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Lefschetz periodic point free self-maps of compact manifolds

Grzegorz Graff [a,*,1], Agnieszka Kaczkowska [a], Piotr Nowak-Przygodzki [a,1], Justyna Signerska [a,b,2]

[a] Faculty of Applied Physics and Mathematics, Gdansk University of Technology, Narutowicza 11/12, 80-233 Gdansk, Poland
[b] Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland

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ABSTRACT

Let f be a self-map of a compact connected manifold M. We characterize Lefschetz periodic point free continuous self-maps of M for several classes of manifolds and generalize the results of Guirao and Llibre [J.L.G. Guirao, J. Llibre, On the Lefschetz periodic point free continuous self-maps on connected compact manifolds, Topology Appl. 158 (16) (2011) 2165–2169].

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1. Introduction

An important but difficult problem in dynamical systems is to identify the periodic point free maps on a given compact manifold. The necessary (but not sufficient) condition for a map to be periodic point free is that the Lefschetz numbers of all its iterates vanish. This motivates the following definition: a map f is called *Lefschetz periodic point free* iff $L(f^m) = 0$ for $m = 1, 2, 3, \ldots$. In [7] a characterization of Lefschetz periodic point free maps on $\mathbb{S}^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{S}^p \times \mathbb{S}^q$ was given. The aim of this paper is to give a more detailed description of the Lefschetz periodic point free maps on the spaces considered in [7] (Section 2), as well as to generalize these results for the large class of manifolds called rational exterior spaces (Section 3).

Our approach in this part of the paper is based on the use of additional information, hidden in the structure of the cohomology ring, which allows one to determine the sequence $\{L(f^m)\}_{m=1}^{\infty}$ (cf. [4]). This method enables us to show that some conditions found in [7] are superfluous, because they are always satisfied. Moreover, it makes it possible to find the explicit formula for $L(f^m)$ which is conceptually simpler than the use of the Lefschetz zeta function applied in [7]. We also correct some incorrect statements in [7] concerning $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

In Section 4 we formulate the necessary and sufficient conditions for a map to be Lefschetz periodic point free in the language of eigenvalues, based only on the definition of Lefschetz number. These conditions enable us to identify some spaces that do not admit Lefschetz periodic point free maps.

* Corresponding author.
E-mail addresses: graff@mif.pg.gda.pl (G. Graff), akaczkowska@mif.pg.gda.pl (A. Kaczkowska), piotrnp@wp.pl (P. Nowak-Przygodzki), j.signerska@impan.pl (J. Signerska).

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In the final part of the paper (Section 5) we illustrate the way in which the purely algebraical condition of being "Lefschetz periodic point free" provides some important information about the structure of periodic points of the map.

2. Lefschetz periodic point free maps of $\mathbb{S}^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{S}^p \times \mathbb{S}^q$: cohomological ring approach

2.1. Lefschetz numbers of iterates

For a compact connected manifold M of dimension n we will consider $H^i(M;\mathbb{Q})$, where $i = 0, 1, \dots, n$, the cohomology groups with coefficients in $\mathbb{Q}$, which are then finite dimensional linear spaces over $\mathbb{Q}$. For a self-map f of M we denote by f^{*i} the linear map induced by f on $H^i(M;\mathbb{Q})$ and by f^* the self-map $\bigoplus_{i=0}^n f^{*i}$ of $\bigoplus_{i=0}^n H^i(M;\mathbb{Q})$. The Lefschetz number $L(f^m)$ of f^m is then equal to

$$L\left(f^m\right) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}\left(f^m\right)^{*i}, \tag{2.1}$$

where $\operatorname{tr}(f^m)^{*i}$ is the trace of the matrix representing $(f^m)^{*i} : H^i(M;\mathbb{Q}) \to H^i(M;\mathbb{Q})$. Notice that if A is a matrix of f^{*i}, then A^m is a matrix of $(f^m)^{*i}$, representing the homomorphism induced on $H^i(M;\mathbb{Q})$ by f^m, the m-th iteration of f (cf. [2,9]). Consequently, when $\operatorname{tr} f^{*i} = \sum_{j=1}^k \lambda_j$, then $\operatorname{tr}(f^m)^{*i} = \sum_{j=1}^k \lambda_j^m$, where the sum is taken over all eigenvalues λ_j in the spectrum of A, counted with multiplicities.

Often the Lefschetz number of f is defined via homology groups, but for our purposes it is more convenient to give the equivalent definition (2.1).

We will make use of the structure of the cohomology ring $\bigoplus_{i=0} H^i(X;\mathbb{Q})$ to obtain additional information concerning the Lefschetz numbers of iterates of f. Let us recall two basic properties of cup product that we will apply later in the paper:

- f^* is a homomorphism of $\bigoplus_{i=0}^n H^i(M;\mathbb{Q})$, i.e.
$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta); \tag{2.2}$$
- the cup product is anticommutative, i.e. if $\alpha \in H^k(M;\mathbb{Q})$ and $\beta \in H^l(M;\mathbb{Q})$, then
$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha. \tag{2.3}$$

2.2. Lefschetz periodic point free maps on $\mathbb{C}P^n$

We consider the complex projective space, denoted $\mathbb{C}P^n$. The cohomology ring over $\mathbb{Q}$ of $\mathbb{C}P^n$ is isomorphic to the quotient polynomial ring $\mathbb{Q}[\alpha]/(\alpha^{n+1})$, where α is the generator of $H^2(\mathbb{C}P^n;\mathbb{Q}) = \mathbb{Q}$ (cf. e.g. [8]). As a consequence, $H^*(\mathbb{C}P^n;\mathbb{Q}) = \bigoplus_{i=0}^n H^{2i}(\mathbb{C}P^n,\mathbb{Q})$ and each $H^{2i}(\mathbb{C}P^n,\mathbb{Q})$ is generated by

$$\alpha^i = \underbrace{\alpha \smile \alpha \smile \cdots \smile \alpha}_{i} \quad \text{for } i \geqslant 0$$

where we adopt the convention that $\alpha^0 = 1$. Now, we are in a position to calculate the sequence of Lefschetz numbers of iterates (cf. similar calculation for $L(f)$ in [9]). As $(f^m)^{*2}$ has the eigenvalue a^m, where $a \in \mathbb{Z}$ is such that $f^{*2}(\alpha) = a\alpha$, we get by the formulas (2.1) and (2.2):

$$L\left(f^m\right) = 1 + \sum_{i=1}^{n} \left(a^m\right)^i = \begin{cases} \frac{1-(a^m)^{n+1}}{1-a^m} & \text{if } a^m \neq 1, \\ n+1 & \text{if } a^m = 1. \end{cases} \tag{2.4}$$

It is obvious that if $a \notin \{-1, 1\}$ then $L(f^m) \neq 0$ for $m = 1, 2, 3, \dots$. On the other hand, if $a \in \{-1, 1\}$ then either for $m = 1$ or for $m = 2$ we get $a^m = 1$ and thus by the second part of formula (2.4) $L(f^m) = n + 1 \neq 0$. As a result we obtain the following:

Proposition 2.1. *There are no Lefschetz periodic point free maps on* $\mathbb{C}P^n$.

2.3. Lefschetz periodic point free maps on $\mathbb{H}P^n$

Let $\mathbb{H}P^n$ denote the n-dimensional quaternionic projective space. The structure of the cohomology ring over $\mathbb{Q}$ is similar to $\mathbb{C}P^n$; however now the generator is four-dimensional (see e.g. [8]):

$$H^*\left(\mathbb{H}P^n;\mathbb{Q}\right) = \bigoplus_{i=0}^{n} H^{4i}\left(\mathbb{H}P^n,\mathbb{Q}\right) \simeq \mathbb{Q}[\alpha]/\left(\alpha^{n+1}\right),$$

where $\alpha \in H^4(\mathbb{H}P^n;\mathbb{Q}) \simeq \mathbb{Q}$.

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Since the generator of $H^{4i}(\mathbb{H}P^n, \mathbb{Q})$ equals α^i and each $(f^m)^{*4}$ has a^m as an eigenvalue (with $f^{*4}(\alpha) = a\alpha$) we find in the analogous way as for $\mathbb{C}P^n$ that the sequence of Lefschetz numbers of iterates of a continuous map $f : \mathbb{H}P^n \to \mathbb{H}P^n$ has exactly the same form as in (2.4).

Again we conclude

Proposition 2.2. *There are no Lefschetz periodic point free maps on $\mathbb{H}P^n$.*

Remark 2.3. There is an error in [7] in Theorems 1 and 2, which state that for $a = 0$ the self-maps of $\mathbb{C}P^n$ or $\mathbb{H}P^n$ could be Lefschetz periodic point free.

2.4. Lefschetz periodic point free maps on spaces of the type $S^p \times S^q$

Now let us consider products of multidimensional spheres. We break the problem into cases as follows

(1) $M = S^p \times S^p$, where p is odd,
(2) $M = S^p \times S^p$, where p is even,
(3) $M = S^p \times S^q$, where p, q even, $p \neq q$,
(4) $M = S^p \times S^q$, where p, q odd, $p \neq q$,
(5) $M = S^p \times S^q$, where one of the p, q is even, the other is odd.

In the first two cases the cohomology groups of M are the following:

$H^0(M; \mathbb{Q}) = \mathbb{Q}$, $H^p(M; \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$, $H^{2p}(M; \mathbb{Q}) = \mathbb{Q}$ and all the other cohomology groups vanish. The induced homomorphism f^{*p} is represented by the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and f^{*2p} is multiplication by a constant $e \in \mathbb{Z}$, where e is the degree of f.

The cohomology ring of M has the form (cf. [8]): $H^*(M; \mathbb{Q}) = \bigoplus_{i \in \{0, p, 2p\}} H^i(M; \mathbb{Q})$ and if α and β are the generators of $H^p(M; \mathbb{Q})$, then $\alpha \cup \beta$ is a generator of $H^{2p}(M; \mathbb{Q})$.

Then by (2.2) and (2.3) and the fact that $\alpha^2 = \beta^2 = 0$ we obtain

$$\begin{aligned} f^{*2p}(\alpha \smile \beta) &= f^{*p}(\alpha) \smile f^{*p}(\beta) = (a\alpha + c\beta) \smile (b\alpha + d\beta) \\ &= ad(\alpha \smile \beta) + bc(\beta \smile \alpha) = \big(ad + (-1)^{p^2} bc\big)(\alpha \smile \beta). \end{aligned} \tag{2.5}$$

Now, we consider the cases (1) and (2) separately.

Case 1. Since p is odd, p^2 is also odd, and thus (2.5) takes the form:

$$f^{*2p}(\alpha \smile \beta) = (ad - bc)(\alpha \smile \beta) = e(\alpha \smile \beta). \tag{2.6}$$

As a result, for p odd $ad - bc = e$. Thus, if λ_1 and λ_2 are the eigenvalues of f^{*p} then $\lambda_1\lambda_2$ is the eigenvalue of f^{*2p}, so

$$\begin{aligned} L(f^m) &= 1 + (-1)^p \operatorname{tr}\big((f^m)^{*p}\big) + (-1)^{2p} \operatorname{tr}\big((f^m)^{*2p}\big) \\ &= 1 + (-1)^p\big(\lambda_1^m + \lambda_2^m\big) + (\lambda_1\lambda_2)^m = \big(1 - \lambda_1^m\big)\big(1 - \lambda_2^m\big). \end{aligned}$$

We obtain the conclusion that f is Lefschetz periodic point free if and only if $\lambda_1 = 1$ or $\lambda_2 = 1$ or, equivalently, $L(f) = 0$. On the other hand,

$$L(f) = 1 - (a + d) + ad - bc.$$

Thus we have

Proposition 2.4. *A map $f : S^p \times S^p \to S^p \times S^p$ where p is odd is Lefschetz periodic point free if and only if $ad - bc = -1 + a + d$, or equivalently, if and only if one of its eigenvalues is equal to* 1.

Remark 2.5. In [7, Theorem 5] besides the condition of Proposition 2.4 there is an additional condition which states that $ad - bc = e$, but it follows from the formula (2.6) that this is always satisfied and thus the condition is superfluous.

Case 2. For p even (2.5) takes the form:

$$f^{*2p}(\alpha \smile \beta) = (ad + bc)(\alpha \smile \beta) = e(\alpha \smile \beta) \tag{2.7}$$

and therefore $ad + bc = e$. We have $L(f^1) = 1 + a + d + (ad + bc)$ and $L(f^2) = 1 + a^2 + 2bc + d^2 + (ad + bc)^2$. Assuming now that $L(f^1) = 0$ and $L(f^2) = 0$ we see that $ad + bc = -1 - a - d$ and $a^2 + d^2 = 0$, hence $a = d = 0$ and $bc = -1$ that is $b = 1$

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and then $c=-1$ or the other way round. In this case one may verify that also $L(f^3)=0$. However, for $m=4$ the matrix for $(f^4)^{*p}$ is of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^4 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

therefore

$$L\big(f^4\big) = 1 + 2 + (-1)^4 = 4$$

and thus f is never Lefschetz periodic point free.

In the cases (3), (4) and (5) the cohomology groups are

$$H^i(M;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i = 0, p, q, p+q; \\ 0 & \text{otherwise.} \end{cases}$$

We will denote by (x) the 1×1 matrix with entry x. Then let $f^{*p}=(a)$, $f^{*q}=(b)$ and $f^{*p+q}=(c)$ be the homomorphisms acting correspondingly on $H^p(M;\mathbb{Q})$, $H^q(M;\mathbb{Q})$ and $H^{p+q}(M;\mathbb{Q})$ as multiplication by integer numbers a, b and c, respectively. However, since for generators $\alpha\in H^p(M;\mathbb{Q})$ and $\beta\in H^q(M;\mathbb{Q})$ the cup product $\alpha\smile\beta$ is a generator of $H^{p+q}(M;\mathbb{Q})$, the formula (2.5) yields that

$$f^{*p+q}(\alpha\smile\beta) = f^{*p}(\alpha)\smile f^{*q}(\beta) = a\alpha\smile b\beta = ab(\alpha\smile\beta) = c(\alpha\smile\beta)$$

and thus $c=ab$.

Case 3. Suppose now that p,q are even and $p\neq q$. Then

$$L\big(f^m\big) = 1 + a^m + b^m + (ab)^m = \big(1+a^m\big)\big(1+b^m\big).$$

Thus for even m, $L(f^m)\neq 0$.

In cases (2) and (3) we have shown that

Proposition 2.6. *Let $f: S^p\times S^q\to S^p\times S^q$, with p and q even (including the case $p=q$), then f is never Lefschetz periodic point free.*

Case 4. Suppose now that $M=S^p\times S^q$, where both p and q are odd and $p\neq q$. Calculation of the Lefschetz number for f^m then gives

$$\begin{aligned} L\big(f^m\big) &= 1 + (-1)^p \operatorname{tr}\big(f^m\big)^{*p} + (-1)^q \operatorname{tr}\big(f^m\big)^{*q} + (-1)^{p+q} \operatorname{tr}\big(f^m\big)^{*p+q} \\ &= 1 - a^m - b^m + (ab)^m = \big(1-a^m\big)\big(1-b^m\big). \end{aligned}$$

We obtain the following conclusion:

Proposition 2.7. *A continuous map $f: S^p\times S^q\to S^p\times S^q$ where p and q are odd, $p\neq q$, is Lefschetz periodic point free if and only if $a=1$ or $b=1$.*

Remark 2.8. In [7, Theorem 4 (i) and (ii)] the corresponding condition is stated in the form (1) $a=1$ and $b=c$ or (2) $a=c$ and $b=1$, where $f^{*p}=(a)$, $f^{*q}=(b)$ and $f^{*p+q}=(c)$. However, since $c=ab$ by the structure of the cohomology ring, these conditions reduce to (1) $a=1$ or (2) $b=1$.

Case 5. It remains to analyze what happens for products of even and odd-dimensional spheres. If p is even and q is odd then, by the same arguments as above

$$L\big(f^m\big) = 1 + a^m - b^m - (ab)^m = \big(1+a^m\big)\big(1-b^m\big)$$

and thus $L(f^m)=0$ for all $m\in\mathbb{N}$ if and only if $b=1$ or $a=b=-1$. In the case $b=1$ necessarily $c=a$, where $f^{*p+q}=(c)$ since always $c=ab$. The second part of the following proposition can be proved in the same way.

Proposition 2.9. *Let $f: S^p\times S^q\to S^p\times S^q$ with $p\neq q$ and $f^{*p}=(a)$ and $f^{*q}=(b)$. Then:*

(1) *If p is even and q is odd, then f is Lefschetz periodic point free if and only if $b=1$ or $a=b=-1$.*
(2) *If p is odd and q is even, then f is Lefschetz periodic point free if and only if $a=1$ or $a=b=-1$.*

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Remark 2.10. In [7, Theorem 4 (iii)–(vi)] there are more conditions which are equivalent to the statement that in Case 5 f is Lefschetz periodic point free. However, they all reduce to the items (1) and (2) of Proposition 2.9. In particular, the condition (iv) (and in the same way (vi)): $a = b$ and $c = 1$ is covered by the items (1) and (2), because $c = ab$ and $a = b$ imply that $a = b = -1$ or $a = b = 1$.

3. The Lefschetz number of maps of rational exterior spaces

In this section we apply the powerful theorem of Duan in [4] to generalize the results concerning products of odd-dimensional spheres.

Let $H^*(X;\mathbb{Q}) = \bigoplus_{r=0}^{n} H^r(X;\mathbb{Q})$ be the cohomology algebra of the finite complex X with multiplication given by the cup product. We will call an element $x \in H^r(X;\mathbb{Q})$ *decomposable* if there are pairs $(x_i, y_i) \in H^{p_i}(X;\mathbb{Q}) \times H^{q_i}(X;\mathbb{Q})$ with p_i, $q_i > 0$, $p_i + q_i = r > 0$ such that $x = \sum_i (x_i \cup y_i)$. For $r \geqslant 1$ let $A^r(X) = H^r(X;\mathbb{Q})/D^r(X)$, where D^r is the linear subspace over $\mathbb{Q}$ consisting of all decomposable elements. For a continuous map $f : X \to X$ let f^* be the induced homomorphism on cohomology algebra and $A(f) = \bigoplus_{r=1}^{n} A^r(f)$ the induced homomorphism on $A(X) = \bigoplus_{r=1}^{n} A^r(X)$.

Definition 3.1. Let $f : X \to X$ and let $I : A(X) \to A(X)$ be the identity morphism. The polynomial

$$A_f(t) := \det\bigl(tI - A(f)\bigr) = \prod_{r\geqslant 1} \det\bigl(tI - A^r(f)\bigr)$$

will be called the *characteristic polynomial* of f. The zeros of this polynomial: $\lambda_1(f), \dots, \lambda_k(f)$, $k = \operatorname{rank}_{\mathbb{Q}} A(X)$ (the dimension of $A(X)$ over $\mathbb{Q}$), will be called the *quotient eigenvalues* of f.

Definition 3.2. Let $\mathcal{R}$ be a commutative ring with identity.

An exterior algebra $\Lambda_{\mathcal{R}}[\alpha_1, \alpha_2, \dots]$ over $\mathcal{R}$ is a free $\mathcal{R}$ – module with a basis of all finite products $\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}$, where the ordering $i_1 < i_2 < \dots < i_k$ holds, and the associative and distributive multiplication defined by the rules: $\alpha_i\alpha_j = -\alpha_j\alpha_j$ for $i \neq j$ and $\alpha_i^2 = 0$. The empty product $1 \in \Lambda_{\mathcal{R}}[\alpha_1, \alpha_2, \dots]$ is the identity element.

In general the exterior algebra $\Lambda_{\mathcal{R}}[\alpha_1, \alpha_2, \dots]$ is the graded tensor product over the ring $\mathcal{R}$ of the one-variable exterior algebras $\Lambda_{\mathcal{R}}[\alpha_i]$, where all the dimensions $|\alpha_i|$ are odd, which we denote

$$\Lambda_{\mathcal{R}}[\alpha_1, \alpha_2, \dots] = \bigotimes_{i}{}_{\mathcal{R}} \Lambda_{\mathcal{R}}[\alpha_i], \quad |\alpha_i|\text{-odd}.$$

Definition 3.3. A connected topological space X is called *rational exterior* if there are some homogeneous elements $x_i \in H^{\mathrm{odd}}(X;\mathbb{Q})$, $i = 1, \dots, k$, such that the inclusions $x_i \hookrightarrow H^*(X;\mathbb{Q})$ give rise to a ring isomorphism $\Lambda_{\mathbb{Q}}[x_1, \dots, x_k] = H^*(X;\mathbb{Q})$.

Among examples of rational exterior spaces are: finite H-spaces (including all finite dimensional Lie groups) and some real Stiefel manifolds and spaces that admit a filtration $X = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} \cdots \xrightarrow{p_{k-1}} X_k \xrightarrow{p_k} X_{k+1} = \{\text{point}\}$, where p_i is the projection of an odd-dimensional sphere bundle (cf. [4]).

Theorem 3.4. *(Duan [4]) Let f be a self-map of a rational exterior space of rank k, and let $\lambda_1, \dots, \lambda_k$ be the quotient eigenvalues of f. Let A denote the matrix of $A(f)$. Then $L(f^m) = \det(I - A^m) = \prod_{i=1}^{k}(1 - \lambda_i^m)$.*

The above theorem has significant consequences in periodic point theory (cf. [1,5]). One of them is the following nice characterization of Lefschetz periodic point free self-maps of rational exterior spaces:

Corollary 3.5. *Any map $f : M \to M$, where M is a rational exterior space is Lefschetz periodic point free if and only if at least one of the quotient eigenvalues equals* 1.

The cohomology ring $H^*(M;\mathbb{Q})$ when M is any product of odd-dimensional spheres is isomorphic to the corresponding exterior algebra, i.e. $H^*(S^{k_1} \times \dots \times S^{k_n};\mathbb{Q}) \simeq \Lambda_{\mathbb{Q}}[\alpha_1, \alpha_2, \dots, \alpha_n]$ with k_i odd and α_i – a generator of $H^{k_i}(S^{k_i};\mathbb{Q}) \simeq \mathbb{Q}$.

Thus, by Corollary 3.5 we obtain in particular a generalization of the results of [7] for odd-dimensional products of spheres.

Example 3.6. Let $M = S^p \times S^p \times S^p$ with p-odd. Then

$$H^i(M;\mathbb{Q}) = \mathbb{Q} \quad \text{for } i = 0, 3p$$

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and

$$H^i(M;\mathbb{Q}) = \mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q} \quad \text{for } i = p, 2p.$$

All the other cohomology groups vanish. Moreover, if α, β, γ are the generators of $H^p(M;\mathbb{Q})$ then $\alpha\smile\beta, \beta\smile\gamma, \alpha\smile\gamma$ are generators of $H^{2p}(M;\mathbb{Q})$ and $\alpha\smile\beta\smile\gamma$ is a generator of $H^{3p}(M;\mathbb{Q})$. Groups $H^{2p}(M;\mathbb{Q})$ and $H^{3p}(M;\mathbb{Q})$ consist only of decomposable elements and in $H^p(M;\mathbb{Q})$ no elements are decomposable. Thus $A(X) = H^p(M;\mathbb{Q})$ and $A(f)$ is represented by the matrix A of f^{*p}. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A, which are also the quotient eigenvalues of f. Theorem 3.4 asserts that $L(f^m) = \det(I - A^m) = (1-\lambda_1^m)(1-\lambda_2^m)(1-\lambda_3^m)$ and thus f is Lefschetz periodic point free if and only if $\lambda_1 = 1$ or $\lambda_2 = 1$ or $\lambda_3 = 1$. What is more, one can easily express these conditions in terms of coefficients of the matrix A, which are coded in the equation $\det(I - A^m) = 0$.

4. Description of Lefschetz periodic point free maps in terms of essential eigenvalues

Let M be an n-dimensional compact connected manifold. For integers $i \geqslant 0$ and $f: M \to M$, let $e_i(\lambda) \neq 0$ be the number of eigenvalues of f^{*i} equal to λ. Define

$$e(\lambda) := \sum_{i=0}^{n} (-1)^i e_i(\lambda).$$

Definition 4.1. ([9]) An eigenvalue $\lambda \neq 0$ is *essential* if $e(\lambda) \neq 0$.

Let $\sigma(f)$ denote the spectrum of f_*. The set of essential eigenvalues will be denoted by $\sigma_{es}(f)$. Notice that only essential eigenvalues contribute to $\{L(f^m)\}_{m=1}^{\infty}$ since

$$L\big(f^m\big) = \sum_{\lambda\in\sigma(f)} e(\lambda)\lambda^m = \sum_{\lambda\in\sigma_{es}(f)} e(\lambda)\lambda^m.$$

Remark 4.2. Observe that if there are no essential eigenvalues, then $L(f^m) = 0$ for all $m \in \mathbb{N}$, so f is Lefschetz periodic point free.

Now we ask whether the converse statement is valid: assume that f is Lefschetz periodic point free, is that true that f has no essential eigenvalues? We will obtain an affirmative answer to that question as follows:

Theorem 4.3. *Let M be a compact connected manifold. $f: M \to M$ is Lefschetz periodic point free if and only if there are no essential eigenvalues of f.*

In the proof of the above theorem we will make use of the following algebraic result (for the proof see e.g. [3]).

Lemma 4.4. *Let A and B be finitely generated $\mathbb{C}$-vector spaces and $u: A \to A$ and $v: B \to B$ linear maps. If $\operatorname{tr}(u^k) = \operatorname{tr}(v^k)$ for every $k \geqslant 1$, then u and v have the same non-zero eigenvalues counted with their multiplicities.*

Proof of Theorem 4.3. We can divide cohomology groups into odd-dimensional and even-dimensional ones:

$$H^{\mathrm{ev}}(M;\mathbb{Q}) := \bigoplus_{i\text{-even}} H^i(M;\mathbb{Q}), \qquad H^{\mathrm{odd}}(M;\mathbb{Q}) := \bigoplus_{i\text{-odd}} H^i(M;\mathbb{Q})$$

and analogously for the induced homomorphisms

$$f^{*\mathrm{ev}}: H^{\mathrm{ev}}(M;\mathbb{Q}) \to H^{\mathrm{ev}}(M;\mathbb{Q}), \qquad f^{*\mathrm{odd}}: H^{\mathrm{odd}}(M;\mathbb{Q}) \to H^{\mathrm{odd}}(M;\mathbb{Q}).$$

Then

$$L(f) = \operatorname{tr} f^{*\mathrm{ev}} - \operatorname{tr} f^{*\mathrm{odd}}.$$

Suppose that $L(f^m) = 0$ for every $m \in \mathbb{N}$. Then $L(f^m) = \operatorname{tr} A_{\mathrm{ev}}^m - \operatorname{tr} A_{\mathrm{odd}}^m = 0$, where A_{ev} and A_{odd} are the matrices representing $f^{*\mathrm{ev}}$ and $f^{*\mathrm{odd}}$, respectively. By Lemma 4.4 we get that the eigenvalues of $f^{*\mathrm{ev}}$ and $f^{*\mathrm{odd}}$ are the same, together with corresponding multiplicities. As a consequence, for any $\lambda \in \sigma(f)$ we have $e(\lambda) = 0$, so there are no essential eigenvalues. $\square$

Notice that to determine if there exist some essential eigenvalues it is enough to consider only the first iterate f, not all the iterates f^m.

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Now we present two applications of Theorem 4.3.

Let us consider a self-map f of a connected compact manifolds with $H^i(M;\mathbb{Q})=0$ for i-odd. Examples of the manifolds satisfying these conditions are $\mathbb{C}P^n$ and $\mathbb{H}P^n$. Then, 1 is an eigenvalue on $H^0(M;\mathbb{Q})$ and obviously it is essential, thus by Theorem 4.3 f cannot be Lefschetz periodic point free. We obtain the following corollary which is much more general than Propositions 2.1 and 2.2.

Corollary 4.5. *If $H^i(M;\mathbb{Q})=0$ for all odd i, then there are no Lefschetz periodic point free maps on M.*

Now, let us consider a self-map f of a connected compact manifold with $H^i(M;\mathbb{Q})=0$ for even i, $i>1$. As M is connected $H^0(M;\mathbb{Q})=\mathbb{Q}$ and 1 is the only eigenvalue on $H^0(M;\mathbb{Q})$. A simple example of such a space is a sphere S^n with n odd. Now, by Theorem 4.3, if f is Lefschetz periodic point free then there are no non-zero eigenvalues different from 1 on $H^{\mathrm{odd}}(M;\mathbb{Q})$ (otherwise they were essential). Furthermore, there must be exactly one eigenvalue equal to 1 on $H^{\mathrm{odd}}(M;\mathbb{Q})$ to make 1 inessential. We then have the following:

Corollary 4.6. *If $H^i(M;\mathbb{Q})=0$ for $i>0$ even, then f is Lefschetz periodic point free if and only if there is exactly one non-zero eigenvalue equal to* 1 *on $H^{\mathrm{odd}}(M;\mathbb{Q})$.*

In particular for $f: S^n\to S^n$, where n is odd, f is Lefschetz periodic point free if and only if f is a map of degree one.

5. Removing periodic points for Lefschetz periodic point free self-maps of simply-connected manifolds

In this section we will demonstrate the method for minimizing the number of periodic points in the homotopy class for Lefschetz periodic point free maps. We consider f a self-map of a compact connected manifold of dimension at least 3.

For a given fixed natural r we define the numbers $MF_r(f)$ and $MF_{\leqslant r}(f)$ in the following way:

$$MF_r(f)=\min\{\#\operatorname{Fix}(g^r):\ g\sim f\},\tag{5.1}$$

$$MF_{\leqslant r}(f)=\min\Big\{\#\bigcup_{k\leqslant r}\operatorname{Fix}(g^k):\ g\sim f\Big\},\tag{5.2}$$

where $\sim$ means that the maps g and f are homotopic.

The following formula was proved by Jezierski in [6, Theorem 5.1] for self-maps of simply-connected manifolds:

$$MF_r(f)=\begin{cases}0 & \text{if } L(f^k)=0 \text{ for all } k|r,\\ 1 & \text{otherwise.}\end{cases}\tag{5.3}$$

We can apply this formula to remove all k periodic points ($k\leqslant r$) in the homotopy class of a map that is Lefschetz periodic point free.

Theorem 5.1. *Let f be a self-map of a compact connected and simply-connected manifold M which is Lefschetz periodic point free. Then, for any fixed r one can find in the homotopy class of f a map with no k-periodic points for $k\leqslant r$.*

Proof. Let r be a fixed natural number. Assume that g_1 is the map that realizes the minimum in the formula (5.2) for r and g_2 in the formula (5.1) but for $r!$. Then

$$\begin{aligned}MF_{\leqslant r}(f)&=\#\bigcup_{k\leqslant r}\operatorname{Fix}(g_1^k)\leqslant\#\bigcup_{k\leqslant r}\operatorname{Fix}(g_2^k)\leqslant\#\bigcup_{k|r!}\operatorname{Fix}(g_2^k)\\&=\#\operatorname{Fix}(g_2^{r!})=MF_{r!}(f).\end{aligned}$$

On the other hand, the condition that f is Lefschetz periodic point free implies by the formula (5.3) that $MF_{r!}(f)=0$. As a consequence $MF_{\leqslant r}(f)=0$, which ends the proof. □

Remark 5.2. In general Theorem 5.1 does not hold for manifolds that are not simply-connected, because then the formula (5.3) is not true.

Remark 5.3. Theorem 5.1 concerns only k-periodic points with $k\leqslant r$ (where r is fixed) i.e. it does not provide any knowledge about the existence of periodic points with higher minimal periods for maps in the homotopy class.

Remark 5.4. We note that the opposite kind of information, establishing the existence of many periodic points, can be obtained by the use of cohomological ring. For example, consider the product of odd-dimensional spheres, $X=S^{d_1}\times S^{d_2}\times\cdots\times S^{d_s}$, where $d_i\geqslant 3$. Let a finite group G act freely on the space X and $f: X\to X$ be an equivariant map. Suppose that there exists a prime $p|\#G$ such that there are no roots of unity of order p^τ ($\tau\geqslant 1$) among quotients eigenvalues of f. Recently, Jezierski and Marzantowicz proved that then f has infinitely many periodic points (Theorem 7.1 in [10]).

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