Discrete Mathematics 339 (2016) 1837–1847

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Strategic balance in graphs[☆,☆☆]

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ARTICLE INFO

Article history:
Received 23 November 2014
Accepted 4 February 2016
Available online 5 March 2016

Keywords:
Global alliance
Alliance partitioning number
Strategic balance

ABSTRACT

For a given graph G, a nonempty subset S contained in $V(G)$ is an *alliance* iff for each vertex $v \in S$ there are at least as many vertices from the closed neighbourhood of v in S as in $V(G) - S$. An alliance is *global* if it is also a dominating set of G. The *alliance partition number* of G was defined in Hedetniemi et al. (2004) to be the maximum number of sets in a partition of $V(G)$ such that each set is an alliance. Similarly, in Eroh and Gera (2008) the *global alliance partition number* is defined for global alliances, where the authors studied the problem for (binary) trees.

In the paper we introduce a new concept of *strategic balance* in graphs: for a given graph G, determine whether there is a partition of vertex set $V(G)$ into three subsets N, S and I such that both N and S are global alliances. We give a survey of its general properties, e.g., showing that a graph G has a strategic balance iff its global alliance partition number equals at least 2. We construct a linear time algorithm for solving the problem in trees (thus giving an answer to the open question stated in Eroh and Gera (2008)) and studied this problem for many classes of graphs: paths, cycles, wheels, stars, complete graphs and complete k-partite graphs. Moreover, we prove that this problem is $\mathcal{NP}$-complete for graphs with a degree bounded by 4 and state an open question regarding subcubic graphs.

1. Introduction

1.1. Problem definition

In the following we consider solely simple connected finite non-empty graphs, and we use standard notation of graph theory. Let G be a graph where $V(G)$ is a set of vertices and $E(G)$ is a set of edges. By Δ we denote the maximum degree in graph. For each vertex $v \in V(G)$ sets $N(v) = \{u \in V(G) : \{u, v\} \in E\}$ and $N[v] = N(v) \cup \{v\}$ are *open* and *closed neighbourhood* of vertex v, respectively. Similarly, for a subset $X \subset V(G)$ sets $N(X) = \bigcup_{u \in X} N(u)$ and $N[X] = N(X) \cup X$ are *open* and *closed neighbourhood* of set X, respectively.

In this paper we study the problem of strategic balance. The problem is connected to the problem of global alliance, which was introduced in [6] and [7]. For a graph G, a nonempty subset S contained in $V(G)$ is an *alliance* if for each vertex $v \in S$

☆ This project has been partially supported by Narodowe Centrum Nauki under contract DEC- 2011/02/A/ST6/00201.

☆☆ The results of this paper were presented at the Seventh Cracow Conference on Graph Theory (September 14–19, 2014, Rytro, Poland).

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http://dx.doi.org/10.1016/j.disc.2016.02.004

there are at least as many vertices from the closed neighbourhood of v in S as in $V(G) - S$, i.e. $|N[v] \cap S| \geq |N[v] - S|$. An alliance is *global* if it is also a dominating set of G, i.e. $N[S] = V$.

Definition 1. For a partition of $V(G)$ into three sets N, S, I where N and S are global alliances the pair $(\{N, S\}, I)$ is a strategic balance. If I is an empty set, the strategic balance is perfect and denoted as $\{N, S\}$.

Intuitively sets N and S can be seen as two sides of the conflict, for example **N**orth and **S**outh. Both sets are safe in terms of the alliance and they have the same global scope i.e. each global alliance has an access to the whole graph (dominating property). Therefore, there is no essential difference between sets N and S. Set I is formed by intermediate vertices which have not chosen any side of the conflict. We cannot assume that they will support or attack any side of the conflict. Sets N and S must be prepared for the worst case thus both consider members of set I as potential enemies.

The question whether there exists a strategic balance in a graph is *Strategic Balance* ($\mathcal{SB}$) problem. Similarly, we define *Perfect Strategic Balance* ($\mathcal{PSB}$) problem as the question whether there exists a perfect strategic balance in a graph.

1.2. Related problems and our contribution

The global alliance problem was naturally motivated as a model of conflict situations. Examples of problems such as alliances between people, countries or plants in botany are mentioned in [2]. The strategic balance problem models the situation when the two sides of the conflict are present.

In [7] the authors defined the *alliance partition number* of graph G to be the maximum number of sets in a partition of $V(G)$ such that each set is an alliance. Similarly, the *global alliance partition number* is defined for global alliances in [3], where the authors studied the problem in trees. By definition, a strategic balance is close to the case when the global alliance partition number equals at least 2. In fact, we proved that these problems are equivalent.

In [3] the authors studied the global alliance partition number problem in trees and observed that this number is equal to 1 or 2 in a tree. They gave partial results regarding the characterization of the trees that cannot have two disjoint global alliances and left the problem of full characterization open. In this paper we give a linear time algorithm that finds the global alliance partition number of a tree, and constructs the partition. We also solved the problem for complete graphs and complete k-partite graphs, as well as basic classes: paths, cycles, wheels and stars.

In the paper [10] the authors proved that the problem of partition into 2 global alliances is $\mathcal{NP}$-complete. We improve this result and show that this problem is $\mathcal{NP}$-complete even for graphs with $\Delta \leq 4$. The case $\Delta \leq 3$ is still open, and we conjecture that it can be solved in polynomial time.

2. Strategic balance and perfect strategic balance

The main result in this section is the equivalence between the existence of a strategic balance and a perfect strategic balance in a graph.

In the following, for the sake of notation simplicity, we define $Av \stackrel{df}{=} N[v] \cap A$. Let G be a graph.

Proposition 1. *For each $A \subseteq B \subseteq V(G)$ and for each $v \in V(G)$ we have $|Bv| \geq |Av|$.*

Proof. Since $A \subseteq B$, we have $|Bv| = |N[v] \cap B| \geq |N[v] \cap A| = |Av|$. □

Proposition 2. *For every two (global) alliances A and B in graph G we have that $A \cup B$ is a (global) alliance in G.*

Proof. Let $v \in A \cup B$, and without loss of generality let $v \in A$. By Proposition 1 we have $|(A\cup B)v| \geq |Av| \geq |(V(G)-A)v| \geq |(V(G) - (A \cup B))v|$. If A and B are the dominating sets of G, then obviously, $A \cup B$ is the dominating set of G. □

Proposition 3. *There is a perfect strategic balance in a graph G iff its global alliance partition number equals at least* 2.

Proof. If there is a perfect strategic balance in a graph, then we have two disjoint global alliances which cover the whole vertex set. Let us assume that we have a partition of vertex set $V(G)$ into global alliances $S_1, \ldots, S_k$, where $k \geq 2$. By Proposition 2 we have that $\bigcup_{i=1}^{k-1} S_i$ is the global alliance, thus $\bigcup_{i=1}^{k-1} S_i$ and S_k is the partition of $V(G)$ into global alliances. □

For a given graph G and a global alliance $N \subseteq V(G)$ the question whether there exists a subset $S' \subseteq (V(G) - N)$ such that there exists a strategic balance $(\{N', S'\}, I)$, where $N \subseteq N'$ is referred to as *Strategic Balance Opponent* ($\mathcal{SBO}$) problem. If we additionally require $I = \emptyset$, we refer to this problem as *Perfect Strategic Balance Opponent* ($\mathcal{PSBO}$) problem.

Lemma 4. *Let $N \subseteq V(G)$ be a global alliance in a graph G. For each $v \in V(G) - N$, if $|Nv| > |(V(G) - N)v|$, then there is no alliance $S \subseteq V(G) - N$ containing vertex v. Moreover, $N \cup \{v\}$ is a global alliance.*

Proof. Let $v \in V(G)-N$ and $|Nv| > |(V(G)-N)v|$. Let us assume, to the contrary, that there is an alliance $S \subseteq V(G)-N$ such that $v \in S$. Hence, by an alliance property we have $|Sv| \geq |(V(G)-S)v|$ and by Proposition 1 we have $|(V(G)-N)v| \geq |Sv|$.

Since $N \subseteq V(G)-S$, then by Proposition 1 we have $|(V(G)-S)v| \geq |Nv|$, thus we get $|(V(G)-S)v| \geq |Nv| > |(V(G)-N)v| \geq |Sv|$, a contradiction.

Let $N' = N \cup \{v\}$. Since N is a dominating set, N' is a dominating set, and since N is an alliance and $|Nv| > |(V(G)-N)v|$, for each $u \in N'$ by Proposition 1 we have $|N'u| \geq |Nu| \geq |(V(G)-N)u| \geq |(V(G)-N')u|$, thus N' is a global alliance. □

Lemma 5. *Let $N \subsetneq V(G)$ be a global alliance in a graph G and let us define $U = \{v \in V(G) - N : |Nv| > |(V(G)-N)v|\}$. We have*

(1) *$N' = N \cup U$ is a global alliance.*
(2) *If $U = \emptyset$, then $S = V(G) - N$ is an alliance.*
(3) *For each alliance $S \subseteq V(G) - N$ we have $S \subseteq V(G) - N'$.*

Proof. (1) By $N \cup U = \bigcup_{u \in U}(N \cup \{u\})$, from Lemma 4 and by Proposition 2 we have that N' is a global alliance.

(2) If $U = \emptyset$, then by definition of U, for every $v \in V(G) - N$, we have $|Nv| \leq |(V(G)-N)v|$, thus $S = V(G) - N$ is an alliance.

(3) Let $u \in U \cap S$, where $S \subseteq V(G) - N$ is an alliance. By Lemma 4 we have a contradiction. □

Lemma 6. *Let $N \subsetneq V(G)$ be a global alliance in a graph G and let us define $N_0 = N$, $U_0 = \emptyset$, $U_i = \{v \in V(G) - N_{i-1} : |N_{i-1}v| > |(V(G)-N_{i-1})v|\}$, and $N_i = N_{i-1} \cup U_i$, for $i = 1, 2, \ldots$. There is $1 \leq k \leq n$, where $n = |V(G)|$, such that $U_k = \emptyset$, and we have:*

(1) *$N_k = N_{k-1} = N_0 \cup U_1 \cup \cdots \cup U_{k-1}$ is a global alliance.*
(2) *$S = V(G) - N_{k-1}$ is an alliance or an empty set.*
(3) *For each alliance $S^* \subseteq V(G) - N$ we have $S^* \subseteq S = V(G) - N_{k-1}$.*

Proof. If $U_i \neq \emptyset$ for some $i \geq 1$, then for each $j = 1, \ldots, i$ we have $|N_j| = |N_{j-1}| + |U_j| > |N_{j-1}|$, hence we get $|N_i| > |N| + i$, thus there is $1 \leq k \leq n$ such that $U_k = \emptyset$. The thesis (1–3) follows by induction from Lemma 5(1–3). □

Theorem 1. *The problem $\mathcal{SBO}$ is equivalent to the problem $\mathcal{PSBO}$ and it can be solved in $O(m)$ time, where $m = |E(G)|$.*

Proof. Let $N \subseteq V(G)$ be a global alliance in a graph G. The decision problem $\mathcal{SBO}$ is as follows: is there a subset $S^* \subseteq (V(G) - N)$ such that there exists a strategic balance $(\{N', S^*\}, I)$, where $N \subseteq N'$? In the problem $\mathcal{PSBO}$ we additionally require that $I = \emptyset$.

Let us assume that there exists a strategic balance $(\{N', S^*\}, I)$, where $N \subseteq N'$. Since $S^* \subseteq V(G) - N' \subseteq V(G) - N$, by Lemma 6(3) we have $S^* \subseteq S = V(G) - N_{k-1}$, and by Lemma 6(2) we have that set S is an alliance. Since S^* is the dominating set of G, S is obviously the dominating set of G, thus by Lemma 6(1) we get that $\{N_{k-1}, V(G) - N_{k-1}\}$ is a perfect strategic balance.

The algorithm solving the problem $\mathcal{SBO}$ proceeds as follows: for a given global alliance $N \subsetneq V(G)$ in G by Lemma 6 we construct N_{k-1} and $V(G) - N_{k-1}$. If $V(G) - N_{k-1}$ is non-empty, then it is an alliance, hence problem $\mathcal{SBO}$ is equivalent to verifying if $V(G) - N_{k-1}$ is the dominating set of G. The time complexity of the optimized algorithm can be bounded by $O(\sum_{v \in V(G)-N} deg(v)) = O(m)$. Note that the calculations of U_i can be optimized solely by updating $|(V(G) - N_0)v| - |N_0 v|$ by -2 if there is $u \in U_j$, where $j < i$ such that $\{u, v\} \in E(G)$. □

Theorem 2. *The problem $\mathcal{SB}$ is equivalent to the problem $\mathcal{PSB}$. Moreover, if $(\{N, S\}, I)$ is a strategic balance in graph G, then there is a perfect strategic balance $\{N', S'\}$ such that $N \subseteq N'$ and $S \subseteq S'$.*

Proof. Let $(\{N, S\}, I)$ be a strategic balance in graph G, thus by Theorem 1 we have a perfect strategic balance $\{N_{k-1}, V(G) - N_{k-1}\}$, where $N_{k-1} = N \cup U_0 \cup \ldots U_{k-1}$, thus by Lemma 6(3) we have $N \subseteq N' = N_{k-1}$ and $S \subseteq S' = V(G) - N_{k-1}$. Sets N' and S' are both dominating sets of graph G as supersets of sets N and S. □

By Proposition 3 and Theorem 2 we have:

Theorem 3. *The problem $\mathcal{SB}$ is equivalent to verifying if the global alliance partition number is at least 2.* □

3. Balance in a strategic balance

One may ask how unbalanced a strategic balance can be. We can measure it as the ratio of the sizes of the bigger global alliance to the smaller one. From [5] we know that the size of the smallest global alliance in a graph is at least $\frac{\sqrt{4n+1}-1}{2}$, where n is the size of a vertex set, and this bound is tight.

Definition 2. For a given graph G we construct graph $SN(G)$ called a supernova of G by appending $\deg(v) + 1$ new pendant vertices to each vertex $v \in V(G)$ (see Fig. 1).

Lemma 7. *For each $k \geq 1$ there exists a perfect strategic balance $\{N, S\}$ in the supernova of the complete graph K_k such that $\frac{|N|}{|S|} = \frac{\sqrt{4n+1}-1}{2}$, where $n = |V(SN(K_k))|$.*

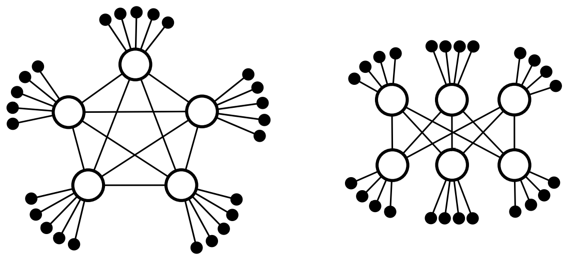

Fig. 1. Supernovas $SN(K_5)$ and $SN(K_{3,3})$.

Proof. Let $N = \{u \in V(SN(K_k)) : \deg(u) = 1\}$ and $S = V(SN(K_k)) - N$. It is easy to notice that $\{N, S\}$ is the required perfect strategic balance. □

Theorem 4. *Let $\{\{N, S\}, I\}$ be a strategic balance in a graph G and let $n = |V(G)|$. The following inequality holds:*

$$\frac{|N|}{|S|} \leq \frac{\sqrt{4n+1}-1}{2}$$

and this bound is tight.

Proof. Let $|S| = k$ and assume that $|N| \geq |S|$ (the case $|S| \geq |N|$ is trivial). By [5] we have $k \geq \frac{\sqrt{4n+1}-1}{2}$ and since $|N| \leq n-k$, we get:

$$\frac{|N|}{|S|} \leq \frac{n-k}{k} = \frac{n}{k} - 1 \leq \frac{2n}{\sqrt{4n+1}-1} - 1 = \frac{\sqrt{4n+1}-1}{2}.$$

By Lemma 7 this bound is tight for the family $\{SN(K_k)\}_{k\geq 1}$. □

Analogously, we may construct the unbalanced perfect strategic balance for bipartite graphs. The ratio of the sizes of the bigger to the smaller global alliance is equal to $\sqrt{\frac{n}{2}+1}$, where n is the number of vertices of $SN(K_{k,k})$. The tight example for this bound is the family $\{SN(K_{k,k})\}_{k\geq 1}$ of supernovas of complete bipartite graphs.

4. $\mathcal{NP}$-completeness of the $\mathcal{PSB}$ problem

In this section we will prove the $\mathcal{NP}$-completeness of the $\mathcal{PSB}$ problem for graphs with $\Delta \leq 4$, thus improving the result given by [10].

Let us denote by $\overline{3SAT}$ the restriction of the classical $3SAT$ problem, defined as follows: $\overline{3SAT}$ is the problem of the satisfiability of a given CNF formula with 2 or 3 literals in each clause and satisfying the condition that for each variable x the total number of clauses with literals x or $\neg x$ is no more than 3. Moreover, we may assume that for each variable both x and $\neg x$ appear in the formula. This problem is known to be $\mathcal{NP}$-complete [4].

Theorem 5. *The $\mathcal{PSB}$ problem is $\mathcal{NP}$-complete for graphs with $\Delta \leq 4$.*

Proof. We construct a polynomial time reduction from $\overline{3SAT}$ to the problem $\mathcal{PSB}$. For a given formula ϕ with n variables $x_1, \ldots, x_n$, such that $\phi = C_1 \wedge \ldots \wedge C_k \wedge C_{k+1} \wedge \ldots \wedge C_{k+l}$, where for $j = 1, \ldots, k$ each clause C_j contains two literals, and for $j = k+1, \ldots, k+l$ each clause C_j contains three literals, we construct a graph $G(\phi)$ with $\Delta \leq 4$ as follows:

(c1) we assign a new vertex c_j to each clause C_j, for $j = 1, \ldots, k+l$,
(c2) if literal x_i or $\neg x_i$ is in a clause C_j, then we add a new vertex $v_{i,j}$ to the graph, for $i = 1, \ldots, n$,
(c3) we join vertices c_j and $v_{i,j}$, for all legal i and j,
(c4) we join all vertices $v_{i,j}$ corresponding to the variable x_i (for $i = 1, \ldots, n$), as follows: if there are only two such vertices (i.e. variable x_i occurs twice in the formula, one as literal x_i and one as literal $\neg x_i$), we join these vertices, otherwise the variable occurs three times, and in that case we form a path with leaves corresponding to the same literal, i.e. $x_i, \neg x_i, x_i$ or $\neg x_i, x_i, \neg x_i$; next we put a new vertex on each edge of the path and obtain a path $v_{i,j_1}, a_{i,1}, v_{i,j_2}$ or $v_{i,j_1}, a_{i,1}, v_{i,j_2}, a_{i,2}, v_{i,j_3}$, respectively,
(c5) we add new vertices b_j, u_j and w_j, for $j = 1, \ldots, k+l$,
(c6) we create a cycle $b_1, w_1, b_2, w_2, \ldots, b_{k+l}, w_{k+l}, b_1$; and next, we put a new vertex on each edge of this cycle: vertex p_j on edge b_jw_j, vertex q_j on edge w_jb_{j+1}, and q_{k+l} on edge $w_{k+l}b_1$,
(c7) we join vertex u_j with vertices c_j and b_j, for $j = 1, \ldots, k+l$,
(c8) we add new vertices y_j and z_j, for $j = 1, \ldots, k$,
(c9) we add three edges $\{w_j, y_j\}, \{y_j, z_j\}, \{z_j, c_j\}$, for $j = 1, \ldots, k$.

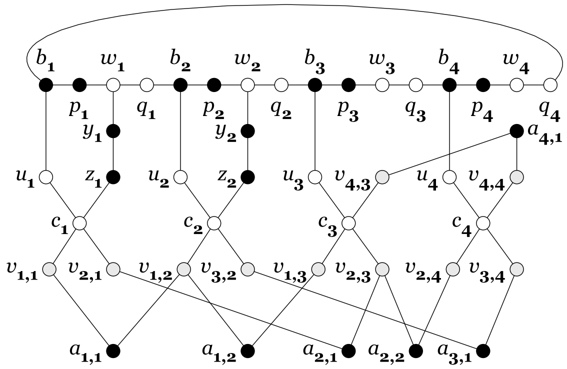

Fig. 2. Graph $G(\phi)$ corresponding to the formula $\phi = (x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4)$.

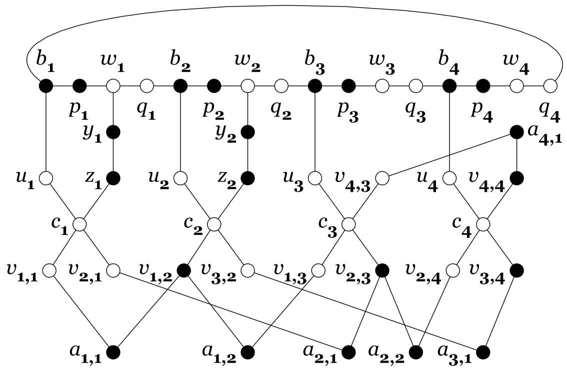

Fig. 3. B is coloured with black, and W is coloured with white. Vertices $v_{i,j}$ (literals) with the value *true* are white, and with *false* are black.

Let us observe that each vertex from $V(G(\phi)) - \{c_1, \ldots, c_{k+l}\}$ has a degree equal to 2 or 3, and since each clause C_j contains 2 or 3 literals, by (c9) we have $\deg(c_j) = 4$, thus $\Delta(G(\phi)) \leq 4$. An example graph $G(\phi)$ for a formula $\phi = (x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4)$ is shown in Fig. 2.

Now, we will show that the formula ϕ is satisfiable if and only if the graph $G(\phi)$ admits a perfect strategic balance.

($\Rightarrow$) Let us assume that the formula ϕ is satisfiable and let f be an assignment of values *true* and *false* to each variable such that $f(\phi)$ is *true*, i.e. each clause contains at least one literal x_i (or $\neg x_i$) for which $f(x_i) = true$ (or $f(\neg x_i) = true$). We construct the perfect strategic balance $\{W, B\}$ in graph G as follows:

(p1) $c_j, q_j, w_j \in W$ and $b_j, p_j, y_j, z_j \in B$, for $j = 1, \ldots, k+l$,
(p2) $a_{i,s} \in B$, for all legal i and s,
(p3) if the literal x_i is in a clause C_j, then $v_{i,j} \in \begin{cases} W & \text{if } f(x_i) = true \\ B & \text{if } f(x_i) = false \end{cases}$
(p4) if the literal $\neg x_i$ is in a clause C_j, then $v_{i,j} \in \begin{cases} W & \text{if } f(\neg x_i) = true \\ B & \text{if } f(\neg x_i) = false \end{cases}$
(p5) $u_j \in W$, for $j = 1, \ldots, k$
(p6) for $j = k+1, \ldots, k+l$, $u_j \in B$ if all other neighbours of c_j are in W (i.e. each literal in C_j is *true*), otherwise, if at least one other neighbour of c_j is in B, then $u_j \in W$.

The strategic balance for the example formula ϕ and the assignment satisfying ϕ ($f(x_1) = f(x_2) = f(x_3) = true, f(x_4) = false$) is shown in Fig. 3.

We will prove that both sets W and B are global alliances in graph $G(\phi)$.

Let us observe the following:

Claim 8. *For each vertex $v \in V(G(\phi))$ of the degree 2 or 3, if v has a neighbour from W and a neighbour from B, then an alliance property holds for v and v is dominated by both W and B.* □

Since the assignment f is legal, i.e. $f(x_i) = \neg f(\neg x_i)$, by (c4) we have that each vertex $a_{i,s}$ has one neighbour from W and one neighbour from B. Thus we may easily observe that for each vertex of degree 2 or degree 3 there is a neighbour from W and a neighbour from B, hence by Claim 8 the alliance property holds for these vertices and they are dominated by both W and B.

Since the assignment f is satisfying the formula ϕ, i.e. each clause C_j contains at least one true literal, by (p3) and (p4) each vertex c_j, for $j = 1, \ldots, k+l$, has at least one neighbour $v_{i,j}$ that is in W. Moreover, if $j \leq k$, then by (p5) we have

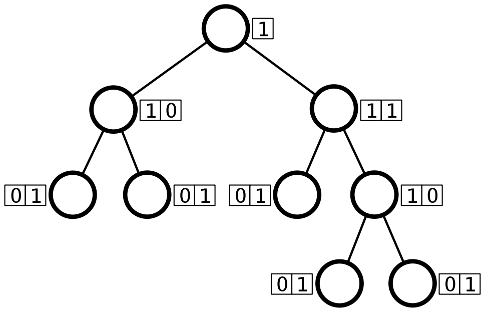

Fig. 4. A tree along with the values s and d.

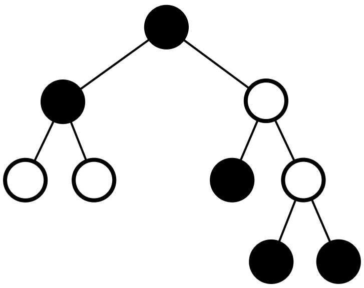

Fig. 5. Strategic balance in a tree.

$u_j \in W$ and by (p1) we have $z_j \in B$, and if $j > k$, then by (p6) we have two cases: if all neighbours of c_j other than u_j are in W, then $u_j \in B$, or if at least one neighbour of c_j other than u_j is in B, then $u_j \in W$. Hence, the vertex c_j has at least two neighbours from W and at least one neighbour from B, thus the alliance property holds for vertex c_j and it is dominated by both W and B.

($\Leftarrow$) Let as assume that there exists a perfect strategic balance $\{W, B\}$ in graph $G(\phi)$. Let us observe the following general property of vertices of degree 2 in a graph with a perfect strategic balance.

Claim 9. *For each vertex $v \in V(G(\phi))$ of degree 2, one of its neighbour belongs to W and the second one belongs to B.* □

First, let us observe that from $deg(p_j) = deg(q_j) = 2$, for $j = 1, \ldots, k + l$, by Claim 9 we have that all vertices b_j are in the same set, analogously for vertices w_j. Let us assume without loss of generality that $b_j \in B$ and $w_j \in W$, and by Claim 9 since $deg(u_j) = 2$, we have that $c_j \in W$, and since $deg(y_j) = 2$, we have that $z_j \in B$, for $j = 1, \ldots, k + l$.

From the alliance property for vertex c_j, by $deg(c_j) = 4$ and $c_j \in W$ we have $|N[c_j] \cap W| \geq 3$, hence by $z_j \in B$ we get that at least one vertex $v_{i,j}$ belongs to W. Let us define the assignment f as follows:

(f1) if the literal x_i is in clause C_j, then $f(x_j) = \begin{cases} true & \text{if } v_{i,j} \in W \\ false & \text{if } v_{i,j} \in B \end{cases}$

(f2) if the literal $\neg x_i$ is in clause C_j, then $f(x_j) = \begin{cases} false & \text{if } v_{i,j} \in W \\ true & \text{if } v_{i,j} \in B. \end{cases}$

We will prove that the assignment f is legal and satisfying the formula ϕ. From $deg(a_{i,s}) = 2$, by Claim 9 we have that both neighbours of $a_{i,s}$ are in different sets W and B. By the construction (c4) we have that one of the neighbours of $a_{i,s}$ is corresponding to x_i and the other is corresponding to $\neg x_i$, thus by (f1) and (f2) the assignment f is legal.

Since at least one vertex $v_{i,j}$ neighbouring with c_j belongs to W, if the literal x_i is in clause C_j and $v_{i,j} \in W$, then by (f1) we have $f(C_j) = f(x_i) = true$, and if the literal $\neg x_i$ is in clause C_j and $v_{i,j} \in W$, then by (f2) we have $f(C_j) = f(\neg x_i) = true$, thus the formula ϕ is satisfied. □

Note, that the distance between each two vertices of the degree 4 (i.e. vertices from $\{c_1, \ldots, c_{k+l}\}$) is equal to 4, hence these vertices form an independent set in G^3 (i.e. the third power of graph G which is the same graph with additional edges between all pairs of vertices of distance at most 3), and thus we have:

Corollary 10. *The $\mathcal{PSB}$ problem is $\mathcal{NP}$-complete for graphs G with $\Delta(G) \leq 4$ and with the property that all vertices of the degree 4 form an independent set in G^3.* □

5. Strategic balance in elementary classes

By Theorem 2 the $\mathcal{SB}$ problem is equivalent to $\mathcal{PSB}$. In this section we analyse the existence of a perfect strategic balance in elementary graph classes: paths, cycles and wheels.

Theorem 6. *There is a perfect strategic balance in a path iff its number of vertices is even.*

Proof. ($\Rightarrow$) Let $\{N, S\}$ be a perfect strategic balance in a path $v_1v_2\dots v_n$. Let us assume without loss of generality that $v_1 \in N$, hence $v_2 \in S$. Analogously to Claim 9, for every $i = 2, \dots, n-1$ we have that $v_{i-1} \in N$ and $v_{i+1} \in S$, or conversely, thus we have $v_2, v_3 \in S$, $v_4, v_5 \in N$, $v_6, v_7 \in S$, and so on and so forth. Since v_{n-1} and v_n have to be in different sets, we get that n is even.

($\Leftarrow$) Take any path $v_1v_2\dots v_{2n}$, where $n \geq 1$. We can easily define two sets as follows: $v_1 \in N$, $v_2, v_3 \in S$, $v_4, v_5 \in N$, $v_6, v_7 \in S$, and so on and so forth. If $n = 2k$, then $v_{4k-2}, v_{4k-1} \in S$, and let $v_{4k} \in N$. If $n = 2k-1$, then $v_{4k-4}, v_{4k-3} \in N$ and let $v_{4k-2} \in S$. It is easy to verify that $\{N, S\}$ is a strategic balance in the path. □

By the proof of Theorem 6 we have the following:

Corollary 11. *For each perfect strategic alliance $\{N, S\}$ of a path of n vertices, we have that both leaves of the path belong to the same partition iff $n = 4k$.* □

Theorem 7. *There is a perfect strategic balance in a cycle iff its number of vertices is divided by* 4.

Proof. ($\Rightarrow$) Let $\{N, S\}$ be a perfect strategic balance in a cycle $v_1v_2\dots v_nv_1$. We can assume without loss of generality that $v_1, v_n \in N$, otherwise, by Claim 9 we can take any two successive vertices belonging to the same set and change the numeration of vertices. By Corollary 11 we have that $n = 4k$.

($\Leftarrow$) If $n = 4k$, then we can define a perfect strategic balance analogously to the proof for paths. □

Theorem 8. *There is a perfect strategic balance in a wheel $W_n = (\{s\} \cup L, E)$ with a central vertex s and $|L| = n$ outer vertices iff $n = 4k + l$, for $k \geq 1$ and $l \in \{-1, 0, 1\}$.*

Proof. ($\Rightarrow$) Let $\{N, S\}$ be a perfect strategic balance in a wheel W_n, with n outer vertices $L = \{v_1, \dots, v_n\}$ and a central vertex s. Let us assume without loss of generality that $s \in N$ and note that graph W_n is dominated by vertex s. Since the alliance property holds for vertex s, we have $|N \cap L| + 1 = |N \cap L| + |\{s\}| = |N[s] \cap N| \geq |N[s] - N| = |N[s] \cap S| = |S \cap L|$. Consider that $C = v_1v_2\dots v_nv_1$ is a cycle of W_n.

By a *compact group of* N we mean any subset of $N \cap L$ of consecutive vertices from a cycle C (i.e. of the form $v_iv_{i+1}\dots$ or $\dots v_nv_1\dots$). Analogously, we define *a compact group of* S. Let us observe that the size (i.e. the number of vertices) of any compact group of S is at least 2, and the size of any compact group of N is at most 2. Let us define by g_1 and g_2 the number of compact groups of N of the size equal to 1 and of the size equal to 2, respectively, and let us define by h_2 and $h_{>2}$ the number of compact groups of S of the size equal to 2 and of the size greater than 2, respectively. Hence, we get $|N \cap L| = g_1 + 2g_2$ and $|S \cap L| \geq 2h_2 + 3h_{>2}$. It is easy to observe that $g_1 + g_2 = h_2 + h_{>2}$, hence we get $1 + g_1 + 2g_2 = 1 + |N \cap L| \geq |S \cap L| \geq 2h_2 + 3h_{>2} = 2g_1 + 2g_2 + h_{>2}$, thus $1 \geq g_1 + h_{>2}$. If $g_1 = 0$ and $h_{>2} = 0$, then $n = 2(g_2 + h_2) = 4g_2$, where $g_2 \geq 1$. If $g_1 = 1$ and $h_{>2} = 0$, then $n = g_1 + 2g_2 + 2h_2 = 4h_2 - 1$, where $h_2 \geq 1$. If $g_1 = 0$ and $h_{>2} = 1$, then $2h_2 + 3 = 1 + 2g_2 = 1 + |N \cap L| \geq |S \cap L| = 2h_2 + x$, where x is the size of the compact group of S of the size greater than 2, hence we get $x = 3$. Thus $n = 2g_2 + 2h_2 + x = 4g_2 + 1$, where $g_2 \geq 1$.

($\Leftarrow$) Take any wheel W_n such that $n = 4k + l$, where $k \geq 1$ and $l \in \{-1, 0, 1\}$. Let us define the perfect strategic balance $\{N, S\}$ as follows:

- if $n = 4k$, then $N = \{s\} \cup \{v_i : i \bmod 4 \in \{1, 2\}, i \leq 4k\}$ and $S = V(W_n) - N$,
- if $n = 4k + 1$, then $N = \{s\} \cup \{v_i : i \bmod 4 \in \{1, 2\}, i \leq 4k\}$ and $S = V(W_n) - N$,
- if $n = 4k - 1$, then $N = \{s\} \cup \{v_i : i \bmod 4 \in \{1, 2\}, i \leq 4k - 3\}$ and $S = V(W_n) - N$. □

6. Strategic balance in complete *k*-partite graphs

In this section we fully characterize the existence of a perfect strategic balance in complete k-partite graphs, for the cases: $k = 1$ (i.e. complete graphs), $k = 2$ (i.e. complete bipartite graphs) and for $k \geq 3$.

Theorem 9. *There is a perfect strategic balance in a complete graph K_n iff n is an even number.*

Proof. If n is an odd number, then every alliance must have at least $(n + 1)/2$ vertices, hence it impossible to find two disjoint alliances in K_n. If n is an even number, then any partition of vertex set into two sets of equal sizes (i.e. $n/2$) is a perfect strategic balance. □

Theorem 10. *There is a perfect strategic balance in a complete bipartite graph $K_{n,m}$ iff $n > 1$ and $m > 1$ or $n = m = 1$.*

Proof. Let us define $K_{n,m} = (V_1 \cup V_2, E)$ where V_1 and V_2 are disjoint independent sets of cardinality n and m, respectively. Since obviously there is no strategic balance in $K_{1,m}$, where $m > 1$, we have that $m = 1$, and in that case we have a trivial solution.

Let us assume that $n > 1$ and $m > 1$. Let us split the partitions V_1 into subsets S_1 and N_1, and V_2 into subsets S_2 and N_2, in such a way that $|S_1| = \lfloor n/2 \rfloor$, $|N_1| = \lceil n/2 \rceil$, $|S_2| = \lceil m/2 \rceil$ and $|N_2| = \lfloor m/2 \rfloor$, thus $\{N_1 \cup N_2, S_1 \cup S_2\}$ is a perfect strategic balance in $K_{n,m}$. □

Corollary 12. *There is a perfect strategic balance in a star S_n iff $n = 1$.* $\square$

Let $K_{n_1,\dots,n_k}$ be a complete k-partite graph ($k \geq 1$) with the vertex set $V = V_1 \cup \cdots \cup V_k$, where $|V_i| = n_i$, for $i = 1, \dots, k$. For any partition of vertex set V into $\{N, S\}$ let us denote by $N_i = N \cap V_i$ and by $S_i = S \cap V_i$, for $i = 1, \dots, k$.

Lemma 13. *For any strategic balance $\{N, S\}$ in a graph $G = K_{n_1,\dots,n_k}$, where $k \geq 2$, we have that $||N_i| - |S_i|| \leq 1$, for $i = 1, \dots, k$.*

Proof. Let us assume, to the contrary, that there is a strategic balance $\{N, S\}$ in G, and there is $i_0 \in \{1, \dots, k\}$ such that $\left|\left|N_{i_0}\right| - \left|S_{i_0}\right|\right| \geq 2$. Let us assume without loss of generality that $\left|N_{i_0}\right| - \left|S_{i_0}\right| \geq 2$. For each vertex $v \in N_{i_0}$ we have $|Nv| \geq |Sv|$, and it is equivalent to $\sum_{i\neq i_0} |S_i| - \sum_{i\neq i_0} |N_i| \leq 1$.

Take any $i_1 \neq i_0$ such that $\left|S_{i_1}\right| > \left|N_{i_1}\right|$, hence $S_{i_1} \neq \emptyset$. For each $v \in S_{i_1}$ we have $|Sv| = 1 + \sum_{i\neq i_1} |S_i| = 1 + \sum_{i\neq i_0} |S_i| - \left|S_{i_1}\right| + \left|S_{i_0}\right|$ and $|Nv| = \sum_{i\neq i_1} |N_i| = \sum_{i\neq i_0} |N_i| - \left|N_{i_1}\right| + \left|N_{i_0}\right|$, hence we have $|Sv| - |Nv| = 1 + (\sum_{i\neq i_0} |S_i| - \sum_{i\neq i_0} |N_i|) - (\left|S_{i_1}\right| - \left|N_{i_1}\right|) + (\left|S_{i_0}\right| - \left|N_{i_0}\right|) < 0$, thus S is not an alliance, a contradiction. As a consequence, we have $|S_i| \leq |N_i|$ for each $i \neq i_0$. Since S is dominating set and $|N_{i_0}| > 0$, there is $i_1 \neq i_0$ such that $S_{i_1} \neq \emptyset$. For each $v \in S_{i_1}$ we have $|Sv| = 1 + \left|S_{i_0}\right| + \sum_{i\notin\{i_0,i_1\}} |S_i| < \left|N_{i_0}\right| + \sum_{i\notin\{i_0,i_1\}} |N_i| = |Nv|$, thus S is not an alliance, a contradiction. $\square$

By Lemma 13 we have:

Lemma 14. *Let $\{N, S\}$ be a perfect strategic balance in a graph $K_{n_1,\dots,n_k}$, where $k \geq 2$. For every $1 \leq j \leq k$ such that $|V_j| > 1$, we have that $N_j \neq \emptyset$ and $S_j \neq \emptyset$.* $\square$

Theorem 11. *For every $k \geq 3$ there is a perfect strategic balance in a complete k-partite graph $G = K_{n_1,\dots,n_k}$ iff the number of partitions of odd cardinality is even or equal to 1.*

Proof. If for every $1 \leq i \leq k$ we have that $n_i = |V_i|$ is an even number, then split each V_i into two subsets N_i and S_i of equal cardinality. Obviously, $\{N_1 \cup \cdots \cup N_k, S_1 \cup \cdots \cup S_k\}$ is a perfect strategic balance.

If there is only one set V_i such that $|V_i|$ is an odd number, then we can split every even partition (i.e. partition of even cardinality) V_j into two subsets N_j and S_j of equal cardinality, and split an odd partition into two subsets such that one of them is equal to $(|V_i| - 1)/2$. Obviously, this is a perfect strategic balance.

Let us assume that the number of partitions of odd cardinality is even, and without loss of generality let $V_1, \dots, V_{2r}$ be set of all partitions of odd cardinality, thus $V_{2r+1}, \dots, V_k$ are even partitions. Now, we split each even partition V_j, for $j = 2r + 1, \dots, k$, into two subsets N_j and S_j of equal cardinality, and each odd partition V_j, for $j = 1, \dots, 2r$, we split into two subsets N_j and S_j such that $|S_j| = |N_j| + 1$, for $j = 1, \dots, r$ and $|S_j| + 1 = |N_j|$, for $j = r + 1, \dots, 2r$. Obviously, $\{N_1 \cup \cdots \cup N_k, S_1 \cup \cdots \cup S_k\}$ is a perfect strategic balance.

Now, let us assume that the number p of odd partitions is odd and greater than 1, and let $\{N, S\}$ be a perfect strategic balance in G. By Lemma 13 we have $|N_i| = |S_i|$ for every even partition, and $||N_i| - |S_i|| \leq 1$ for every odd partition. Since p is odd, we can assume without loss of generality that $q = |i : i \in \{1, \dots, k\} \wedge |S_i| > |N_i|| > p/2$. If $q = p$, then obviously for each $v \in N$ the alliance property does not hold. If $q < p$, then there is an odd partition such that $|S_i| \leq |N_i|$, thus for $v \in N_i$ the alliance property does not hold, thus we get a contradiction. $\square$

7. Polynomial time algorithm for the $\mathcal{PSB}$ problem in trees

In this section we present a dynamic programming algorithm for the $\mathcal{PSB}$ problem in trees which runs in $\mathcal{O}(n)$-time, where n is the size of a vertex set of a tree. In [3] the authors studied the global alliance partition number problem for trees and they observed that this number is equal to 1 or 2 in a tree. They gave some partial results regarding the characterization of the trees that cannot have two disjoint global alliances and left the problem of full characterization open. In this section we give a polynomial time algorithm that finds the global alliance partition number of a tree, and constructs the partition.

7.1. A dynamic algorithm for the $\mathcal{PSB}$ problem in trees

To construct the strategic balance (or give a negative answer to the $\mathcal{PSB}$ problem) we use the bottom-up-bottom technique in accordance to the defined orientation of T. First, we orient all edges of T in an *in-tree* manner with a root, i.e. we choose any vertex r as root and orient all edges of the tree T towards the root r. We start from the leafs to assign two bits of information to each vertex of the tree. If the construction is possible we start from the root r and we construct the $\mathcal{PSB}$ using assigned bits of information.

By T_v we denote a subtree of T rooted at v and consisting of all (oriented) edges that lead to the vertex v. For each vertex $v \in V(T) - \{r\}$ there is exactly one oriented edge outcoming from a vertex v towards r, let us denote this edge by $e_v = \{v, r_v\}$. For each vertex $v \in V(T) - \{r\}$ we define T_v^* to be a tree T_v with an attached edge e_v, i.e. $T_v^* = (V(T_v) \cup \{r_v\}, E(T_v) \cup \{e_v\})$ rooted at vertex r_v. By $C(v)$ we denote the set of all children (vertices) of vertex v i.e. $C(r)$ is a set of all vertices adjacent to r and for each $v \in V(T) - \{r\}$ set $C(v)$ consist of all vertices adjacent to v and different from r_v.

For a tree T rooted at vertex r an *almost perfect strategic balance* is a partition of $V(T)$ into two sets, such that one of them is a global alliance and the second one is an alliance, which dominates $V(T) - \{r\}$.

We assign two bits of information to each vertex $v \in V(T) - \{r\}$:

- $s_v = 1$ iff there exists an almost perfect strategic balance in the subtree T_v^*, such that v and r_v are in the same alliance,
- $d_v = 1$ iff there exists an almost perfect strategic balance in the subtree T_v^*, such that v and r_v are in different alliances.

For each leaf $l \in V(T) - \{r\}$ we have $s_l = 0$ and $d_l = 1$. For each $v \in V(T)$ and $i, j \in \{0, 1\}$ we define $C_v^{ij} = \{w \in C(v) : s_w = i \text{ and } d_w = j\}$.

Proposition 15. *For each $v \in V(T) - \{r\}$ which is not a leaf and whose all children have assigned values s and d we have:*

1. $s_v = 1$ *iff* $C_v^{00} = \emptyset$ *and* $C_v^{01} \cup C_v^{11} \neq \emptyset$ *and* $2 + \left|C_v^{10} \cup C_v^{11}\right| \geq \left|C_v^{01}\right|$,
2. $d_v = 1$ *iff* $C_v^{00} = \emptyset$ *and* $\left|C_v^{10} \cup C_v^{11}\right| \geq \left|C_v^{01}\right|$.

Proof. Proof of the statement (1):

($\Rightarrow$) Let $\{N, S\}$ be an almost perfect strategic balance in the subtree T_v^*. We can assume without loss of generality that $v \in N$ and $r_v \in N$. Vertex v is not a leaf thus $C(v) \neq \emptyset$. For each child $c \in C(v)$ we consider subtree T_c^*. The pair $\{V(T_c^*) \cap N, V(T_c^*) \cap S\}$ is an almost perfect strategic balance in the subtree T_c^*. Therefore, at least one of the values s_c and d_c is equal to 1, thus we have that $C_v^{00} = \emptyset$. Set S dominates each vertex except for the vertex r_v thus there exist a vertex $c \in C(v)$ for which $d_c = 1$, therefore, $C_v^{01} \neq \emptyset$ or $C_v^{11} \neq \emptyset$. Finally, by the alliance condition for the vertex v we have $2 + \left|C_v^{10} \cup C_v^{11}\right| \geq 2 + |C(v) \cap N| \geq |C(v) \cap S| \geq \left|C_v^{01}\right|$.

($\Leftarrow$) We have $C_v^{00} = \emptyset$, thus for each vertex $c \in C(v)$ there is an almost perfect strategic balance in the subtree T_c^*. We construct an almost perfect strategic balance $\{N, S\}$ in the following way:

1. $v, r_v \in N$,
2. $C_v^{01} \subseteq S$,
3. $C_v^{10} \subseteq N$,
4. if $C_v^{01} = \emptyset$, then $\left|C_v^{11} \cap S\right| = 1$ and $\left|C_v^{11} \cap N\right| = \left|C_v^{11}\right| - 1$,
5. if $C_v^{01} \neq \emptyset$, then $C_v^{11} \subseteq N$,
6. for each vertex $c \in C(v) \cap N$ which is not a leaf we can repeat presented steps,
7. for each vertex $c \in C(v) \cap S$ which is not a leaf we can apply the steps defined for assigning the value d_c (see proof of the statement (2) with swapped N and S).

The alliance condition holds for vertex r_v because $2 = |N[r_v] \cap N| \geq |N[r_v] - N| = 0$. By the construction if $C_v^{01} = \emptyset$, then $|N[v] \cap N| = 2 + \left|C_v^{10} \cup C_v^{11}\right| - 1 \geq 1 = |N[v] - N|$, thus the alliance condition for the vertex v holds. Similarly, if $C_v^{01} \neq \emptyset$, then $|N[v] \cap N| = 2 + \left|C_v^{10} \cup C_v^{11}\right| \geq \left|C_v^{01}\right| = |N[v] - N|$, thus again the alliance condition for the vertex v holds. By the 4th or the 5th step of the construction we have that vertex v is dominated by the set S. We have $C_v^{00} = \emptyset$ and by the 3rd, 4th and 5th step of the construction we have that if $c \in C(v) \cap N$, then $s_c = 1$ and by the induction vertex c is dominated by the set S and the alliance condition for this vertex holds. Similarly, if $c \in C(v) \cap S$, then $d_c = 1$ and we can use corresponding part of the proof for assigning d value to confirm that vertex c is dominated by the set N and the alliance condition for this vertex holds. By the induction, for each vertex $u \in V(T_v^*) - \{\{r_v\} \cup \{v\} \cup C(v)\}$ which is not a leaf we have that u is dominated by both sets N and S and the alliance property for this vertex holds. Finally, for each leaf other than the root r_v we use the 2nd step of the construction, thus we have dominating property for both sets N and S as well as the alliance condition. Therefore, $\{N, S\}$ is an almost perfect strategic balance.

Proof of the statement (2):

($\Rightarrow$) Let $\{N, S\}$ be an almost perfect strategic balance in the subtree T_v^*. We can assume without loss of generality that $v \in N$ and $r_v \in S$. Vertex v is not a leaf thus $C(v) \neq \emptyset$. For each child $c \in C(v)$ we consider subtree T_c^*. The pair $\{V(T_c^*) \cap N, V(T_c^*) \cap S\}$ is an almost perfect strategic balance in the subtree T_c^*. Therefore, at least one of the values s_c and d_c is equal to 1, thus we have that $C_v^{00} = \emptyset$. By the alliance condition for the vertex v we have $\left|C_v^{10} \cup C_v^{11}\right| \geq |C(v) \cap N| \geq |C(v) \cap S| \geq \left|C_v^{01}\right|$.

($\Leftarrow$) We have $C_v^{00} = \emptyset$, thus for each vertex $c \in C(v)$ there is an almost perfect strategic balance in the subtree T_c^*. We construct an almost perfect strategic balance $\{N, S\}$ in the following way:

1. $v \in N, r_v \in S$,
2. $C_v^{01} \subseteq S$,
3. $C_v^{10} \cup C_v^{11} \subseteq N$,
4. for each vertex $c \in C(v) \cap S$ which is not a leaf we can repeat presented steps (with swapped N and S),
5. for each vertex $c \in C(v) \cap N$ which is not a leaf we can apply the steps defined for assigning the value s_c (see proof of the statement (1)).

The alliance condition holds for vertex r_v because $1 = |N[r_v] \cap S| = |N[r_v] - S| = 1$. Moreover vertex r_v is dominated by both sets N and S. By the construction $|N[v] \cap N| = 1 + \left|C_v^{10} \cup C_v^{11}\right| \geq 1 + \left|C_v^{10}\right| = |N[v] - N|$, thus the alliance condition for the vertex v holds. Vertex v is dominated by the set S as $r_v \in S$. We have $C_v^{00} = \emptyset$ and by the 2nd and 3rd step of the construction we have that if $c \in C(v) \cap S$, then $d_c = 1$ and by the induction vertex c is dominated by the set N and the alliance condition for this vertex holds. Similarly, if $c \in C(v) \cap N$, then $s_c = 1$ and we can use corresponding part of the proof for assigning s value to confirm that vertex c is dominated by the set S and the alliance condition for this vertex holds. By the induction, for each vertex $u \in V(T_v^*) - \{r_v\} \cup \{v\} \cup C(v)$ which is not a leaf we have that u is dominated by both sets N and S and the alliance property for this vertex holds. Finally, for each leaf other than the root r_v we use the 2nd step of the construction, thus we have dominating property for both sets N and S as well as the alliance condition. Therefore, $\{N, S\}$ is an almost perfect strategic balance. □

Proposition 16. *A perfect strategic balance in a tree T exists iff $C_r^{00} = \emptyset$ and $C_r^{01} \cup C_r^{11} \neq \emptyset$ and $1 + \left|C_r^{10} \cup C_r^{11}\right| \geq \left|C_r^{01}\right|$,*

Proof. ($\Rightarrow$) Let $\{N, S\}$ be a perfect strategic balance in the tree T rooted at vertex r. We can assume without loss of generality that $r \in N$. Set S dominates the root r thus $C(v) \neq \emptyset$. For each child $c \in C(r)$ we consider subtree T_c^*. The pair $\{V(T_c^*) \cap N, V(T_c^*) \cap S\}$ is an almost perfect strategic balance in the subtree T_c^*. Therefore, at least one of the values s_c and d_c is equal to 1, thus we have that $C_r^{00} = \emptyset$. As the set S dominates the root r there exist a vertex $c \in C(r)$ for which $d_c = 1$, therefore, $C_r^{01} \neq \emptyset$ or $C_r^{11} \neq \emptyset$. Finally, by the alliance condition for the root r we have $1 + \left|C_r^{10} \cup C_r^{11}\right| \geq 1 + |C(r) \cap N| \geq |C(r) \cap S| \geq \left|C_r^{01}\right|$.

($\Leftarrow$) We have $C_r^{00} = \emptyset$, thus for each vertex $c \in C(r)$ there is an almost perfect strategic balance in the subtree T_c^*. We construct a perfect strategic balance $\{N, S\}$ in the following way:

1. $r \in N$,
2. $C_r^{01} \subseteq S$,
3. $C_r^{10} \subseteq N$,
4. if $C_r^{01} = \emptyset$, then $\left|C_r^{11} \cap S\right| = 1$ and $\left|C_r^{11} \cap N\right| = \left|C_r^{11}\right| - 1$,
5. if $C_r^{01} \neq \emptyset$, then $C_r^{11} \subseteq N$,
6. for each vertex $c \in C(r) \cap N$ which is not a leaf we can apply the steps defined for assigning the value s_c in the proof of the statement (1) in the Proposition 15,
7. for each vertex $c \in C(r) \cap S$ which is not a leaf we can apply the steps defined for assigning the value d_c in the proof of the statement (2) in the Proposition 15 (with swapped N and S).

It is enough to verify the domination and the alliance property for the root r. By the 4th or 5th step of the construction we have that set S dominates the root r. By the 2nd, 3rd, 4th and 5th step of the construction if $C_r^{01} = \emptyset$, then $|N[r] \cap N| = 1 + \left|C_r^{10} \cup C_r^{11}\right| - 1 \geq 1 = |N[r] - N|$, thus the alliance condition for the root r holds. Similarly, if $C_r^{01} \neq \emptyset$, then $|N[r] \cap N| = 1 + \left|C_r^{10} \cup C_r^{11}\right| \geq \left|C_r^{01}\right| = |N[v] - N|$, thus again the alliance condition for the root r holds. For now we can follow the proof of the Proposition 15 to verify that $\{N, S\}$ is a perfect strategic balance. □

7.2. *Construction of the strategic balance*

If the values s and d are set on every vertex and there exists a perfect strategic balance, it is easy to make a construction. We start from the root r with any assignment and dominate it by one of its children (we prefer the one from the set C_r^{01}). From the rest of the children we greedily make as many children, in the same alliance as parent, as possible. For any other vertex v which is not a leaf we have two options. If its parent is in a different alliance, again, if possible, we greedily make the children of v alliances. If the parent of v is in the same alliance as v, first, we dominate it (again we prefer vertex from set C_v^{01}) and then, again, we greedily make as many children, in the same alliance as parent, as possible. For each leaf l we choose an opposite alliance to its parent. See Figs. 4 and 5 as an example of the construction.

Since for each vertex v we calculate the values s and d based on its children's values, and there is at most $\deg(v)$ neighbours for each vertex, we get that the complexity of this algorithm can be bounded by $O(\sum_{v \in V(T)} deg(v)) = O(n)$.

8. Final remarks and open problems

In the paper we introduced the strategic balance concept and gave a survey of its general properties, e.g., showing that a graph has a strategic balance iff its global alliance partition number equals at least 2, which means that every strategic balance can be extended to a perfect strategic balance, i.e. partition into two disjoint global alliances. We constructed an algorithm for solving the problem for trees working in time $\mathcal{O}(n)$, thus giving an answer to the open question stated in [3], and studied this problem for certain common classes of graphs: paths, cycles, wheels, stars, complete graphs and complete k-partite graphs. Moreover, we proved that this problem is $\mathcal{NP}$-complete for graphs with a degree bounded by 4.

8.1. Strategic balance in cubic graphs

Based on Theorem 5, we know that the problem of verifying the existence of strategic balance in graphs is $\mathcal{NP}$-complete for graphs with $\Delta \leq 4$. We considered the problem for subcubic and cubic graphs. The problem of the existence of a perfect strategic balance in a cubic graph is equivalent to the existence of two disjoint total dominating sets in this graph.

Proposition 17. *Let G be a cubic graph. The partition $\{N, S\}$ is a perfect strategic balance in G iff N and S are disjoint total dominating sets of G.* $\square$

We have conducted numerical calculations to verify how many cubic graphs do not have a strategic balance. We used the nauty package [9] to generate all connected cubic graphs up to 20 vertices, and we found only one cubic graph which has no strategic balance. The graph in question is the so-called Heawood graph [1], which has 14 vertices and is obtained from the Fano plane as the graph of incidences between points and lines in that geometry.

In [8] Henning and Southey showed that every cubic graph has a dominating set and a total dominating set which are disjoint. However, the full characterization of cubic and subcubic graphs which have no partition into two total dominating sets remains still open and appears to be a very challenging problem.

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