SUCCESSIVE ITERATIVE METHOD FOR HIGHER-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING STIELTJES INTEGRAL BOUNDARY CONDITIONS

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Abstract: In this paper, the existence of positive solutions to fractional differential equations with delayed arguments and Stieltjes integral boundary conditions is discussed. The convergence of successive iterative method of solving such problems is investigated. This allows us to improve some recent works. Some numerical examples illustrate the results.

Keywords: fractional differential equations with delayed arguments, Stieltjes integral boundary conditions, successive iterative method, convergence, existence of positive solutions

1. Introduction

Put $J=[0,1]$, $J_0=(0,1)$, $\mathbb{R}_+=[0,\infty)$. In this paper, we are interested in the existence of positive solutions to the boundary value problem:

$$\begin{cases} D^q x(t)+f\big(t,x(\alpha(t),x(\alpha(t))\big)=0, & t\in J_0,\ n-1<q\le n,\ n\ge 3 \\ x^{(i)}(0)=0, & i=0,1,\cdots,n-2 \\ \big[D^k x(t)\big]_{t=1}=\lambda[x], & k \text{ is fixed and } k\in[0,n-2] \end{cases} \tag{1}$$

where D^q is the standard Riemann-Liouville derivative. Here, λ denotes a linear functional given by:

$$\lambda[x]=\int_0^\eta x(t)d\Lambda(t), \quad \eta\in(0,1] \tag{2}$$

involving the Stieltjes integral with a suitable function Λ of bounded variation. The functional is not assumed to be positive for all positive x.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by some applications in various fields of sciences and engineering. Recently, many papers have dealt with the existence of positive solutions to the boundary value problems of fractional differential equations. To the author's knowledge, there are several papers in the literature concerning the existence of research of positive solutions for some high order nonlinear fractional differential equations with integral boundary value conditions.

In [1], the authors studied the following higher-order boundary value problem of fractional differential equations

$$\begin{cases} D^q x(t) + f\big(t, x(t)\big) = 0, \ t \in J_0, \quad n-1 < q \le n, \ n \ge 3 \\ x(0) = x'(0) = \cdots = x^{n-2}(0) = 0 \\ x^{(k)}(1) = p\displaystyle\int_0^\eta x(s)ds, \quad p \ge 0, \ \eta \in (0,1] \end{cases} \tag{3}$$

where integer k is fixed and $k \in \{0, 1, \cdots, n-2\}$. They showed the existence of positive solutions, by using the fixed point index theory. In problem (3), k is the integer, but in considered problem (1), k is any fixed number from the interval $[0, n-2]$.

In [2, 3], the authors investigated the above problem for $n = 4$, $k = 0$, by using Krasnoselskii's fixed point theorem in a cone or the fixed point index theory. In [4], the authors discussed the above problem for $p = 1$, $n = 4$, $k = 0$, by constructing two iterative sequences showing their convergence. See also Example 1 connected with the above mentioned papers. Some papers have studied the equation from (3) with the boundary condition $x(1) = p\int_0^\eta x(s)ds$ replaced by $x(1) = \lambda[x]$ or a special case of it, see for example, [5–9]. For the case where q is an integer, see for example [10]. In [11], the author studied problem (1) with $\lambda[x] = 0$, $k \in [1, n-2]$, by using Krasnoselskii's fixed point theorem in a cone. Note that, in all the above cited papers, fractional differential equations without deviating arguments α have been investigated.

Motivated by the above works, in this paper, we establish new existence results for positive solutions to quite general boundary value problems of type (1), by using the method of successive iterations. Some error estimations are given, see Theorem 1 and Example 3. It is worth indicating that the Stieltjes integral boundary condition in problem (1) covers m-point boundary conditions and integral boundary conditions too. We do not suppose that $\lambda[x] \ge 0$ for all $x \ge 0$, but we allow $d\Lambda$ to be a signed measure, so we extend the m-point case to allow some coefficients a_i of both signs, see Examples 2 and 3.

2. Preliminaries

By $D^q x$, we denote the Riemann-Liouville fractional derivative of order $q > 0$, and by $I^q x$, the Riemann-Liouville fractional integral of order $q > 0$, see [12–14], so

$$D^q x(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{-q+n-1} x(s)ds, \quad n = [q]+1,\ q > 0,\ t < 1$$
$$D^n x(t) = y^{(n)}(t), \quad n \in \{1,2,3,\cdots\} \tag{4}$$
$$I_1^q x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} x(s)ds, \quad q > 0,\ t < 1$$

where $[q]$ means the integer part of q.

We require the following assumptions:

$H_1:$ $f \in C(J \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C(J,J)$, $\alpha(t) \le t$,

$H_2:$ Λ is a function of bounded variation on $[0,\eta]$, $\eta \in (0,1]$, $n-1 < q \le n$, k is fixed, $k \in [0, n-2]$, $n \ge 3$ and

$$\Delta = \Gamma(q) - \Gamma(q-k)\int_0^\eta t^{q-1} d\Lambda(t) \neq 0, \quad \mathcal{G}(s) = \int_0^\eta G_1(t,s) d\Lambda(t)$$
$$G_1(t,s) = \frac{1}{\Gamma(q)}\begin{cases} t^{q-1}(1-s)^{q-k-1} - (t-s)^{q-1}, & \text{if } s \le t \\ t^{q-1}(1-s)^{q-k-1}, & \text{if } t \le s \end{cases} \tag{5}$$

Consider the following boundary value problem:

$$\begin{cases} D^q u(t) + y(t) = 0, & t \in J_0,\ n-1 < q \le n,\ n \ge 3 \\ u^{(i)}(0) = 0, & i = 0,1,\cdots,n-2 \\ \left[D^k u(t)\right]_{t=1} = \lambda[u], & k \text{ is fixed and } k \in [1, n-2] \end{cases} \tag{6}$$

Lemma 1. *Assume that Assumption H_2 holds. Let $y \in L(J_0, \mathbb{R})$. Then, problem (6) has the unique solution given by the following formula*

$$u(t) = \int_0^1 G(t,s) y(s) ds \tag{7}$$

where

$$G(t,s) = G_1(t,s) + \frac{\Gamma(q-k)}{\Delta}\mathcal{G}(s) t^{q-1} \tag{8}$$

Proof. The general solution of (6) is given by

$$u(t) = -I^q y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n} \tag{9}$$

Indeed, $c_2 = c_3 = \cdots = c_n = 0$ in view of conditions $u^{(i)}(0) = 0$, $i = 0,1,\cdots,n-2$, so

$$u(t) = -I^q y(t) + c_1 t^{q-1} \tag{10}$$

Hence, in view of the property $D^k I^q = I^{q-k}$,

$$D^k u(t) = -D^k I^q y(t) + c_1 D^k[t^{q-1}] =$$
$$= -\frac{1}{\Gamma(q-k)}\int_0^t (t-s)^{q-k-1} y(s) ds + c_1 \frac{\Gamma(q)}{\Gamma(q-k)} t^{q-k-1} \tag{11}$$

This and condition $[D^k u(t)]_{t=1} = \lambda[u]$ give

$$-\frac{1}{\Gamma(q-k)}\int_0^1 (1-s)^{q-k-1}y(s)ds + c_1\frac{\Gamma(q)}{\Gamma(q-k)} = \lambda[u] \tag{12}$$

Finding from this c_1 and substituting to (10) we obtain

$$u(t) = \frac{\Gamma(q-k)}{\Gamma(q)}t^{q-1}\lambda[u] + \int_0^1 G_1(t,s)y(s)ds \tag{13}$$

In the next step, we have to eliminate $\lambda[u]$ from (13). If u is a solution of (13), then

$$\lambda[u] = \frac{\Gamma(q)}{\Delta}\int_0^1 \mathcal{G}(s)y(s)ds \tag{14}$$

because

$$\begin{aligned}
&\int_0^\eta \left(\int_0^1 G_1(t,s)y(s)ds\right)d\Lambda(t) = \\
&\quad = \frac{1}{\Gamma(q)}\int_0^\eta \left(\int_0^1 t^{q-1}(1-s)^{q-k-1}y(s)ds - \int_0^t (t-s)^{q-1}y(s)ds\right)d\Lambda(t) \\
&\quad = \frac{1}{\Gamma(q)}\int_0^1 \left(\int_0^\eta t^{q-1}d\Lambda(t)\right)(1-s)^{q-k-1}y(s)ds \\
&\qquad - \frac{1}{\Gamma(q)}\int_0^\eta \left(\int_s^\eta (t-s)^{q-1}d\Lambda(t)\right)y(s)ds = \int_0^1 \mathcal{G}(s)y(s)ds
\end{aligned} \tag{15}$$

Substituting it to formula (13) we finally get the assertion of this lemma. ■

Remark 1. Note that G is the Green function of problem (1).

Remark 2. In view of Assumption H_2, it is easy to see that there exists a positive constant d such that

$$0 \le G(t,s) \le dt^{q-1} \tag{16}$$

3. The main results

Define the operator T, by

$$Tu(t) = \int_0^1 G(t,s)f\big(s,u(s),u(\alpha(s))\big)ds \tag{17}$$

In view of Lemma 1, problem (1) is equivalent to the operator equation $x = Tx$, and we show that it has a solution.

Let

$$P = \big\{x \in C(J):\ x(t) \ge 0,\ t \in J\big\} \tag{18}$$

Indeed, P is a cone. We define a sub-cone of P, by

$$D = \big\{x \in P:\ \text{there exists a positive constant } K \text{ such that } x(t) \le Kt^{q-1}\big\} \tag{19}$$

Theorem 1. *Let Assumptions H_1, H_2 hold with $\Delta > 0$, $\mathcal{G}(s) \geq 0$, $s \in [0,1]$. Moreover, we assume that* $\max_{t\in J} f(t,0,0) > 0$, *and*

(i) there exist nonnegative constants A, B such that

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq A|v_1 - u_1| + B|v_2 - u_2| \tag{20}$$

(ii) $d(A+B) < q$, where d is taken from Remark 2.

Then the sequence of functions defined by

$$u_0(t) = 0, \quad u_{n+1}(t) = Tu_n(t), \quad t \in J,\ n = 0,1,\cdots \tag{21}$$

converges uniformly to the unique solution $u^* \in D$. Moreover, we have the error estimation

$$|u^*(t) - u_n(t) \leq Kt^{q-1}\rho^n, \quad n = 0,1,\cdots \tag{22}$$

where

$$\rho = \frac{d(A+B)}{q}, \quad K = \frac{Md}{1-\rho}, \quad M = \max_{t\in J} f(t,0,0) \tag{23}$$

Proof. Indeed, $u_n \in C(J,\mathbb{R}_+)$. Put

$$w_n(t) = |u_n(t) - u_{n-1}(t)|, \quad n = 1,2,\cdots \tag{24}$$

Then,

$$\begin{aligned} w_{n+1}(t) &= \big|Tu_n(t) - Tu_{n-1}(t)\big| \\ &\leq \int_0^1 G(t,s)\big|f\big(s,u_n(s),u_n(\alpha(s))\big) - f\big(s,u_{n-1}(s),u_{n-1}(\alpha(s))\big)\big|ds \end{aligned} \tag{25}$$

so using condition (i),

$$w_{n+1}(t) \leq dt^{q-1}\int_0^1 \big[Aw_n(s) + Bw_n(\alpha(s))\big]ds, \quad t \in J,\ n = 1,2,\cdots \tag{26}$$

Indeed,

$$\begin{aligned} w_1(t) &= |Tu_0(t)| = \int_0^1 G(t,s)f(s,0,0)ds \leq Mdt^{q-1} \\ w_2(t) &\leq dt^{q-1}\int_0^1 \big[Aw_1(s) + Bw_1(\alpha(s))\big]ds \leq Mdt^{q-1}\rho \end{aligned} \tag{27}$$

because $0 \leq G(t,s) \leq dt^{q-1}$, $\alpha(s) \leq s$.

Now, we have to prove that

$$w_n(t) \leq Mdt^{q-1}\rho^{n-1} \equiv z_n(t), \quad n = 1,2,\cdots \tag{28}$$

Assume that (28) holds for some integer $n = m > 1$. Using (26) and $w_m(\alpha(s)) \leq z_m(s)$, we obtain

$$\begin{aligned} w_{m+1}(t) &\leq dt^{q-1}\int_0^1 \big[Aw_m(s) + Bw_m(\alpha(s))\big]ds \\ &\leq d(A+B)t^{q-1}\int_0^1 z_m(s)ds = z_{m+1}(t) \end{aligned} \tag{29}$$

This and the mathematical induction show that (28) holds.

Now, we have to show that the sequence $\{u_n\}$ is convergent. First, we note that

$$u_n(t)=u_0(t)+\sum_{j=1}^{n}\big[u_j(t)-u_{j-1}(t)\big], \quad n=1,2,\cdots \tag{30}$$

In view of (28), we see that

$$\sum_{j=1}^{\infty} w_j(t) \le Mdt^{q-1}\sum_{j=1}^{\infty}\rho^{j-1} = Mdt^{q-1}\frac{1}{1-\rho} < \frac{Md}{1-\rho} = K \tag{31}$$

Using the Weierstrass test, this shows that the series

$$u_0(t)+\sum_{j=1}^{\infty}\big[u_j(t)-u_{j-1}(t)\big] \tag{32}$$

is uniformly convergent. This asserts that the sequence $\{u_n\}$ is uniformly convergent too. Indeed, $u^*(t)=\lim_{n\to\infty}u_n(t)$ and $u^*\in C(J,\mathbb{R}_+)$. Obviously, taking the limit $n\to\infty$, we see that $u^*(t)=Tu^*(t)$, $t\in J$, so $u^*\in C(J,\mathbb{R}_+)$ is a solution of problem (1).

Moreover,

$$0\le u_n(t)\le \sum_{j=1}^{n} w_j(t) \le \sum_{j=1}^{\infty} w_j(t) \le Kt^{q-1}, \quad n=1,2,\cdots \tag{33}$$

This proves that u_n, $u^*\in D$, $n=0,1,\cdots$.

Now, we have to prove that u^* is a unique solution of (1) in D. Suppose that v is another solution distinct from u^* and $v\in D$. Put $V(t)=|u^*(t)-v(t)|$. Then

$$\begin{aligned} V(t) &= \big|Tv(t)-Tu^*(t)\big| \\ &\le \int_0^1 G(t,s)\big|f\big(s,v(s),v(\alpha(s))\big)-f\big(s,u^*(s),u^*(\alpha(s))\big)\big|\,ds \\ &\le dt^{q-1}\int_0^1\big[AV(s)+BV(\alpha(s))\big]ds \end{aligned} \tag{34}$$

so

$$V(t)\le 2dt^{q-1}K(A+B)\int_0^1 s^{q-1}ds = 2Kt^{q-1}\rho \tag{35}$$

because

$$V(t)\le 2Kt^{q-1} \tag{36}$$

This and the previous relation on V give

$$V(t)\le 2Kt^{q-1}\rho^2 \tag{37}$$

Repeating it, we can show, by induction, that

$$V(t)\le 2Kt^{q-1}\rho^n, \quad n=1,2,\cdots \tag{38}$$

so $\rho^n\to 0$. This immediately shows that u^* is the unique solution of (1) in D.

Now, we need to obtain the error estimation. Put $Z_n(t)=|u^*(t)-u_n(t)|$. Then

$$\begin{aligned} Z_{n+1}(t) &= |Tu^*(t)-T_n(t)| \\ &\le \int_0^1 G(t,s)\left|f\big(s,u^*(s),u^*(\alpha(s))\big)-f\big(s,u_n(s),u_n(\alpha(s))\big)\right|ds \\ &\le dt^{q-1}\int_0^1\left[AZ_n(s)+BZ_n(\alpha(s))\right]ds, \quad n=0,1,\cdots \end{aligned} \tag{39}$$

Hence

$$Z_1(t)\le dt^{q-1}\int_0^1\left[Au^*(s)+Bu^*(\alpha(s))\right]ds\le Kt^{q-1}\rho \tag{40}$$

because $u^*(t)\le Kt^{q-1}$.

By induction in n, we can show that $Z_n(t)\le Kt^{q-1}\rho^n$, so

$$|u^*(t)-u_n(t)|\le Kt^{q-1}\rho^n, \quad n=0,1,\cdots \tag{41}$$

This ends the proof. ■

4. Examples

Example 1. Let $d\Lambda(t)=c\,dt$, $c\ge 0$. Indeed, $\mathcal{G}(s)\ge 0$, $s\in[0,1]$. Moreover, $\Delta>0$ provided that

$$0\le c\frac{\eta^q}{q}<\frac{\Gamma(q)}{\Gamma(q-k)} \tag{42}$$

If $k\in\{0,1,\cdots,n-2\}$, then (42) takes the form

$$0\le c\frac{\eta^q}{q}<1 \text{ if } k=0 \tag{43}$$

$$0\le c\frac{\eta^q}{q}<\prod_{i=1}^{k}(q-i) \text{ if } k\in\{1,2,\cdots,n-2\} \tag{44}$$

Condition (43) appeared in papers [2, 3] for $n=4$; in [4] for $c=1$, $n=4$. The case with $c\ge 0$ has also been discussed in paper [1] for $k\in\{0,1,\cdots,n-2\}$ with conditions (43) and (44).

Example 2. Let

$$\lambda[x]=\sum_{i=1}^{m}a_i x(\eta_i),\ a_i\in\mathbb{R},\ i=1,2,\cdots,m,\ 0<\eta_1<\eta_2<\cdots<\eta_m\le\eta<1 \tag{45}$$

Then

$$\mathcal{G}(s)=\sum_{i=1}^{m}a_iG_1(\eta_i,s) \tag{46}$$

so

$$\mathcal{G}(s)=\frac{1}{\Gamma(q)}\begin{cases}\sum_{i=1}^{m}a_iW_1(\eta_i,s),\ 0\le s\le\eta_1\\ a_1W_2(\eta_1,s)+\sum_{i=2}^{m}a_iW_1(\eta_i,s),\ \eta_1<s\le\eta_2\\ \cdots,\ \cdots\\ \sum_{i=1}^{m-1}a_iW_2(\eta_i,s)+a_mW_1(\eta_m,s),\ \eta_{m-1}<s\le\eta_m\\ \sum_{i=1}^{m}a_iW_2(\eta_i,s),\ \eta_m<s\le 1\end{cases}\tag{47}$$

where

$$\begin{aligned}W_1(t,s)&=\frac{1}{\Gamma(q)}\left[t^{q-1}(1-s)^{q-k-1}-(t-s)^{q-1}\right]\text{ if } s\le t\\ W_2(t,s)&=\frac{1}{\Gamma(q)}t^{q-1}(1-s)^{q-k-1}\text{ if } t\le s\end{aligned}\tag{48}$$

We need $\mathcal{G}(s)\ge 0$, $s\in[0,1]$ which is automatic if $a_i\ge 0$, $i=1,2,\cdots,m$, because H_1 and H_2 are nondecreasing. Now, we also need $\Delta>0$, which holds if

$$0\le\sum_{i=1}^{m}a_i\eta_i^{q-1}<\frac{\Gamma(q)}{\Gamma(q-k)}\tag{49}$$

Example 3. Now, we consider the sign changing case. Let $\lambda[x]=2^{q-1}x\left(\frac{\eta}{2}\right)-x(\eta)$, $\eta\in(0,1)$. Then $\mathcal{G}(s)=2^{q-1}G_1\left(\frac{\eta}{2},s\right)-G_1(\eta,s)$, so

$$\mathcal{G}(s)=\begin{cases}2^{q-1}W_1\left(\frac{\eta}{2},s\right)-W_1(\eta,s),\ 0\le s<\frac{\eta}{2}\\ 2^{q-1}W_2\left(\frac{\eta}{2},s\right)-W_1(\eta,s),\ \frac{\eta}{2}\le s<\eta\\ 2^{q-1}W_2\left(\frac{\eta}{2},s\right)-W_2(\eta,s),\ \eta\le s\le 1\end{cases}\tag{50}$$

where W_1 and W_2 are defined as in Example 2. It is easy to verify that $\mathcal{G}(s)\ge 0$, $s\in[0,1]$. Moreover,

$$\int_0^{\eta}t^{q-1}d\Lambda(t)=2^{q-1}\left(\frac{\eta}{2}\right)^{q-1}-\eta^{q-1}=0\tag{51}$$

so $\Delta=\Gamma(q)>0$.

5. Numerical example

Consider the following problem

$$\begin{cases}D^qx(t)=g(t)+a\sin x(t)+bx(\sqrt{t})\equiv f\big(t,x(t),x(\alpha(t))\big),\ t\in(0,1)\\ x(0)=x'(0)=0,\ \left[D^kx(t)\right]_{t=1}=c\int_0^{\eta}x(s)ds\end{cases}\tag{52}$$

with $g\in C\big([0,1],\mathbb{R}_+\big)$, $g(0)=1$, $q=\frac{7}{2}$ $a,b,c\in\mathbb{R}_+$, $a+b=5$, $k=\frac{1}{2}$, $\eta=\frac{3}{4}$. Hence

$$\Delta=\Gamma(q)-c\Gamma(q-k)\int_0^{\eta}t^{q-1}dt=\frac{15}{8}\sqrt{\pi}-c\frac{27\sqrt{3}}{224}\tag{53}$$

If we assume that

$$c < \frac{140\sqrt{\pi}}{9\sqrt{3}} \approx 15.918 \tag{54}$$

then $\Delta > 0$. Moreover, $M = 1$, $A + B = 5$, and

$$\begin{gathered} \mathcal{G}(s) \le \frac{9\sqrt{3}}{280\sqrt{\pi}}\, c, \quad G_1(t,s) \le \frac{8}{15\sqrt{\pi}} t^{\frac{5}{2}} \\ G(t,s) \le \left(\frac{8}{15\sqrt{\pi}} + \frac{168\sqrt{3}\,c}{35(140\pi - 9\sqrt{3\pi}\,c} \right) t^{\frac{5}{2}} = dt^{\frac{5}{2}}, \quad d = d(c) \end{gathered} \tag{55}$$

Assume that condition (54) holds. Then, in view of Theorem 1, problem (52) has a unique positive solution $u^* \in D$. This solution is the limit of the sequence of functions defined by

$$u_0(t) = 0, \quad u_{n+1}(t) = Tu_n(t), \quad t \in J,\ n = 0, 1, \cdots \tag{56}$$

and we have the error estimation

$$|u^*(t) - u_n(t)| \le K t^{\frac{5}{2}} \rho^n, \quad n = 0, 1, \cdots \tag{57}$$

with

$$\rho = \frac{10d}{7}, \quad K = \frac{d}{1-\rho} \tag{58}$$

The estimation (57) depends on c, because $d = d(c)$. Now, we are going to discuss the estimation (57) for $c = 1$, $c = 5$ and $c = 10$.

1. Put $c = 1$. Then, $d \approx 0.321$, $\rho \approx 0.459$, $K \approx 0.593$. Moreover,

$$0 \le u^*(t) \le 0.593 t^{\frac{5}{2}}, \quad \left|u^*(t) - u_5(t)\right| \le 0.002 t^{\frac{5}{2}} \tag{59}$$

2. Put $c = 5$. Then, $d \approx 0.439$, $\rho \approx 0.627$, $K \approx 1.176$. Moreover,

$$0 \le u^*(t) \le 1.176 t^{\frac{5}{2}}, \quad \left|u^*(t) - u_5(t)\right| \le 0.114 t^{\frac{5}{2}}, \quad \left|u^*(t) - u_{10}(t)\right| \le 0.003 t^{\frac{5}{2}} \tag{60}$$

3. Put $c = 10$. Then, $d \approx 0.594$, $\rho \approx 0.848$, $K \approx 3.912$. Moreover,

$$\begin{gathered} 0 \le u^*(t) \le 3.912 t^{\frac{5}{2}}, \quad \left|u^*(t) - u_5(t)\right| \le 1.715 t^{\frac{5}{2}}, \quad \left|u^*(t) - u_{10}(t)\right| \le 0.752 t^{\frac{5}{2}} \\ \left|u^*(t) - u_{20}(t)\right| \le 0.145 t^{\frac{5}{2}}, \quad \left|u^*(t) - u_{50}(t)\right| \le 0.001 t^{\frac{5}{2}} \end{gathered} \tag{61}$$

Conclusion. The results presented above demonstrate that we have quite good estimation for $c = 1$ finding only the iterations u_n from 1 to 5, however, the iterations u_n from 1 do 50 are needed to get a similar order of estimation for $c = 10$.

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