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Coronas and Domination Subdivision Number of a Graph

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Abstract In this paper, for a graph G and a family of partitions $\mathcal{P}$ of vertex neighborhoods of G, we define the general corona $G \circ \mathcal{P}$ of G. Among several properties of this new operation, we focus on application general coronas to a new kind of characterization of trees with the domination subdivision number equal to 3.

Keywords Domination · Domination subdivision number · Tree · Corona

Mathematics Subject Classification 05C69 · 05C05 · 05C99

1 Introduction

In this paper, we follow the notation and terminology of [7]. Let $G = (V(G), E(G))$ be a (finite, simple, undirected) graph of order $n = |V(G)|$. For a vertex v of G, its

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neighborhood, denoted by $N_G(v)$, is the set of all vertices adjacent to v. The cardinality of $N_G(v)$, denoted by $d_G(v)$, is called the *degree* of v. A vertex v is a *leaf* of G if $d_G(v) = 1$. Every neighbor of a leaf is called a *support* vertex. A *strong support* vertex is a vertex adjacent to at least two leaves.

A subset D of $V(G)$ is said to be *dominating* in G if every vertex belonging to $V(G) - D$ has at least one neighbor in D. The cardinality of the smallest dominating set in G, denoted by $\gamma(G)$, is called the *domination number* of G. A subset S of vertices in G is called a 2-*packing* if every two distinct vertices belonging to S are at distance greater than 2.

The *corona* of graphs G_1 and G_2 is a graph $G = G_1 \circ G_2$ resulting from the disjoint union of G_1 and $|V(G_1)|$ copies of G_2 in which each vertex v of G_1 is adjacent to all vertices of the copy of G_2 corresponding to v.

For a graph G, the *subdivision* of an edge $e = uv$ with a new vertex x is an operation which leads to a graph G' with $V(G') = V(G) \cup \{x\}$ and $E(G') = (E(G) - \{uv\}) \cup \{ux, xv\}$. The graph obtained from G by the replacing every edge $e = uv$ with a path (u, x_1, x_2, v), where x_1 and x_2 are new vertices, is called the 2-*subdivision* of G and is denoted by $S_2(G)$.

For a graph G and a family $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V(G)\}$, where $\mathcal{P}(v)$ is a partition of the set $N_G(v)$, by $G \circ \mathcal{P}$, we denote the graph in which

$$V(G \circ \mathcal{P}) = \{(v, 1)\colon v \in V(G)\} \cup \bigcup_{v \in V(G)} \{(v, A)\colon A \in \mathcal{P}(v)\}$$

and

$$\begin{aligned} E(G \circ \mathcal{P}) = & \bigcup_{v \in V(G)} \{(v, 1)(v, A)\colon A \in \mathcal{P}(v)\} \\ & \cup \bigcup_{uv \in E(G)} \{(v, A)(u, B)\colon (u \in A) \wedge (v \in B)\}. \end{aligned}$$

The family $\mathcal{P}$ is called a vertex neighborhood partition of G and the graph $G \circ \mathcal{P}$ is called a $\mathcal{P}$-*corona* (or shortly *general corona*) of G. The set $\{(v, 1)\colon v \in V(G)\}$ of vertices of $G \circ \mathcal{P}$ is denoted by $Ext(G \circ \mathcal{P})$ and its elements are called the *external* vertices. Those vertices of $G \circ \mathcal{P}$ which are not external, are said to be *internal*.

Example 1 Let G be the graph shown in Fig. 1a and let $\mathcal{P} = \{\mathcal{P}(a), \mathcal{P}(b), \mathcal{P}(c), \mathcal{P}(d), \mathcal{P}(e)\}$, where $\mathcal{P}(a) = \{N_G(a)\} = \{\{b, c\}\}$, $\mathcal{P}(b) = \{N_G(b)\} = \{\{a, c, d, e\}\}$, $\mathcal{P}(c) = \{N_G(c)\} = \{\{a, b\}\}$, $\mathcal{P}(d) = \{N_G(d)\} = \{\{b\}\}$, $\mathcal{P}(e) = \{N_G(e)\} = \{\{b\}\}$. Then the $\mathcal{P}$-corona $G \circ \mathcal{P}$ is the graph G_1 given in Fig. 1b and in fact it is the corona $G \circ K_1$.

Now if $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V(G)\}$ and $\mathcal{P}(v)$ is the family of all 1-element subsets of $N_G(v)$, that is $\mathcal{P}(a) = \{\{b\}, \{c\}\}$, $\mathcal{P}(b) = \{\{a\}, \{c\}, \{d\}, \{e\}\}$, $\mathcal{P}(c) = \{\{a\}, \{b\}\}$, $\mathcal{P}(d) = \{\{b\}\}$, $\mathcal{P}(e) = \{\{b\}\}$, then $G \circ \mathcal{P}$ is the graph G_2 shown in Fig. 1c and in this case it is the 2-subdivision $S_2(G)$ of G.

Finally, let us consider—for an example—the case where $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V(G)\}$ and $\mathcal{P}(a) = \{\{b, c\}\}$, $\mathcal{P}(b) = \{\{a\}, \{c, e\}, \{d\}\}$, $\mathcal{P}(c) = \{\{a\}, \{b\}\}$, $\mathcal{P}(d) = \{\{b\}\}$,

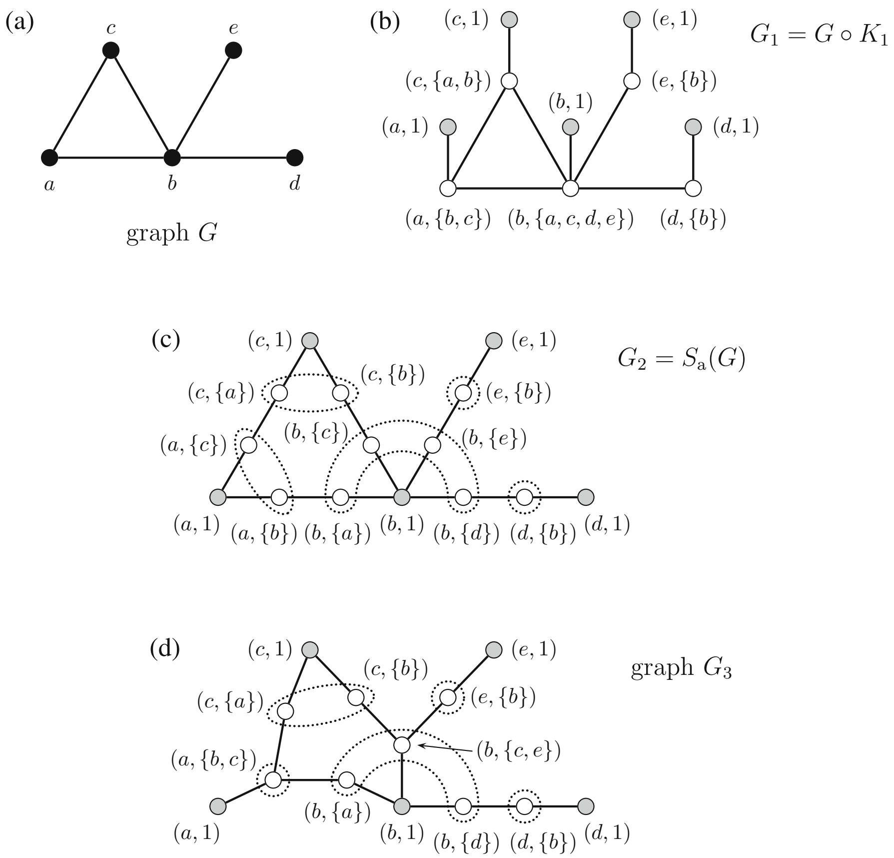

Fig. 1 Graph G and its exemplary coronas

$\mathcal{P}(e) = \{\{b\}\}$. In this case, $G \circ \mathcal{P}$ is the graph G_3 shown in Fig. 1d. This graph is an example of possible general coronas of G which are "between" the corona $G \circ K_1$ and the 2-subdivision $S_2(G)$.

From the definition of general corona, it obviously follows (as we have seen in the above example) that

(a) if $\mathcal{P}(v) = \{N(v)\}$ for every $v \in V(G)$, then $G \circ \mathcal{P}$ is the corona $G \circ K_1$ (and the vertices of G are internal vertices in $G \circ K_1$);
(b) if $\mathcal{P}(v) = \{\{u\}: u \in N_G(v)\}$ for every $v \in V(G)$, then $G \circ \mathcal{P}$ is the 2-subdivision $S_2(G)$ (and the vertices of G are external vertices of $S_2(G)$).

Let H be a subgraph of a graph G. The *contraction* of H to a vertex is the replacement of H by a single vertex k. Each edge that joined a vertex $v \in V(G) - V(H)$ to a vertex in H is replaced by an edge with endpoints v and k.

Let $\mathcal{P} = \{\mathcal{P}(v): v \in V(G)\}$ and $\mathcal{P}' = \{\mathcal{P}'(v): v \in V(G)\}$ be two vertex neighborhood partitions of G. We say that $\mathcal{P}'$ is a *refinement* of $\mathcal{P}$ and write $\mathcal{P}' \prec \mathcal{P}$ if for every

$v \in V(G)$ and every $A \in \mathcal{P}'(v)$ there exists $B \in \mathcal{P}(v)$ such that $A \subseteq B$. If $\mathcal{P}' \prec \mathcal{P}$, then the general corona $G \circ \mathcal{P}'$ is said to be *refinement* of $G \circ \mathcal{P}$. In this case, we write $G \circ \mathcal{P}' \prec G \circ \mathcal{P}$ and say that $G \circ \mathcal{P}'$ has been obtained from $G \circ \mathcal{P}$ by *splitting* some of its internal vertices. On the other hand, $G \circ \mathcal{P}$ can be obtained from $G \circ \mathcal{P}'$ contracting some of its internal vertices. For example, G_2 from Fig. 1 is refinement of G_3 and G_3 is refinement of G_1, so $G_2 \prec G_3 \prec G_1$. Notice that in general, a graph $G \circ \mathcal{P}$ can be treated as a graph obtained from corona $G \circ K_1$, where we split every support vertex v according to the partition $\mathcal{P}(v)$ of $N_G(v)$. Let us again consider the graphs G, G_1, G_2 and G_3 from Fig. 1. The graph $G_2 = S_2(G)$ can be obtained from $G \circ K_1$ by splitting support vertex into maximum possible number of vertices. Moreover, if in $G \circ K_1$ we split the vertex $(c, \{a, b\})$ into two vertices: $(c, \{a\})$ and $(c, \{b\})$, the vertex $(b, \{a, c, d, e\})$ into three vertices: $(b, \{a\})$, $(b, \{c, e\})$, $(b, \{d\})$, and we leave other support vertices unchanged, then we obtain G_3. On the other hand, G_3 can be obtained from $G_2 = S_2(G)$ contracting $(a, \{c\})$ and $(a, \{b\})$, and also $(b, \{c\})$ and $(b, \{e\})$

The contraction (splitting) of internal vertices is called an *internal contraction* (*splitting*). We have the following observations:

Observation 2 *Let T be a tree with at least three vertices. Then, the following properties are equivalent:*

1. *T is a general corona of a tree.*
2. *There exists a tree T' such that T is obtained from the 2-subdivision $S_2(T')$ by a sequence of internal contractions.*
3. *There exists a tree T' such that T is obtained from the corona $T' \circ K_1$ by a sequence of internal splittings.* □

Observation 3 *If G is a general corona of a tree, then $Ext(G)$ is a dominating 2-packing of G containing all leaves of G.*

Proof It follows from the following three facts: The distance between any two external vertices of G is at least three. Next, every internal vertex of G is adjacent to an external vertex. Finally, every leaf of G belongs to $Ext(G)$. □

Observation 4 *Let G and H be general coronas of some trees. If they share only one vertex which is an external vertex in each of them, then $G \cup H$ is a general corona.*

Proof Assume that G and H are general coronas of some trees T_1 and T_2, say $G = T_1 \circ \mathcal{P}_1$ and $H = T_2 \circ \mathcal{P}_2$ for some neighborhood partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of T_1 and T_2, respectively. Let $(v, 1)$ be the only common external vertex of G and H. Then the trees T_1 and T_2 share only v and the union $T = T_1 \cup T_2$ is a tree. Now, let $\mathcal{P}$ be the family $\{\mathcal{P}(x)\colon x \in V(T)\}$, where $\mathcal{P}(v) = \mathcal{P}_1(v) \cup \mathcal{P}_2(v)$, $\mathcal{P}(x) = \mathcal{P}_1(x)$ for $x \in V(T_1) - \{v\}$, and $\mathcal{P}(x) = \mathcal{P}_2(x)$ for $x \in V(T_2) - \{v\}$. Then $G \cup H$ is a $\mathcal{P}$-corona of T, that is, $G \cup H = T \circ \mathcal{P}$, see Fig. 2. □

Observation 5 *Let G be a general corona of a tree and let $(v, 1)$ be an external vertex of G. If we contract two distinct neighbors of $(v, 1)$, then the resulting graph is also a general corona of a tree.*

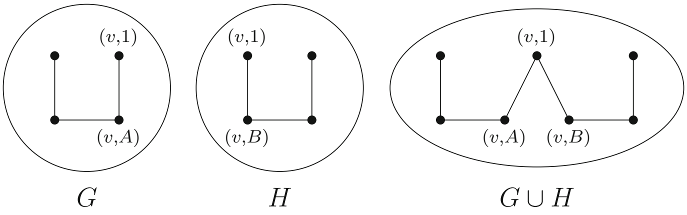

Fig. 2 Graphs G, H and $G \cup H$

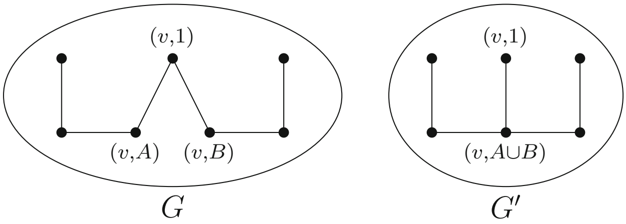

Fig. 3 Graphs G and G'

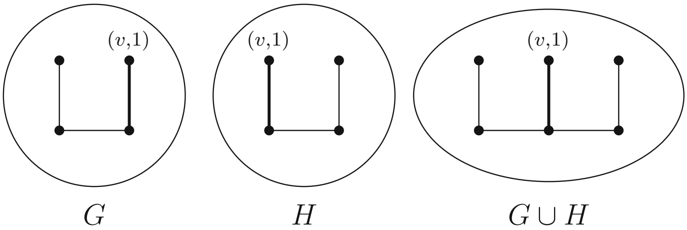

Fig. 4 Graphs G, H and $G \cup H$

Proof Assume that $G = T \circ \mathcal{P}$ for some tree T and its neighborhood partition $\mathcal{P}$. Let (v, A) and (v, B) be distinct neighbors of $(v, 1)$. Then the graph G', obtained from G by the contraction of (v, A) and (v, B), is a $\mathcal{P}'$-corona of T, where $\mathcal{P}'(v) = (\mathcal{P}(v) - \{A, B\}) \cup \{A \cup B\}$, and $\mathcal{P}'(x) = \mathcal{P}(x)$ if $x \in V(T) - \{v\}$, see Fig. 3. □

From Observations 4 and 5, we immediately have the next observation (see Fig. 4 for an illustration).

Observation 6 *Let G and H be general coronas of some trees. If they share only one edge such that exactly one of its end vertices is an external vertex in each of G and H, then the union $G \cup H$ is a general corona.*

2 Trees with Domination Subdivision Number 3

The *domination subdivision number* of a graph G, denoted by $\mathrm{sd}(G)$, is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. Since the domination number of the graph K_2 does not increase when its edge is subdivided, we consider the subdivision numbers for connected graphs of order at least 3. The domination subdivision number was defined by Velammal [8] and since then it has been widely studied, see [2–6] to mention just a few.

It was shown in [8] that the domination subdivision number of a tree is either 1, 2, or 3. Let $\mathcal{S}_i$ be the family of trees with domination subdivision number equal to i for $i \in \{1, 2, 3\}$. Some characterizations of the classes $\mathcal{S}_1$ and $\mathcal{S}_3$ were given in [2] and [1], respectively. In particular, the following constructive characterization of $\mathcal{S}_3$ was given in [1].

Let the label of a vertex v be denoted by $l(v)$ and $l(v) \in \{A, B\}$. Now, let $\mathcal{F}$ be the family of labeled trees that (i) contains P_4, where leaves have label A and support vertices have label B, and (ii) is closed under the following two operations, which extend a labeled tree $T \in \mathcal{F}$ by attaching a labeled path to a vertex $v \in V(T)$ in such a way that:

- If $l(v) = A$, then we add a path (x, y, z) (with labels $l(x) = l(y) = B$ and $l(z) = A$) and an edge vx.
- If $l(v) = B$, then we add a path (x, y) (with labels $l(x) = B$ and $l(y) = A$) and an edge vx.

The following characterization of trees belonging to the class $\mathcal{S}_3$ was given in [1].

Theorem 7 *The next three statements are equivalent for a tree T with at least three vertices:*

1. *T belongs to the class $\mathcal{S}_3$.*
2. *T has a unique dominating 2-packing containing all leaves of T.*
3. *T belongs to the family $\mathcal{F}$.*

Now we are in position to give a new characterization of trees belonging to the class $\mathcal{S}_3$. Namely, we shall show that all these graphs precisely are general coronas of trees.

Lemma 8 *If a tree T is a general corona, then T belongs to $\mathcal{S}_3$.*

Proof From Observation 3, the set of external vertices of T is a dominating 2-packing containing all leaves of T and, consequently, by Theorem 7, $T \in \mathcal{S}_3$. □

Lemma 9 *If a tree T belongs to $\mathcal{S}_3$, then T is a general corona.*

Proof We use induction on n, the number of vertices of a tree. The smallest tree belonging to $\mathcal{S}_3$ is a path P_4 and, obviously, P_4 is a $\mathcal{P}$-corona of P_2. Let $T \in \mathcal{S}_3$ be a tree on n vertices, $n > 4$. We will show that T is a general corona. Let $P = (v_0, v_1, \ldots, v_k)$ be the longest path in T. From the choice of P, since T does not have a strong support vertex (by Theorem 7), it follows that $k \geqslant 4$ and $d_T(v_1) = 2$. We consider two cases: $d_T(v_2) = 2$, $d_T(v_2) > 2$.
Case 1: $d_T(v_2) = 2$. Let T_1 and T_2 denote subtrees $T[\{v_0, v_1, v_2, v_3\}]$ and $T - \{v_0, v_1, v_2\}$, respectively. By Theorem 7, the tree T has a dominating 2-packing S containing all leaves of T and certainly $\{v_0, v_3\} \subseteq S$. Consequently, $S - \{v_0\}$ is a

dominating 2-packing in T_2 containing all leaves of T_2. Again by Theorem 7, the tree T_2 belongs to $\mathcal{S}_3$. Thus, by induction, T_2 is a general corona. Since v_3 belongs to $S - \{v_0\}$, by Observation 3 and Theorem 7, $v_3 \in Ext(T_2)$. Obviously $T_1 = P_4$ is a general corona. Because v_3 is also an external vertex in T_1, and trees T_1 and T_2 do not share any other vertex, $T = T_1 \cup T_2$ is a general corona by Observation 4.

Case 2: $d_T(v_2) > 2$. In this case, again by Theorem 7, the tree T has a dominating 2-packing S containing all leaves of T. Let v' be the unique neighbor of v_2 belonging to S. Since S is a 2-packing containing all leaves of T, v' is not a support vertex in T. Thus, from the choice of P, it follows that either v' is a leaf or $v' = v_3$. In both cases, let T_1 and T_2 be subtrees $T[\{v_0, v_1, v_2, v'\}]$ and $T - \{v_0, v_1\}$, respectively. It is easy to observe that $S - \{v_0\}$ is a dominating 2-packing in T_2 containing all leaves of T_2. Now, again by Theorem 7, the tree T_2 belongs to $\mathcal{S}_3$. Thus, by induction, T_2 is a general corona. Since v' belongs to $S - \{v_0\}$, by Observation 3 and Theorem 7, $v' \in Ext(T_2)$. Certainly $T_1 = P_4$ is a general corona and v' is external vertex in T_1. In addition, T_1 and T_2 share only the edge v_2v'. Consequently, by Observation 6, the tree $T = T_1 \cup T_2$ is a general corona. □

Taking into account Observation 2, Theorem 7, Lemmas 8 and 9 we have the summary result.

Theorem 10 *Let T be a tree with at least three vertices. Then, the following properties are equivalent:*

1. *The domination subdivision number of T is equal to 3 $(i.e., T \in \mathcal{S}_3)$.*
2. *T has a unique dominating 2-packing containing all leaves of T.*
3. *T belongs to the family $\mathcal{F}$.*
4. *T is a general corona of a tree.*
5. *There exists a tree T' such that T is obtained from the 2-subdivision $S_2(T')$ by a sequence of internal contractions.*
6. *There exists a tree T' such that T is obtained from the corona $T' \circ K_1$ by a sequence of internal splittings.*

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