usual we identify rows and columns of R with vectors from \mathbb{R}^d . If $R \in \mathcal{M}_d(\mathbb{R})$ and $K \in \mathcal{M}_d(\mathbb{N}_0)$, we denote

$$|K| = (|K^1|, \dots, |K^d|), \quad |R| = (|R^1|, \dots, |R^d|),$$

$$K! = K^1! \dots K^d! = \prod_{i,j=1}^d K_j^i!, \quad R^K = R^{1K^1} \dots R^{dK^d} = \prod_{i,j=1}^d R_j^{iK_j^i},$$

with the convention $0^0 = 1$. Given $R \in \mathcal{M}_d(\mathbb{R})$, a multiindex $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ and a vector $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, it is easy to check that

$$(1.1) \quad (Rt)^n = \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{n!}{K!} R^K t^{|K^T|}$$

(here T stands for transposition) . Putting $t = (1, \dots, 1) \in \mathbb{R}^d$ in the above formula we obtain

$$(1.2) \quad |R|^n = \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{n!}{K!} R^K.$$

Let $H_n, n \geq 0$, be the Hermite polynomial on \mathbb{R} of degree n , i.e.

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2}), \quad x \in \mathbb{R}.$$

Hermite polynomials on \mathbb{R}^d are defined as tensor products of Hermite polynomials on \mathbb{R} , namely for $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we put

$$H_n(x) = \prod_{i=1}^d H_{n_i}(x_i) \quad \text{and} \quad h_n(x) = \prod_{i=1}^d h_{n_i}(x_i),$$



where $h_{n_i}(x_i) = \frac{1}{\sqrt{n_i!}} H_{n_i}(x_i)$. It is known that the collection $\{h_n\}_{n \in \mathbb{N}_0^d}$ forms an orthonormal basis in $L^2(\mu)$. The polynomials H_n divided by $n!$ are the coefficients of the expansion in powers of $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ of the function $w_t(x) = \exp(-\|t\|^2/2 + (t, x)_d)$. In fact, we have

$$w_t(x) = \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} H_n(x), \quad t, x \in \mathbb{R}^d.$$

PROPOSITION 1.2. *Let $R \in \mathcal{M}_d(\mathbb{R})$ be a symmetric matrix such that $\|R\|_\infty < 1$. Then*

$$(1.3) \quad (P_R H_n)(x) = \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{|K^T|!}{K!} R^K H_{|K|}(x), \quad x \in \mathbb{R}^d.$$

Proof. By the definition of the operator P_R and of the generating function w_t of Hermite polynomials we have

$$\begin{aligned} (P_R w_t)(x) &= \int_{\mathbb{R}^d} \exp\left[-\frac{\|t\|^2}{2} + (t, Rx + \sqrt{I - R^2} y)_d\right] d\mu(y) \\ &= \exp\left[-\frac{\|t\|^2}{2} + (t, Rx)_d\right] \int_{\mathbb{R}^d} \exp[(\sqrt{I - R^2} t, y)_d] d\mu(y) \\ &= \exp\left[-\frac{\|t\|^2}{2} + (t, Rx)_d\right] \exp\left[\frac{1}{2}((I - R^2)t, t)_d\right] \\ &= \exp\left[(Rt, x)_d - \frac{1}{2}\|Rt\|^2\right] = \sum_{n \in \mathbb{N}_0^d} \frac{(Rt)^n}{n!} H_n(x). \end{aligned}$$

Hence and from (1.1) we conclude that

$$\begin{aligned} (P_R w_t)(x) &= \sum_{n \in \mathbb{N}_0^d} \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{R^K}{K!} t^{|K^T|} H_{|K|}(x) \\ &= \sum_{n \in \mathbb{N}_0^d} \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{R^K}{K!} t^{|K^T|} H_{|K|}(x) = \sum_{n \in \mathbb{N}_0^d} \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{R^K}{K!} t^n H_{|K|}(x) \\ &= \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{|K^T|!}{K!} R^K H_{|K|}(x). \end{aligned}$$



On the other hand we have

$$(P_R w_t)(x) = \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} (P_R H_n)(x),$$

and finally

$$(P_R H_n)(x) = \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{|K^T|!}{K!} R^K H_{|K|}(x). \quad \blacksquare$$

We observe that for normalized (in $L^2(\mu)$) Hermite polynomials h_n the formula (1.3) has the form

$$(1.4) \quad (P_R h_n)(x) = \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{\sqrt{|K^T|!} \sqrt{|K|!}}{K!} R^K h_{|K|}(x), \quad x \in \mathbb{R}^d.$$

We can now formulate the following generalization of the classical Gebelein inequality (see [BC], [DK], [G])

THEOREM 1.1. *Let $R \in \mathcal{M}_d(\mathbb{R})$ be a symmetric matrix such that $\|R\|_\infty < 1$. Then for $f \in L^2(\mu)$ such that $\int_{\mathbb{R}^d} f \, d\mu = 0$ we have*

$$\|P_R f\|_{L^2(\mu)} \leq \|R\|_\infty \|f\|_{L^2(\mu)}.$$

Proof. Fix $f \in L^2(\mu)$ with $\int_{\mathbb{R}^d} f \, d\mu = 0$. Expanding f with respect to the orthonormalized Hermite system $\{h_n\}_{n \in \mathbb{N}_0^d}$ and using (1.4) we obtain

$$\begin{aligned} \|P_R f\|_{L^2(\mu)}^2 &= \int_{\mathbb{R}^d} \left| \sum_{n \in \mathbb{N}_0^d} (f, h_n)_\mu (P_R h_n) \right|^2 d\mu \\ &= \int_{\mathbb{R}^d} \left| \sum_{n \in \mathbb{N}_0^d} (f, h_n)_\mu \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{\sqrt{|K^T|!} \sqrt{|K|!}}{K!} R^K h_{|K|} \right|^2 d\mu \\ &= \int_{\mathbb{R}^d} \left| \sum_{n \in \mathbb{N}_0^d} \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{\sqrt{|K^T|!} \sqrt{|K|!}}{K!} R^K (f, h_{|K^T|})_\mu h_n \right|^2 d\mu \\ &= \sum_{n \in \mathbb{N}_0^d} \left(\sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{\sqrt{|K^T|!} \sqrt{|K|!}}{K!} R^K (f, h_{|K^T|})_\mu \right)^2 \\ &\leq \sum_{n \in \mathbb{N}_0^d} \left(\sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{\sqrt{|K^T|!} \sqrt{|K|!}}{K!} \bar{R}^K |(f, h_{|K^T|})_\mu| \right)^2, \end{aligned}$$



where $\bar{R} = [|R_j^i|]$. Hence and by the Schwarz inequality, the observation that $R^K = R^{K^T}$, $K! = K^T!$, and (1.2), we conclude that

$$\begin{aligned} \|P_R f\|_{L^2(\mu)}^2 &\leq \sum_{n \in \mathbb{N}_0^d} \left(\sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{n!}{K!} \bar{R}^K \right) \left(\sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \frac{|K^T!}{K^T!} \bar{R}^{K^T} (f, h_{|K^T|})_\mu^2 \right) \\ &\leq \sum_{n \in \mathbb{N}_0^d} |\bar{R}|^n \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K|=n}} \bar{R}^{K^T} (f, h_{|K^T|})_\mu^2 \\ &\leq \|R\|_\infty \sum_{n \in \mathbb{N}_0^d} \sum_{\substack{K \in \mathcal{M}_d(\mathbb{N}_0) \\ |K^T|=n}} \frac{n!}{K^T!} \bar{R}^{K^T} (f, h_n)_\mu^2 \\ &= \|R\|_\infty \sum_{n \in \mathbb{N}_0^d} |\bar{R}|^n (f, h_n)_\mu^2 \leq \|R\|_\infty^2 \|f\|_{L^2(\mu)}^2. \blacksquare \end{aligned}$$

3. Applications of Gebelein’s inequality. Suppose the normalized Gaussian sequence $X = (X_i, i = 1, 2, \dots)$ of random vectors in \mathbb{R}^d is given. In particular $X_i \in N(0, I)$ for each $i \geq 1$. It is assumed that the matrices $R_{i,j} = E(X_i X_j)$ are symmetric for $i, j = 1, 2, \dots$ and satisfy the following hypothesis:

$$(1.5) \quad \|R_{i,j}\|_\infty < 1, \quad i, j = 1, 2, \dots, \quad C = \sup_{i \geq 1} \sum_{j \geq 1} \|R_{i,j}\|_\infty < \infty.$$

By the Frobenius Theorem (see [HLP]) and Theorem 1.1 we get the estimate

$$\text{Var} \left(\sum_{i=1}^n f_i(X_i) \right) \leq C \sum_{i=1}^n \text{Var}(f_i(X_i)), \quad n = 1, 2, \dots,$$

where $f_i \in L^2(\mu)$, $i = 1, 2, \dots$ (see [B], [BC], [V]). Using this inequality and adopting the methods from [B] and [BC] we obtain the following two statements:

LEMMA 1.1 (Borel–Cantelli Lemma). *Let $X = (X_n, n = 1, 2, \dots)$ be a Gaussian sequence with $X_i \in N(0, I)$ for $i \geq 1$ and suppose that X satisfies (1.5). Then for every sequence of Borel sets $(A_n, n = 1, 2, \dots)$ in \mathbb{R}^d such that*

$$\sum_{n=1}^\infty P\{X_n \in A_n\} = \infty$$

we have $P\{X_n \in A_n \text{ i.o.}\} = 1$. \blacksquare

THEOREM 1.2 (Strong Law of Large Numbers). *Let $X = (X_i, i = 1, 2, \dots)$ be a Gaussian sequence with $X_i \in N(0, I)$ for $i \geq 1$ and suppose that X*

satisfies (1.5). Then for $f \in L^1(\mu)$ we have

$$\frac{1}{n} \sum_{i=1}^n f(X_i) \xrightarrow[n \rightarrow \infty]{} Ef(X_1) \quad a.s. \quad \blacksquare$$

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