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MULTIPLE SOLUTIONS TO THIRD-ORDER DIFFERENTIAL EQUATIONS WITH DERIVATIVE DEPENDENCE AND DEVIATING ARGUMENTS

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Abstract: In this paper, we give some new results for multiplicity of positive (nonnegative) solutions for third-order differential equations with derivative dependence, deviating arguments and Stieltjes integral boundary conditions. We discuss our problem with advanced argument α and arbitrary $\beta \in C([0,1],[0,1])$, see problem (2). It means that argument β can change the character on $[0,1]$, so β can be delayed in some set $\bar{J} \subset [0,1]$ and advanced in $[0,1] \setminus \bar{J}$. Four examples illustrate the main results.

Keywords: boundary value problems with deviating arguments, Stieltjes integral boundary conditions, cone, existence of positive solutions, fixed point theorem

1. Introduction

Put $J = [0,1]$, $\mathbb{R}_+ = [0,\infty)$ and

$$\mathcal{F}x(t) = f\big(t, x(\alpha(t)), x'(\beta(t))\big) \tag{1}$$

In this paper, we shall study the existence of multiple nonnegative solutions to the following problems:

$$\begin{cases} x'''(t) + h(t)\mathcal{F}x(t) = 0, & t \in J_0 \equiv (0,1) \\ x(0) = 0, \quad x'(1) = \lambda_1[x], & x'(0) = \lambda_2[x] \end{cases} \tag{2}$$

where λ_1 and λ_2 denote linear functionals on $C(J,\mathbb{R})$ given by:

$$\lambda_1[x] = \int_0^1 x(t)\,d\mathcal{A}(t), \quad \lambda_2[x] = \int_0^1 x(t)\,d\mathcal{B}(t) \tag{3}$$

involving Stieltjes integrals with suitable functions $\mathcal{A}$ and $\mathcal{B}$ of bounded variation on J. It is important to indicate that it is not assumed that $\lambda_1[x]$ and $\lambda_2[x]$

are positive to all positive x. The measures $d\mathcal{A}$, $d\mathcal{B}$ can be signed measures, see Examples 1–4.

Recently, the existence of multiple positive solutions for differential equations has been studied extensively, for details, see for example [1−36]. In many papers, problems without deviating arguments have been investigated, see for example, [1−7, 13, 17−19, 22, 24, 25, 27−34]. Problems with derivative dependence have been discussed for example in papers: [4, 15−18, 22, 24, 27, 33, 34]. From the list [1−36], only papers [8−12, 14−16, 23, 26, 35] concern positive solutions to problems with deviating arguments. Note that Boundary Conditions (BCs) in differential problems have important influence on the existence of the results obtained. It is important to indicate that Stieltjes BCs are quite general since they can include both sums and integrals, so multipoint BCs and integral BCs are special cases of Stieltjes BCs. Differential equations with Stieltjes BCs have been discussed in some papers, see for example [6, 7, 13−16, 18, 28−32]. It is worth mentioning that occurring measures in Stieltjes BCs can be signed.

A standard approach in studying the existence of positive (nonnegative) solutions of boundary value problems for differential equations is to find the corresponding Green's function G and rewrite this problem as an equivalent fixed-point problem for a Hammerstein integral operator T of the form

$$Tx(t) = \int_0^1 G(t,s) f\big(s, x(s)\big)\, ds \tag{4}$$

in the cone $P = \{x \in C[0,1]:\ x \geq 0\}$. When seeking multiple positive solutions it is convenient to work with a smaller cone, namely

$$\min_{[\xi,\bar{\xi}]} \big|x(t)\big| \geq \bar{\rho} \max_{t \in J} \big|x(t)\big| \tag{5}$$

with $[\xi,\bar{\xi}] \subset [0,1]$ and a constant $\bar{\rho} \in (0,1]$. Usually, we need to find a nonnegative function κ and a constant $\bar{\rho} \in (0,1]$ such that $G(t,s) \leq \kappa(s)$ for $t,s \in J$; and $G(t,s) \geq \bar{\rho}\kappa(t)$ for $t \in [\eta,\bar{\eta}] \subset [0,1]$ and $s \in J$ (see for example [28−31]) to work with inequality (5). When we have problems with delayed or advanced arguments, then, instead of $[\xi,\bar{\xi}]$ we have to use interval $[0,\eta] \subset [0,1)$ or $[\eta,1] \subset (0,1]$, respectively. It shows that the approach from papers [28−31] needs some modification to problems with delayed or advanced arguments. For problems with advanced arguments, when $\alpha(t) \geq t$, we have to find a constant $\rho \in (0,1)$ to work with the inequality

$$\min_{[\eta,1]} \big|x(t)\big| \geq \rho \max_{t \in J} \big|x(t)\big| \tag{6}$$

with a fixed $\eta \in (0,1)$.

Problems with advanced arguments α have been investigated, for example, in papers [8−12, 15, 16, 21, 26, 35] to second-order differential equations with corresponding BCs, and in [14] to third-order differential equations. In [15], the Stieltjes BCs had the form:

$$x(0)=\lambda_1[x], \quad x(1)=\xi x(\delta)+\lambda_2[x], \quad \xi>0,\ \delta\in(0,1) \tag{7}$$

with the assumption $0<\xi+\lambda_2[p]<1$ for $p(t)=1$; while in paper [14],

$$x(0)=x''(0)=0, \quad x(1)=\xi x(\delta)+\lambda_1[x] \tag{8}$$

with a similar assumption $0<\xi\delta+\lambda_1[p]<1$, $\xi>0$, $\delta\in(0,1)$ for $p(t)=1$. It means that in [15] the terms $\xi x(\delta)$ and $\lambda_2[x]$ are separated with $\xi>0$, so $\xi x(\delta)$ cannot be included in $\lambda_2[x]$ while if $\alpha(t)=t$, then, someone can put $\xi=0$ which means that in the case without advanced arguments the term $\xi x(\delta)$ could be included in $\lambda_2[x]$. There is a similar situation in [14]. It is important to mention that in Stieltjes BCs of problem (2) the term of the type $\xi x(\delta)$ can be included in $\lambda_1[x]$ or $\lambda_2[x]$ to obtain new results, so this situation is more general then, discussed in [14].

Motivated by [28–31], in this paper, we apply the fixed point theorem due to Avery-Peterson to obtain sufficient conditions for the existence of multiple positive solutions to problems of type (2) with advanced arguments α. Function f appearing in problem (2) depends also on argument $\beta\in C(J,J)$ which can be both of an advanced, delayed or mixed type, too. Function f depends also on the first order derivative. It is important to indicate that problems of type (2) have been discussed with signed measures of dA,dB appearing in Stieltjes integrals of functionals λ_1,λ_2. Exactly speaking, BCs in problem (2) with functionals λ_1, λ_2 cover some nonlocal BCs, for example:

$$\lambda_1[x]=\mu_1 x(\bar{\xi}), \quad \lambda_2[x]=\mu_2 x(\bar{\eta}), \quad \mu_1,\mu_2\ge 0,\ \bar{\xi},\bar{\eta}\in(0,1) \tag{9}$$

$$\begin{cases} \lambda_1[x]=\sum_{i=1}^{m} a_i x(\xi_i), \quad \lambda_2[x]=\sum_{j=1}^{r} b_j x(\eta_i) \\ 0<\xi_1<\cdots<\xi_m<1, \quad 0<\eta_1<\cdots<\eta_r<1 \end{cases} \tag{10}$$

$$\lambda_1[x]=\int_0^1 g_1(t)x(t)dt, \quad \lambda_2[x]=\int_0^1 g_2(t)x(t)dt \tag{11}$$

for some constants a_i,b_i and some functions g_1,g_2. In our paper, the assumption, that the measures $d\mathcal{A},d\mathcal{B}$ in the definitions of λ_1,λ_2 are positive, is not needed. More precisely, one needs to choose the above constants a_i, b_j and functions g_1, g_2 in such a way that Assumption H_1 holds. It means that some coefficients of a_i, b_j can be negative, and functions g_1, g_2 can change sign on J, see Examples [1–4].

The organization of this paper is as follows. In Section 2, we present some necessary lemmas connected with our main results. In Section 3, we first present some definitions and a theorem of Avery and Peterson which is useful in our research. Also in Section 3, we discuss the existence of multiple positive (nonnegative) solutions to problems with advanced argument α, by using the above mentioned Avery-Peterson theorem. Four examples are given in the next section to verify theoretical results.

2. Some lemmas

Consider the following system:

$$\begin{cases} u'''(t)+z(t)=0, & t\in J_0 \\ u(0)=0, \quad u'(1)=\lambda_1[u], & u'(0)=\lambda_2[u], \end{cases} \tag{12}$$

where $z\in L^1(J,\mathbb{R})$.

Let us introduce Assumption:

H_0 : $\mathcal{A}$ and $\mathcal{B}$ are functions of bounded variation and

$$\Delta\equiv(1-B_1)(1-A_2)+B_2(1-A_1)\neq 0 \tag{13}$$

for

$$\begin{aligned} A_1&=\int_0^1 t\,d\mathcal{A}(t), \quad A_2=\frac{1}{2}\int_0^1 t^2 d\mathcal{A}(t), \quad \mathcal{L}_1(s)=\int_0^1 L(t,s)d\mathcal{A}(t),\\ B_1&=\int_0^1 t\,d\mathcal{B}(t), \quad B_2=\frac{1}{2}\int_0^1 t^2 d\mathcal{B}(t), \quad \mathcal{L}_2(s)=\int_0^1 L(t,s)d\mathcal{B}(t) \end{aligned} \tag{14}$$

with

$$L(t,s)=\frac{1}{2}\begin{cases} s[t(1-t)+t-s], & 0\le s\le t\\ (1-s)t^2, & t\le s\le 1 \end{cases} \tag{15}$$

We require the following result.

Lemma 1. *Let Assumption H_0 hold and let $z\in L^1(J,\mathbb{R})$. Then, problem (12) has a unique solution u given by*

$$u(t)=\int_0^1 \mathcal{L}(t,s)z(s)ds \tag{16}$$

with

$$\begin{aligned} \mathcal{L}(t,s)=&\frac{1}{\Delta}\left[\frac{t^2}{2}(1-B_1)+tB_2\right]\mathcal{L}_1(s)\\ &+\frac{1}{\Delta}\left[\frac{t^2}{2}(A_1-1)+t(1-A_2)\right]\mathcal{L}_2(s)+L(t,s) \end{aligned} \tag{17}$$

Proof. Integrating the differential equation for u in (12) three times from t to 1, we have

$$u(t)=C_1(1-t)^2+C_2(1-t)+C_3+\frac{1}{2}\int_t^1(s-t)^2 z(s)ds, \quad C_1,C_2,C_3\in\mathbb{R} \tag{18}$$

Using the BC: on $u'(1)$ we see that $C_2=-\lambda_1[u]$, so

$$u(t)=C_1(1-t)^2-(1-t)\lambda_1[u]+C_3+\frac{1}{2}\int_t^1(s-t)^2 z(s)ds \tag{19}$$

Moreover, using BC: $u'(0)=\lambda_2[u]$, we see that

$$C_1=\frac{1}{2}\left(\lambda_1[u]-\lambda_2[u]-\int_0^1 sz(s)ds\right) \tag{20}$$

so

$$u(t)=\left(t-1+\frac{(1-t)^2}{2}\right)\lambda_1[u]-\frac{(1-t)^2}{2}\lambda_2[u]+C_3-\int_0^1 G(t,s)z(s)ds \tag{21}$$

with

$$G(t,s)=\frac{1}{2}\begin{cases} s(1-t)^2, & 0\le s\le t\\ (1-s)(s-t^2), & t\le s\le 1\end{cases} \tag{22}$$

Now using the condition: $u(0)=0$, we see that

$$C_3=\frac{1}{2}\lambda_1[u]+\frac{1}{2}\lambda_2[u]+\int_0^1 G(0,s)z(s)ds \tag{23}$$

so

$$u(t)=\frac{1}{2}t^2\lambda_1[u]+\frac{1}{2}\left(2t-t^2\right)\lambda_2[u]+\int_0^1 L(t,s)z(s)ds. \tag{24}$$

Now, we have to eliminate $\lambda_1[u],\lambda_2[v]$ from (24). If u is a solution of (24), then,

$$\begin{cases} \lambda_1[u]=A_2\lambda_1[u]+(A_1-A_2)\lambda_2[u]+\displaystyle\int_0^1 \mathcal{L}_1(s)z(s)ds\\ \lambda_2[u]=B_2\lambda_1[u]+(B_1-B_2)\lambda_2[u]+\displaystyle\int_0^1 \mathcal{L}_2(s)z(s)ds\end{cases} \tag{25}$$

Solving this system with respect to $\lambda_1[u],\lambda_2[v]$ and then, substituting to (24) we have the conclusion of this lemma. This ends the proof. ■

Remark 1. Note that $\mathcal{L}$ is the Green function of problem (2).

Lemma 2. *Function L has the following properties:*

(i) L *is continuous for* $t,s\in J$ *and* $L(t,s)\ge 0,\ t,s\in J$,

(ii) $\max\limits_{t\in J} L(t,s)=\frac{1}{2}s(1-s)\equiv k(s),\ s\in J$,

(iii) $\min\limits_{t\in[\eta,1]} L(t,s)\ge \eta^2 k(s),\ \eta\in(0,1),\ s\in J$.

Proof. Parts (i) and (ii) are true. Note that

$$\min_{t\in[\eta,1]} L(t,s)=\frac{1}{2}\begin{cases} s\left[\eta(1-\eta)+\eta-s\right], & s\le\eta\\ (1-s)\eta^2, & \eta\le s\le 1\end{cases} \tag{26}$$

Let

$$\rho(s)=\begin{cases} \dfrac{\eta(1-\eta)+\eta-s}{1-s}, & s\in[0,\eta]\\ \dfrac{\eta^2}{s}, & s\in[\eta,1]\end{cases} \tag{27}$$

Note that ρ is continuous on J and $\min\limits_{s\in J}\rho(s)=\eta^2$. It proves that

$$\min_{t\in[\eta,1]} L(t,s)=\rho(s)k(s)\ge\eta^2 k(s),\quad s\in J \tag{28}$$

This ends the proof. ■

Let us introduce the assumption:

H_1 : $\mathcal{A}$ and $\mathcal{B}$ are functions of bounded variation and: $\Delta>0$, $\mathcal{L}_1\ge 0$, $\mathcal{L}_2\ge 0$, $A_1\ge 1$, $0\le A_2\le 1$, $0\le B_1\le 1$, $B_2\ge 0$, where Δ, $\mathcal{L}_1$, $\mathcal{L}_2$, A_1, A_2, B_1 and B_2 are defined as in Assumption H_0.

Lemma 3. *Let Assumption H_1 hold. Then, the following relation*

$$\eta^2\mathcal{D}(s)\le\min_{t\in[\eta,1]}\mathcal{L}(t,s)\le\mathcal{D}(s),\quad s\in J \tag{29}$$

is satisfied, where

$$\mathcal{D}(s)=k(s)+\frac{1}{2\Delta}(1-B_1+2B_2)\mathcal{L}_1(s)+\frac{1}{2\Delta}(1+A_1-2A_2)\mathcal{L}_2(s) \tag{30}$$

with k defined as in Lemma 2.

Proof. Using the definition of $\mathcal{L}$ and Lemma 2, we see that

$$\max_{t\in J}\mathcal{L}(t,s)=\mathcal{D}(s),\quad t\in J \tag{31}$$

Moreover,

$$\begin{aligned}\min_{t\in[\eta,1]}\mathcal{L}(t,s)&=\frac{1}{\Delta}\left[\frac{\eta^2}{2}(1-B_1)+\eta B_2\right]\mathcal{L}_1(s)\\&\quad+\frac{1}{\Delta}\left[\frac{\eta^2}{2}(A_1-1)+\eta(1-A_2)\right]+\eta^2k(s)\\&\ge\eta^2\mathcal{D}(s)\end{aligned} \tag{32}$$

This ends the proof. ■

Let $E=C^1(J,\mathbb{R})$ with the norm $\|u\|=\max(\|u\|_0,\|u'\|_0)$, where $\|u\|_0=\max_{t\in J}|u(t)|$. Define the cone K by

$$K=\left\{\begin{array}{c}u\in E: u(t)\ge0,\ t\in J,\quad \lambda_1[u]\ge0,\quad \lambda_2[u]\ge0\\ \min_{[\eta,1]}u(t)\ge\eta^2\|u\|_0\end{array}\right\} \tag{33}$$

It is obvious that K is a cone in E.

Now we define the operator T as

$$Tu(t)=\int_0^1\mathcal{L}(t,s)h(s)\mathcal{F}u(s)ds \tag{34}$$

It is clear that the existence of a positive solution for problem (2) is equivalent to the existence of nontrivial fixed point of T.

Let us introduce the following assumptions:

H_2: $f\in C(J\times\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$, α, $\beta\in C(J,J)$,
H_3: $h\in L^1(J_0,\mathbb{R}_+)$ and h does not vanish identically on any subinterval of J_0,
H_4: $\alpha(t)\ge t$ in J.

Lemma 4. *Let Assumptions H_1–H_3 hold. Then, $T:K\to K$.*

Proof. First, it is clear that in view of Assumption H_1, the Green function $\mathcal{L}$ is nonnegative on $J\times J$, so $Tu(t)\ge0$, $t\in J$.

Note that

$$\int_0^1\mathcal{L}(t,s)d\mathcal{A}(t)=\frac{1}{\Delta}\big[(1-B_1)A_2+B_2A_1\big]\mathcal{L}_1(s)+\frac{1}{\Delta}(A_1-A_2)\mathcal{L}_2(s)+\mathcal{L}_1(s)\ge0 \tag{35}$$

by Assumption H_1. Hence $\mathcal{L}_1[Tu]\ge0$, by Assumptions H_2, H_3. By a similar way, we obtain $\mathcal{L}_2[Tu]\ge0$.

Finally, using Lemma 3, we see that

$$\min_{[\eta,1]} Tu(t) = \min_{[\eta,1]} \int_0^1 \mathcal{L}(t,s)h(s)\mathcal{F}u(s)ds \geq \eta^2 \|Tu\|_0 \tag{36}$$

It shows $T: K \to K$. This ends the proof. ■

Remark 2. Let $u \in C^1(J,\mathbb{R})$. If $u(0)=0$ or $u(1)=0$, then, $\|u\|_0 \leq \|u'\|_0$. Indeed,

$$\begin{aligned} u(t) &= \int_0^t u'(s)ds \leq \|u'\|_0 \quad \text{if } u(0)=0 \\ u(t) &= -\int_t^1 u'(s)ds \leq \|u'\|_0 \text{ if } u(1)=0 \end{aligned} \tag{37}$$

■

3. Nonnegative solutions to problem (2) with advanced arguments

Now, we present the necessary definitions from the theory of cones in Banach spaces.

Definition 1. Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $ku \in P$ for all $u \in P$ and all $k \geq 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

Definition 2. A map Φ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\Phi: P \to \mathbb{R}_+$ is continuous and

$$\Phi\big(tx + (1-t)y\big) \geq t\Phi(x) + (1-t)\Phi(y) \tag{38}$$

for all $x, y \in P$ and $t \in [0,1]$.

Similarly, we say the map φ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\varphi: P \to \mathbb{R}_+$ is continuous and

$$\varphi\big(tx + (1-t)y\big) \leq t\varphi(x) + (1-t)\varphi(y) \tag{39}$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 3. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Let φ and Θ be nonnegative continuous convex functionals on P, let Φ be a nonnegative continuous concave functional on P, and let Ψ be a nonnegative continuous functional on P. Then, for positive numbers a,b,c,d, we define the following sets:

$$\begin{aligned} P(\varphi,d) &= \{x\in P:\ \varphi(x)<d\} \\ P(\varphi,\Phi,b,d) &= \{x\in P:\ b\le\Phi(x),\ \varphi(x)\le d\} \\ P(\varphi,\Theta,\Phi,b,c,d) &= \{x\in P:\ b\le\Phi(x),\ \Theta(x)\le c,\ \varphi(x)\le d\} \\ R(\varphi,\Psi,a,d) &= \{x\in P:\ a\le\Psi(x),\ \varphi(x)\le d\} \end{aligned} \tag{40}$$

We will use the following fixed point theorem of Avery and Peterson to establish multiple positive solutions to problem (2).

Theorem 1 (see [3]). *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative continuous convex functionals on P, let Φ be a nonnegative continuous concave functional on P, and let Ψ be a nonnegative continuous functional on P satisfying $\Psi(kx)\le k\Psi(x)$ for $0\le k\le 1$, such that for some positive numbers $\overline{M}$ and d,*

$$\Phi(x)\le\Psi(x) \quad \text{and} \quad \|x\|\le\overline{M}\varphi(x) \tag{41}$$

for all $x\in\overline{P(\varphi,d)}$. Suppose

$$T:\overline{P(\varphi,d)}\to\overline{P(\varphi,d)} \tag{42}$$

is completely continuous and there exist positive numbers a,b,c with $a<b$, such that

(S_1): $\{x\in P(\varphi,\Theta,\Phi,b,c,d):\Phi(x)>b\}\ne 0$ *and* $\Phi(Tx)>b$ *for* $x\in P(\varphi,\Theta,\Phi,b,c,d)$;
(S_2): $\Phi(Tx)>b$ *for* $x\in P(\varphi,\Phi,b,d)$ *with* $\Theta(Tx)>c$,
(S_3): $0\notin R(\varphi,\Psi,a,d)$ *and* $\Psi(Tx)<a$ *for* $x\in R(\varphi,\Psi,a,d)$ *with* $\Psi(x)=a$.

Then, T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\varphi,d)}$, such that

$$\varphi(x_i)\le d, \quad \text{for} \quad i=1,2,3 \tag{43}$$

$$b<\Phi(x_1), \quad a<\Psi(x_2), \quad \text{with} \quad \Phi(x_2)<b \tag{44}$$

and

$$\Psi(x_3)<a \tag{45}$$

We apply Theorem 1 with the cone K instead of P and let $\overline{P}_r=\{x\in K: \|x\|\le r\}$. Now, we define the nonnegative continuous concave functional Φ on K by

$$\Phi(x)=\min_{[\eta,1]}|x(t)| \tag{46}$$

Note that $\Phi(x)\le\|x\|_0$. Put $\Psi(x)=\Theta(x)=\|x\|_0$, $\varphi(x)=\|x'\|_0$.

Now, we can formulate the main result of this section.

Theorem 2. *Let Assumptions H_1–H_4 hold and let $\lambda_1[p]\geq 0$, $\lambda_2[p]\geq 0$ for $p(t)=1$, $t\in J$. In addition, we assume that there exist positive constants a,b,c,d, $a<b$ and such that*

$$\mu>\max(D_3,D_4),\quad 0<\nu<\eta^2 D_4 \tag{47}$$

with

$$\begin{aligned} D_i&=\int_0^1 \mathcal{L}_i(s)h(s)ds,\quad i=1,2\\ D_3&=\frac{1}{\Delta}[(1-B_1+B_2)D_1+(A_1-A_2)D_2]+\int_0^1 h(s)ds\\ D_4&=\frac{1}{2}\int_0^1 s(1-s)h(s)ds+\frac{1}{2\Delta}(1-B_1+2B_2)D_1+\frac{1}{2\Delta}(1+A_1-2A_2)D_2\end{aligned} \tag{48}$$

and

$(A_1):$ $f(t,u,v)\leq \frac{d}{\mu}$ *for* $(t,u,v)\in J\times[0,d]\times[-d,d]$,
$(A_2):$ $f(t,u,v)\geq \dfrac{b}{\nu}$ *for* $(t,u,v)\in[\eta,1]\times[b,\frac{b}{\rho}]\times[-d,d]$ *with* $\rho=\eta^2$,
$(A_3):$ $f(t,u,v)\leq \frac{a}{\mu}$ *for* $(t,u,v)\in J\times[0,a]\times[-d,d]$.

Then, problem (2) has at least three nonnegative solutions x_1,x_2,x_3 satisfying $\|x_i'\|_0\leq d$, $i=1,2,3$,

$$b\leq\Phi(x_1),\quad a<\|x_2\|_0\quad\text{with}\quad\Phi(x_2)<b,\quad\|x_3\|_0<a \tag{49}$$

Proof. Basing on the definitions of T, we see that $T\overline{P}$ is equicontinuous on J, so T is completely continuous. In view of Remark 2, we have

$$\|x\|=\max\big(\|x\|_0,\|x'\|_0\big)=\|x'\|_0=\varphi(x) \tag{50}$$

Let $x\in\overline{P(\varphi,d)}$, so $\varphi(x)=\|x'\|_0\leq d$. By Remark 2, $\|x\|_0\leq\|x'\|_0\leq d$, so $0\leq x(t)\leq d$, $t\in J$. Assumption (A_1) implies $f\big(t,x(\alpha(t)),x'(\beta(t))\big)\leq\frac{d}{\mu}$. Moreover,

$$\begin{aligned}(Tx)'(t)&=\frac{1}{\Delta}\int_0^1\big[t(1-B_1)+B_2\big]\mathcal{L}_1(s)h(s)f\big(s,x(\alpha(s)),x'(\beta(s))\big)ds\\ &\quad+\frac{1}{\Delta}\int_0^1\big[t(A_1-1)+1-A_2\big]\mathcal{L}_2(s)h(s)f\big(s,x(\alpha(s)),x'(\beta(s))\big)ds\\ &\quad+\frac{d}{dt}\int_0^1 L(t,s)h(s)f\big(s,x(\alpha(s)),x'(\beta(s))\big)ds\\ &\leq\frac{d}{\mu}\left\{\frac{1}{\Delta}\big[(1-B_1+B_2)D_1+(A_1-A_2)D_2\big]+\int_0^1 h(s)ds\right\}\end{aligned} \tag{51}$$

for $t\in J$. This and the assumption on μ prove that

$$\varphi(Tx)=\|(Tx)'\|_0<d \tag{52}$$

This shows that $T:\overline{P(\varphi,d)}\to\overline{P(\varphi,d)}$.

Now we need to show that condition (S_1) is satisfied. Take

$$x_0(t)=\frac{1}{2}\left(b+\frac{b}{\rho}\right),\quad t\in J \text{ with } \rho=\eta^2 \tag{53}$$

Then, $x_0(t)>0$, $t\in J$, and

$$\begin{aligned}\lambda_1[x_0]&=\frac{1}{2}\left(b+\frac{b}{\rho}\right)\lambda_1[p]\geq 0\\ \lambda_2[x_0]&=\frac{1}{2}\left(b+\frac{b}{\rho}\right)\lambda_2[p]\geq 0\end{aligned} \tag{54}$$

for $p(t)=1$, $t\in J$. Moreover,

$$\begin{aligned}&\Theta(x_0)=\|x_0\|_0=\frac{b(\rho+1)}{2\rho}<\frac{b}{\rho}=c\\ &\Phi(x_0)=\min_{[\eta,1]}x_0(t)=\frac{b(\rho+1)}{2\rho}>b=\frac{b}{\rho}\rho>\rho\|x_0\|_0\\ &\varphi(x_0)=0<d\end{aligned} \tag{55}$$

This proves that

$$\left\{x_0\in P(\varphi,\Theta,\Phi,b,\frac{b}{\rho},d):\ b<\Phi(x_0)\right\}\neq\emptyset \tag{56}$$

Let $b\leq x(t)\leq\frac{b}{\rho}$ for $t\in[\eta,1]$. Then, $\eta\leq t\leq\alpha(t)\leq 1$, so $b\leq x(\alpha(t))\leq\frac{b}{\rho}$, $t\in[\eta,1]$. Assumption (A_2) implies $f\big(t,x(\alpha(t)),x'(\beta(t))\big)\geq\frac{b}{\nu}$. This and Lemma 3 show

$$\begin{aligned}\Phi(Tx)&=\min_{[\eta,1]}(Tx)(t)=\min_{[\eta,1]}\int_0^1\mathcal{L}(t,s)h(s)f\big(s,x(\alpha(s)),x'(\beta(s))\big)ds\\ &\geq\eta^2\int_0^1\mathcal{D}(s)h(s)f\big(s,x(\alpha(s)),x'(\beta(s))\big)ds\\ &\geq\frac{b}{\nu}\eta^2\int_0^1\mathcal{D}(s)h(s)ds>b\end{aligned} \tag{57}$$

This proves that condition (S_1) holds.

Now we need to prove that condition (S_2) is satisfied. Take $x\in P(\varphi,\Phi,b,d)$ and $\|Tx\|_0>\frac{b}{\rho}=c$ with $\rho=\eta^2$. Then,

$$\Phi(Tx)=\min_{[\eta,1]}(Tx)(t)\geq\rho\|Tx\|_0>\rho\frac{b}{\rho}=b \tag{58}$$

so condition (S_2) holds.

Indeed, $\varphi(0)=0<a$, so $0\notin R(\varphi,\Psi,a,d)$. Suppose that $x\in R(\varphi,\Psi,a,d)$ with $\Psi(x)=\|x\|_0=a$. In view of Lemma 3 and Assumption (A_3), we obtain

$$\begin{aligned}\Psi(Tx)&=\max_{t\in J}(Tx)(t)\leq\int_0^1\mathcal{D}(s)h(s)f\big(s,x(\alpha(s)),x'(\beta(s))\big)ds\\ &\leq\frac{a}{\mu}\int_0^1\mathcal{D}(s)h(s)ds<a\end{aligned} \tag{59}$$

This shows that condition (S_3) is satisfied.

Since all the conditions of Theorem 1 are satisfied, problem (2) has at least three nonnegative solutions x_1, x_2, x_3 such that $\|x_i'\|_0 \le d$ for $i = 1,2,3$, and

$$b \le \min_{[\eta,1]} x_1(t), \quad a < \|x_2\|_0 \quad \text{with} \quad \min_{[\eta,1]} x_2(t) < b, \quad \|x_3\|_0 < a \tag{60}$$

This ends the proof. ■

Remark 3. Indeed, Theorem 2 holds true for $\alpha(t) = t$, $t \in J$.

4. Examples

First, we consider some examples connected with Assumption H_1.

Example 1. Take $d\mathcal{A}(t) = (at-1)dt$, $d\mathcal{B}(t) = (1-bt)dt$, $a > 1$, $b > 1$. Note that the measures change the sign, $d\mathcal{A}$ is increasing while $d\mathcal{B}$ is decreasing. It is easy to show that

$$\begin{aligned} A_1 &= \frac{1}{6}(2a-3), \quad A_2 = \frac{1}{24}(3a-4) \\ B_1 &= \frac{1}{6}(3-2b), \quad B_2 = \frac{1}{24}(4-3b) \\ \mathcal{L}_1(s) &= \frac{1}{24}s(1-s)[-as^2 + s(4-a) + 5a - 8] \\ \mathcal{L}_2(s) &= \frac{1}{24}s(1-s)[bs^2 + s(b-4) - 5b + 8] \end{aligned} \tag{61}$$

Note that

$$\begin{aligned} A_1 \ge 1,\ 0 \le A_2 \le 1 \quad &\Leftrightarrow \quad \frac{9}{2} \le a \le \frac{28}{3} \\ 0 \le B_1 \le 1,\ B_2 \ge 0 \quad &\Leftrightarrow \quad 1 \le b \le \frac{4}{3} \end{aligned} \tag{62}$$

Put $a = 5$, $b = \frac{4}{3}$. Then,

$$\begin{aligned} &A_1 = \frac{7}{6}, \quad A_2 = \frac{11}{24}, \quad \mathcal{L}_1(s) = \frac{1}{24}s(1-s)[17 - 5s^2 - s] \ge 0, \quad s \in [0,1] \\ &B_1 = \frac{1}{18}, \quad B_2 = 0, \quad \mathcal{L}_2(s) = \frac{1}{18}s(1-s)^3 \ge 0, \quad s \in [0,1] \\ &\Delta = (1-B_1)(1-A_2) + B_2(1-A_1) = \frac{221}{432} \end{aligned} \tag{63}$$

It shows that Assumption H_1 holds. ■

Example 2. Take $d\mathcal{A}(t) = (at-1)dt$, $d\mathcal{B}(t) = (bt^2-1)dt$, $a > 1$, $b > 1$. Note that the measures change the sign and are increasing. It is easy to show that

$$\begin{aligned} A_1 &= \frac{1}{6}(2a-3), \quad A_2 = \frac{1}{24}(3a-4) \\ B_1 &= \frac{1}{4}(b-2), \quad B_2 = \frac{1}{30}(3b-5) \\ \mathcal{L}_1(s) &= \frac{1}{24}s(1-s)[-as^2 + s(4-a) + 5a - 8], \\ \mathcal{L}_2(s) &= \frac{1}{60}s(1-s)[-b(s^2+s^3) + (10-b)s + 9b - 20] \end{aligned} \tag{64}$$

Note that

$$
\begin{aligned}
&A_1 \geq 1, \quad 0 \leq A_2 \leq 1 \quad \Leftrightarrow \quad \frac{9}{2} \leq a \leq \frac{28}{3} \\
&0 \leq B_1 \leq 1, \quad B_2 \geq 0 \quad \Leftrightarrow \quad 2 \leq b \leq 6
\end{aligned} \tag{65}
$$

Put $a=5$, $b=\frac{20}{9}$. Then,

$$
\begin{aligned}
&A_1 = \frac{7}{6}, \quad A_2 = \frac{11}{24}, \quad \mathcal{L}_1(s) = \frac{1}{24}s(1-s)[17-5s^2-s] \geq 0, \quad s \in [0,1] \\
&B_1 = B_2 = \frac{1}{18}, \quad \mathcal{L}_2(s) = \frac{1}{54}s^2(1-s)[7-2(s+s^2)] \geq 0, \quad s \in [0,1] \\
&\Delta = (1-B_1)(1-A_2) + B_2(1-A_1) = \frac{217}{432}
\end{aligned} \tag{66}
$$

It shows that Assumption H_1 holds. ■

Example 3. Take $0<\delta_1<\delta_2<1$. Let $\lambda_2[x]=\gamma_1 x(\delta_1)+\gamma_2 x(\delta_2)$. Moreover, we assume that

$$
\begin{cases}
0 \leq \gamma_1\delta_1 + \gamma_2\delta_2 \leq 1, \quad 0 \leq \gamma_1\delta_1^2 + \gamma_2\delta_2^2 \\
\gamma_1(2\delta_1 - \delta_1^2 - s) + \gamma_2(2\delta_2 - \delta_2^2 - s) \geq 0, \quad s \leq \delta_1 \\
\gamma_1(1-s)\delta_1^2 + \gamma_2 s(2\delta_2 - \delta_2^2 - s) \geq 0, \quad \delta_1 \leq s \leq \delta_2
\end{cases} \tag{67}
$$

In this case, we have:

$$
\begin{aligned}
B_1 &= \gamma_1\delta_1 + \gamma_2\delta_2, \quad B_2 = \frac{1}{2}(\gamma_1\delta_1^2 + \gamma_2\delta_2^2) \\
\mathcal{L}_2(s) &= \gamma_1 L(\delta_1, s) + \gamma_2 L(\delta_2, s) \\
&= \frac{1}{2}\begin{cases}
s\left[\gamma_1(2\delta_1 - \delta_1^2 - s) + \gamma_2(2\delta_2 - \delta_2^2 - s)\right], & s \leq \delta_1 \\
\gamma_1(1-s)\delta_1^2 + \gamma_2 s(2\delta_2 - \delta_2^2 - s), & \delta_1 \leq s \leq \delta_2 \\
(1-s)\left[\gamma_1\delta_1^2 + \gamma_2\delta_2^2\right], & s \geq \delta_2
\end{cases}
\end{aligned} \tag{68}
$$

see Assumption H_0.

Note that if we take $\gamma_1,\gamma_2>0$, then, usually inequalities (67) hold and this case is not so interesting as the case when $\gamma_1\gamma_2<0$. Therefore, we take $\gamma_1=-1$, $\gamma_2=1$ and $\delta_1=\frac{1}{4}$, $\delta_2=\frac{3}{4}$. It is easy to verify that (67) holds for such δ_1,δ_2 and γ_1,γ_2.

For example, we can define λ_1 by

$$
\lambda_1[x] = \int_0^1 x(t)(5t-1)\,dt \tag{69}
$$

Then,

$$
A_1 = \frac{7}{6}, \quad A_2 = \frac{11}{24}, \quad \mathcal{L}_1(s) = \frac{1}{24}s(1-s)[17-5s^2-s] \geq 0, \quad s \in [0,1] \tag{70}
$$

by Example 1. Moreover, $B_1=\frac{1}{2}$, $B_2=\frac{1}{4}$ and

$$
\Delta = (1-B_1)(1-A_2) + B_2(1-A_1) = \frac{11}{48} \tag{71}
$$

It shows that Assumption H_1 holds.

The next example is connected with problem (2).

Example 4. Consider the problem:

$$\begin{cases} x'''(t)+h(t)f\big(t,x(\alpha(t)),x'(\beta(t))\big), & t\in(0,1) \\ x(0)=0, \quad x'(1)=\int_0^1 x(s)(5s-1)ds \quad x'(0)=\int_0^1 x(s)\left(1-\frac{4}{3}s\right)ds \end{cases} \tag{72}$$

where

$$f(t,u,v)\begin{cases} 10^{-4}(t+\cos^2 v)+\frac{1}{2}u^2, & t\in[0,1],\ u\in[0,2],\ |v|\le d \\ 10^{-4}(t+\cos^2 v)+2, & t\in[0,1],\ u\ge 2,\ |v|\le d \end{cases} \tag{73}$$

$$h(t)=\frac{D}{\sqrt{t}}, \quad D>0, \quad \eta=\frac{3}{4} \tag{74}$$

We see that

$$\lambda_1[x]=\int_0^1 x(t)(5t-1)dt, \quad \lambda_2[x]=\int_0^1 x(t)\left(1-\frac{4}{3}t\right)dt \tag{75}$$

so

$$d\mathcal{A}(t)=(5t-1)dt, \quad d\mathcal{B}(t)=\left(1-\frac{4}{3}t\right)dt \tag{76}$$

and the measures change the sign. The argument α should be advanced on J. For example, we can take $\alpha(t)=\sqrt{t}$ or $\alpha(t)=\sqrt[4]{t}$. As β we can take any function $\beta\in C(J,J)$, so, for example,

$$\beta_1(t)=\sqrt{t}(1-t) \quad \text{or} \quad \beta_2(t)=\begin{cases} \sqrt{t}, & [0,\frac{1}{4}] \\ \frac{1}{16}(4t+7), & [\frac{1}{4},\frac{3}{4}] \\ \frac{1}{2}(3t-1), & [\frac{3}{4},1] \end{cases} \tag{77}$$

in the place of β. Note that β_1 is advanced on $[0,t_1]$ and delayed on $[t_1,1]$ for $t_1=\left(\frac{\sqrt{5}-1}{2}\right)^2\approx 0.38$. Similarly, β_2 is advanced on $[0,\frac{7}{12}]$ and delayed on $[\frac{7}{12},1]$.

In view of Example 1, it is easy to calculate:

$$D_1=\frac{323}{1890}D, \quad D_2=\frac{16}{2835}D, \quad D_3=\frac{161738}{69615}D, \quad D_4=\frac{20744}{69615}D \tag{78}$$

and $\lambda_1[p]=\frac{3}{2}$, $\lambda_2[p]=\frac{1}{3}$ for $p(t)=1$. Basing on the above,

$$\mu>\max(D_3,D_4)=D_3\approx 2.32332D, \quad 0<\nu<\eta^2 D_4\approx 0.16761D \tag{79}$$

Put $a=\frac{1}{10}$, $b=2$, $d=31$, so $c=\frac{b}{\eta^2}=\frac{32}{9}$. Let $D=6$, $\mu=15$, $\nu=1$. Then,

$$\begin{aligned} &f(t,u,v)<\frac{d}{\mu}=\frac{31}{15}, && t\in J,\ 0\le u\le d,\ |v|\le d \\ &f(t,u,v)\ge\frac{b}{\nu}=2 && t\in[\eta,1],\ 2\le u\le\frac{32}{9},\ |v|\le d \\ &f(t,u,v)<\frac{a}{\mu}=\frac{1}{150}, && t\in J,\ 0\le u\le a,\ |v|\le d \end{aligned} \tag{80}$$

We see that all assumptions of Theorem 2 hold, so problem (72) has at least three positive solutions.

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