



## Weakly Convex Domination Subdivision Number of a Graph

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**Abstract.** A set  $X$  is *weakly convex* in  $G$  if for any two vertices  $a, b \in X$  there exists an  $ab$ -geodesic such that all of its vertices belong to  $X$ . A set  $X \subseteq V$  is a *weakly convex dominating set* if  $X$  is weakly convex and dominating. The *weakly convex domination number*  $\gamma_{\text{wcon}}(G)$  of a graph  $G$  equals the minimum cardinality of a weakly convex dominating set in  $G$ . The *weakly convex domination subdivision number*  $\text{sd}_{\gamma_{\text{wcon}}}(G)$  is the minimum number of edges that must be subdivided (each edge in  $G$  can be subdivided at most once) in order to increase the weakly convex domination number. In this paper we initiate the study of weakly convex domination subdivision number and establish upper bounds for it.

### 1. Introduction

Throughout this paper,  $G$  is a simple connected graph with vertex set  $V(G)$  and edge set  $E(G)$  (briefly  $V$  and  $E$ ). For every vertex  $v \in V(G)$ , the *open neighborhood* of  $v$ ,  $N_G(v) = N(v)$ , is the set  $\{u \in V(G) \mid uv \in E(G)\}$  and its *closed neighborhood* is the set  $N_G[v] = N[v] = N(v) \cup \{v\}$ . The *open neighborhood* of a set  $S \subseteq V$  is the set  $N_G(S) = N(S) = \cup_{v \in S} N(v)$ , and the *closed neighborhood* of  $S$  is the set  $N_G[S] = N[S] = N(S) \cup S$ . The *degree* of a vertex  $v$  is  $d_G(v) = |N_G(v)|$ . A *leaf* is a vertex of degree one and a *universal vertex* is a vertex of degree  $|V(G)| - 1$ . We denote the number of leaves in a graph  $G$  by  $\ell(G)$ . The minimum and maximum degrees of  $G$  are respectively denoted by  $\delta(G)$  and  $\Delta(G)$ . The *private neighborhood* of a vertex  $u \in D$  with respect to a set  $D \subseteq V$ , is the set  $PN_G[u, D] = N_G[u] - N_G[D - \{u\}]$ . If  $v \in PN_G[u, D]$ , then we say that  $v$  is a private neighbor of  $u$  with respect to the set  $D$ . For a set  $S$  of vertices of  $G$  we denote by  $G[S]$  the subgraph induced by  $S$  in  $G$ . The *distance*  $d_G(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $uv$ -path in  $G$ . A  $uv$ -path of length  $d_G(u, v)$  is called a  *$uv$ -geodesic*. The largest distance between any pair of vertices  $u, v$  in  $G$  is the *diameter* of  $G$ , denoted by  $\text{diam}(G)$ . The *girth*  $g(G)$  of a graph  $G$  is the length of a shortest cycle in  $G$ . The *edge connectivity number*  $\kappa'(G)$  of  $G$  is the minimum number of edges whose removal results in a disconnected graph. For every connected graph,  $\kappa'(G) \leq \delta(G)$ .

A set  $A \subset V$  is a *dominating set* of  $G$  if  $N_G[A] = V$ , and is a *connected dominating set* if  $N_G[A] = V$  and the induced subgraph  $G[A]$  is connected. The (*connected*) *domination number*  $\gamma(G)$  ( $\gamma_c(G)$ ) is the minimum

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cardinality of a (connected) dominating set of  $G$ , and a (connected) dominating set of minimum cardinality is called a  $\gamma(G)$ -set ( $\gamma_c(G)$ -set).

A set  $X$  is *weakly convex* in  $G$  if for any two vertices  $a, b \in X$  there exists an  $ab$ -geodesic such that all of its vertices belong to  $X$ . A set  $X \subseteq V$  is a *weakly convex dominating set* if  $X$  is weakly convex and dominating. The *weakly convex domination number* of a graph  $G$ , denoted by  $\gamma_{wcon}(G)$ , equals to the minimum cardinality of a weakly convex dominating set in  $G$ . Weakly convex domination number was first introduced by Jerzy Topp, Gdańsk University of Technology, 2002.

In application, network design for example, if a parameter  $\mu(G)$  is important to study, then it is important to know the effect that modifications of  $G$  have on  $\mu(G)$ . For example, vertices can be deleted and edges can be deleted or added. In network design, deleting a vertex or an edge may represent component's failure. From the other perspective, networks can be made fault-tolerant by providing redundant communication link (adding edges). The effects on the domination number of a graph, when  $G$  is modified by deleting a vertex or deleting or adding an edge, have been investigated extensively (see chapter 7 of [16]). In particular, the effects on the weakly convex domination number of a graph, when  $G$  is modified by deleting a vertex or deleting or adding an edge, have been investigated in [19].

Alternatively, one can consider how many modifications must take place before a parameter changes. Along these lines, Fink et al. [12], defined the *bondage number* of a graph to equal the minimum number of edges whose removal increases the domination number. On the other hand, Kok and Mynhardt [17] defined the *reinforcement number* of a graph to equal the minimum number of edges which must be added to a graph in order to decrease the domination number. Considering a different type of graph modification, Velammal [20] defined the domination subdivision number  $sd_\gamma(G)$  to be the minimum number of edges that must be subdivided (where each edge in  $G$  can be subdivided at most once) in order to increase the domination number. The domination subdivision number has been studied by several authors (see for instance [1, 11, 14, 15]). A similar concepts related to connected domination were studied in [10], to total domination in [14], to Roman domination in [2], to rainbow domination in [5, 13], and to 2-domination in [3]. It is known that the domination subdivision parameters can take arbitrarily large values [2, 5, 9, 10, 14] and an interesting problem is to find good bounds on these parameters in terms of other parameters of  $G$ . For instance, it has been proved that for any connected graph  $G$  of order  $n$ ,  $sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1$  [7],  $sd_{\gamma_t}(G) \leq 2n/3$  [8],  $sd_{\gamma_c}(G) \leq \lfloor \frac{n}{2} \rfloor$  [10],  $sd_{\gamma_R}(G) \leq \lceil \frac{n}{2} \rceil - 1$  [2] and  $sd_{\gamma_{t2}}(G) \leq n - \Delta(G) + 2$  [13].

The (*weakly convex, connected*) *domination subdivision number*  $sd_\gamma(G)$  ( $sd_{\gamma_{wcon}}(G)$ ,  $sd_{\gamma_c}(G)$ ) of a graph  $G$  is the minimum number of edges that must be subdivided (where each edge in  $G$  can be subdivided at most once) in order to increase the (*weakly convex, connected*) domination number. (We say that an edge  $e = uv \in E(G)$  is *subdivided* with a vertex  $x$  if the edge  $uv$  is deleted, but a new vertex  $x$  is added, along with two new edges  $ux$  and  $vx$ . The vertex  $x$  is called a *subdivision vertex* and obtained graph is denoted by  $G_e$ ). Since the (*weakly convex, connected*) domination number of the graph  $K_2$  does not change when its only edge is subdivided, we consider weakly convex domination subdivision number for all graphs  $G$  satisfying  $\Delta(G) \geq 2$ .

For any unexplained terms see [16].

Our purpose in this paper is to initialize the study of the weakly convex domination subdivision number  $sd_{\gamma_{wcon}}(G)$ . In particular, we establish some sharp upper bounds on  $sd_{\gamma_{wcon}}(G)$ .

Next result shows that subdividing an edge can decrease or increase the weakly convex domination number.

**Theorem 1.1.** The differences  $\gamma_{wcon}(G) - \gamma_{wcon}(G_e)$  and  $\gamma_{wcon}(G_e) - \gamma_{wcon}(G)$  can be arbitrarily large.

*Proof.* First we show that for some edge  $e$  the difference  $\gamma_{wcon}(G) - \gamma_{wcon}(G_e)$  can be arbitrarily large. Let  $k \geq 3$  and let  $G'$  be the graph obtained from a  $(2k + 1)$ -cycle

$$C_{2k+1} = (v_1, v_2, \dots, v_{2k+1})$$

(where  $v_1v_2, v_2v_3, \dots, v_{2k}v_{2k+1}, v_{2k+1}v_1$  are edges of this cycle) by adding the edge  $v_kv_{k+2}$ , the edges  $v_iv_{2k-i+1}$  for  $i = 1, \dots, k - 2$  and adding the pendant edges  $v_iv_i$  for  $i = 1, \dots, k + 1, 2k + 1$ . For any  $\gamma_{wcon}(G')$ -set  $D$

of  $G'$ , we must have  $|D \cap \{v_i, w_i\}| \geq 1$  for  $i = 1, \dots, k + 1, 2k + 1$ . Since  $D$  is weakly convex, we deduce that  $v_i \in D$ . In particular,  $v_{k+1}, v_{2k+1} \in D$ . Since  $v_{k+1}, v_{k+2}, \dots, v_{2k+1}$  is the only  $v_{k+1}v_{2k+1}$ -geodesic in  $G'$ , we have  $v_i \in D$  for each  $i = 1, 2, \dots, 2k + 1$  and hence  $\gamma_{\text{wcon}}(G') \geq 2k + 1$ . On the other hand,  $V(C_{2k+1})$  is obviously a weakly convex dominating set of  $G'$  and so  $\gamma_{\text{wcon}}(G') = 2k + 1$ . Let  $G'_e$  be a graph obtained from  $G'$  by subdividing the edge  $e = v_{2k+1}v_{2k}$ . It is easy to see that  $\gamma_{\text{wcon}}(G'_e) = k + 2$  (note that the support vertices of  $G'_e$  form a  $\gamma_{\text{wcon}}(G'_e)$ -set). The case  $k = 3$  is illustrated in Figure 1.

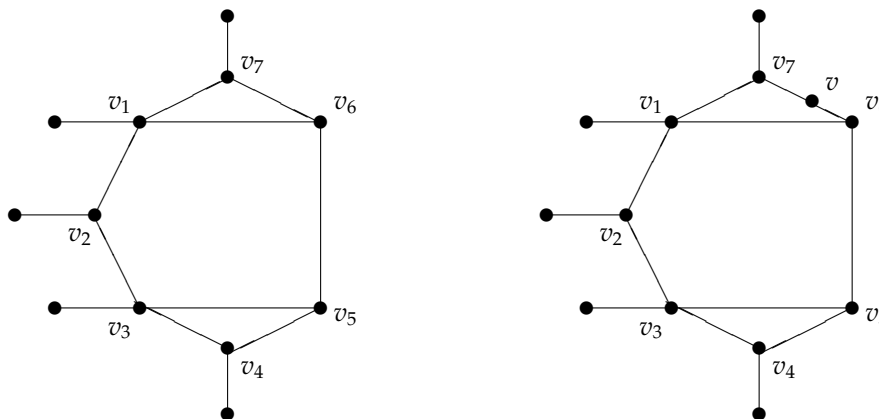


Figure 1: Graph  $G'$  and  $G'_e$  for  $k = 3$ .

Now we show that  $\gamma_{\text{wcon}}(G_e) - \gamma_{\text{wcon}}(G)$  can be arbitrarily large for some edge  $e$ . Let  $k \geq 1$  be an integer and let  $G''$  be obtained from a  $(2k + 2)$ -cycle  $C_{2k+2} = (v_1, v_2, \dots, v_{2k+2})$  by adding the edges  $v_i v_{2k+2-i}$  for  $i = 1, \dots, k$  and adding the pendant edges  $v_i w_i$  for  $i = 1, \dots, k + 1, 2k + 2$ . As above we can see that  $\gamma_{\text{wcon}}(G'') = k + 2$  (note that the support vertices form a minimum weakly convex dominating set of  $G''$ ). After subdividing the edge  $e = v_1 v_{2k+2}$  we obtain  $G''_e$ , for which  $\gamma_{\text{wcon}}(G''_e) = 2k + 3$  (all of the vertices except leaves form a minimum weakly convex dominating set for  $G''_e$ ). Figure 2 demonstrates the case  $k = 3$ .  $\square$

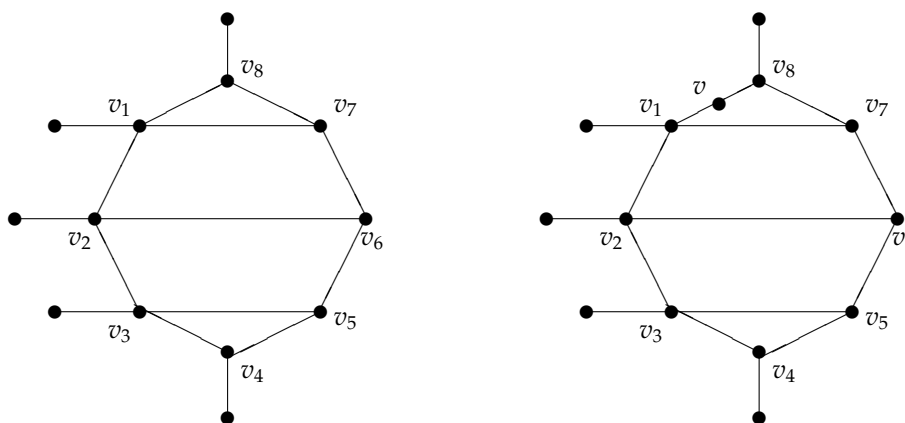


Figure 2: Graph  $G''$  and  $G''_e$  for  $k = 3$ .

In the next theorem we give an upper bound for weakly convex domination number.

**Theorem 1.2.** Let  $G$  be a connected graph of order  $n \geq 3$  with  $\delta(G) > \lfloor \frac{n}{2} \rfloor$ . Then

$$\gamma_{\text{wcon}}(G) \leq \max\{3, 2 \lfloor \frac{n}{2} \rfloor - \delta(G)\}.$$

*Proof.* Let us denote  $c = \delta(G) - \lfloor \frac{n}{2} \rfloor$ . If  $\text{diam}(G) \geq 3$ , then let  $x$  and  $y$  be the vertices such that  $d_G(x, y) = 3$ . Then  $1 + d_G(x) + 1 + d_G(y) \leq n$  that implies  $2\lfloor \frac{n}{2} \rfloor + 2 < 2\delta(G) + 2 \leq n$ , which is impossible. Thus  $\text{diam}(G) \leq 2$ . If  $\text{diam}(G) = 1$ , then  $\gamma_{\text{wcon}}(G) = 1$ . Suppose now that  $\text{diam}(G) = 2$ . Let  $v \in V(G)$  be a vertex with minimum degree  $\delta(G)$  and let  $N(v) = \{v_1, v_2, \dots, v_{\delta(G)}\}$ . Since  $\text{diam}(G) = 2$  and  $d_G(v) = \delta(G)$ ,  $V(G) - N[v] \neq \emptyset$ . Assume first there is a vertex  $u \in V(G) - N[v]$  such that  $V(G) - N[v] \subseteq N[u]$ . Since  $\text{diam}(G) = 2$ ,  $u$  and  $v$  have a common neighbor, say  $w$ . In the case,  $\{u, v, w\}$  is clearly a weakly convex dominating set of  $G$  and hence  $\gamma_{\text{wcon}}(G) \leq 3$ . Now let  $V(G) - N[v] \not\subseteq N[u]$  for every  $u \in V(G) - N[v]$ . It follows that  $|V(G) - N[v]| \geq 2$ . Let  $x \in V(G) - N[v]$ . Then

$$d_G(v) + d_G(x) - |N(v) \cap N(x)| + 3 \leq n.$$

Since  $d_G(x) \geq \delta(G)$ ,  $d_G(v) \geq \delta(G)$ ,  $\delta(G) = \lfloor \frac{n}{2} \rfloor + c$  and  $2\lfloor \frac{n}{2} \rfloor \geq n - 1$  we have

$$|N(v) \cap N(x)| \geq 2c + 2.$$

Hence each vertex in  $V(G) - N[v]$  is adjacent to at least  $2c + 2$  vertices in  $N(v)$ . Moreover, the set  $N(v) - \{v_1, v_2, \dots, v_{\lfloor \frac{n}{2} \rfloor - c - 1}\}$  consists of  $2c + 1$  vertices. Hence, each vertex in  $V(G) - N[v]$  is adjacent to at least one of the vertices in  $\{v_1, \dots, v_{\lfloor \frac{n}{2} \rfloor - c - 1}\}$ . This implies that the set  $\{v, v_1, \dots, v_{\lfloor \frac{n}{2} \rfloor - c - 1}\}$  is a weakly convex dominating set of  $G$  and so  $\gamma_{\text{wcon}}(G) \leq \lfloor \frac{n}{2} \rfloor - c = 2\lfloor \frac{n}{2} \rfloor - \delta(G)$ . This completes the proof.  $\square$

## 2. Bounds on Weakly Convex Domination Subdivision Number

In this section, we establish some upper bounds on weakly convex domination subdivision number. Given  $S, T \subseteq V(G)$ , we write  $[S, T]$  for the set of edges having one end-point in  $S$  and the other in  $T$ . An edge cut is an edge set of the form  $[S, \bar{S}]$ , where  $S$  is a nonempty proper subset of  $V(G)$  and  $\bar{S}$  denotes  $V(G) - S$ .

**Theorem 2.1.** For any connected graph  $G$  of order  $n \geq 3$ ,  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \kappa'(G)$ .

*Proof.* Let  $[S, \bar{S}]$  be an edge cut of  $G$  of size  $\kappa'(G)$ , and let  $G_1$  and  $G_2$  be the components of  $G - [S, \bar{S}]$ . Assume  $G'$  is the graph obtained from  $G$  by subdividing the edges of  $[S, \bar{S}]$  and  $T$  be the set of all subdivision vertices. Let  $D$  be a minimum weakly convex dominating set of  $G'$  and  $D_i = D \cap V(G_i)$  for  $i = 1, 2$ . If  $D \cap T = \emptyset$ , then  $D_i \neq \emptyset$  for  $i = 1, 2$ , and  $D = D_1 \cup D_2$ . Now for vertices  $x_1 \in D_1$  and  $x_2 \in D_2$ , any  $x_1x_2$ -geodesic intersects  $T$  implying that  $D \cap T \neq \emptyset$  which leads to a contradiction. Therefore  $D \cap T \neq \emptyset$ . Since  $D$  is weakly convex set of  $G'$  and  $d_G(x, y) \leq d_{G'}(x, y)$  for the vertices  $x, y \in D - T$ , the set  $D - T$  is a weakly convex dominating set of  $G$ . Moreover  $|D - T| < \gamma_{\text{wcon}}(G')$ . This yields  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \kappa'(G)$  and the proof is completed.  $\square$

According to Theorem 1.1, the subdividing an edge may decrease the weakly convex domination number. Hence, it is not immediately obvious that the weakly convex domination subdivision number is defined for all connected graphs  $G$  with  $\Delta(G) \geq 2$ . However, since every connected graph of order at least 3 has an edge cut, we conclude from Theorem 2.1 that the weakly convex domination number is well-defined for all connected graphs  $G$  with  $\Delta(G) \geq 2$ .

Moreover, from Theorem 2.1 we also obtain two corollaries.

**Corollary 2.2.** If there exists a cut edge in  $G$ , then  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ .

**Corollary 2.3.** For any connected simple graph  $G$  of order  $n \geq 3$ ,  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \delta(G)$ .

Next result presents the necessary condition for a graph to have  $\text{sd}_{\gamma_{\text{wcon}}}(G) > 1$ .

**Proposition 2.4.** If  $\text{sd}_{\gamma_{\text{wcon}}}(G) > 1$ , then every edge of  $G$  belongs to a cycle  $C_3, C_4$  or  $C_5$ .

*Proof.* Assume that  $G$  has an edge  $e$  such that  $e$  does not belong neither to a 3-cycle nor to 4-cycle nor to 5-cycle. We will show that  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ . If  $e$  is a cut-edge, then from Corollary 2.2,  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ . Suppose now  $e$  belongs to a cycle. Let  $C$  be a smallest cycle containing  $e$ . From our assumption  $C = C_p$ , where  $p \geq 6$ . Let us subdivide the edge  $e = uv$  with a vertex  $w$  and let  $D'$  be a  $\gamma_{\text{wcon}}(G_e)$ -set. If  $\{u, v\} \subseteq D'$ , then  $w \in D'$

and clearly  $D' - \{w\}$  is a weakly convex dominating set in  $G$  that implies  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ . Let  $|D' \cap \{u, v\}| = 1$ . Assume, without loss of generality, that  $\{u, v\} \cap D' = \{u\}$ . First let  $w \in D'$ . Suppose  $v'$  is the neighbor of  $v$  on  $C$  other than  $u$ . Since  $p \geq 6$ ,  $v' \notin D'$ . So  $v'$  is dominated by  $v'' \in D'$ . Then  $d_{G_e}(w, v'') \leq 3$ . Since  $v$  and  $v'$  does not belong to  $D'$  and  $D'$  is weakly convex, there is another  $ww''$ -path, say  $P_1$ . Then the induced subgraph  $G[V(P_1) - \{w\} \cup \{v, v', v''\}]$  gives a cycle of length at most 5, a contradiction. Now let  $w \notin D'$ . Then  $v$  is dominated by  $z \in D'$  and  $d_{G_e}(u, z) \leq 3$ . Since  $w \notin D'$  and  $v \notin D'$  and  $D'$  is weakly convex, there is another  $uz$ -path, say  $P_2$ , of length at most 3. Then the induced subgraph  $G[V(P_2) \cup \{v\}]$  is a cycle of length at most 5, a contradiction.  $\square$

In [10] the following Proposition was shown.

**Proposition 2.5.** [10] If  $G$  is a connected graph of order  $n \geq 3$ , then  $\text{sd}_{\gamma_c}(G) \leq \gamma_c(G) + 1$ .

We prove similar relation for weakly convex domination. Let  $\alpha'(G)$  be the maximum number of edges in a matching in  $G$ .

**Proposition 2.6.** If  $G$  contains a matching  $M$  such that  $\gamma_{\text{wcon}}(G) < |M|$ , then  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq |M|$ . In particular, if  $\alpha'(G) > \gamma_{\text{wcon}}(G)$ , then  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \gamma_{\text{wcon}}(G) + 1$ .

*Proof.* Let  $G'$  be obtained by subdividing every edge of  $M$ . Each weakly convex dominating set of  $G'$  has order at least  $|M|$ . Hence  $\gamma_{\text{wcon}}(G') > \gamma_{\text{wcon}}(G)$  and thus  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq |M|$ . If  $\alpha'(G) > \gamma_{\text{wcon}}(G)$ , then  $G$  contains a matching  $M$  of size  $\gamma_{\text{wcon}}(G) + 1$ , which leads to the result.  $\square$

**Theorem 2.7.** If  $G$  is a connected graph of order  $n \geq 3$ , then

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \gamma_{\text{wcon}}(G) + 1.$$

*Proof.* The result is immediate for  $n = 3, 4, 5$ . Let  $n \geq 6$ . If  $\delta(G) \leq \gamma_{\text{wcon}}(G) + 1$ , then by Corollary 2.3 we have  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \delta(G) \leq \gamma_{\text{wcon}}(G) + 1$ . If  $\gamma_{\text{wcon}}(G) > \lfloor \frac{n}{2} \rfloor$ , then by Theorem 1.2,  $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$  and again by Corollary 2.3 we have  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \gamma_{\text{wcon}}(G)$ . Moreover, if  $\gamma_c(G) = \gamma_{\text{wcon}}(G)$ , then from Proposition 3.2 and Proposition 2.5 we obtain  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \text{sd}_{\gamma_c}(G) \leq \gamma_c(G) + 1 = \gamma_{\text{wcon}}(G) + 1$ . For  $\alpha'(G) > \gamma_{\text{wcon}}(G)$  the result follows from Proposition 2.6.

In the remaining cases we assume that  $\gamma_{\text{wcon}}(G) \leq \lfloor \frac{n}{2} \rfloor$ ,  $\delta(G) \geq \gamma_{\text{wcon}}(G) + 1$ ,  $\alpha'(G) \leq \gamma_{\text{wcon}}(G)$  and  $\gamma_c(G) < \gamma_{\text{wcon}}(G)$ . It is known from [6] that the matching number of every graph is at least  $\min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$ . Hence, since  $\delta(G) > \alpha'(G)$ , we have  $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$ . This implies  $\gamma_{\text{wcon}}(G) = \lfloor \frac{n}{2} \rfloor$ . So  $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$ , what gives  $\text{diam}(G) \leq 2$ . Hence every  $\gamma_c(G)$ -set is also  $\gamma_{\text{wcon}}(G)$ -set, a contradiction with  $\gamma_c(G) < \gamma_{\text{wcon}}(G)$ .  $\square$

In [4] and [18] the following results were shown.

**Proposition 2.8.** ([4]) For any connected simple graph  $G$  of order  $n \geq 3$  with  $g(G) \geq 5$ ,  $\gamma(G) \geq \delta(G)$ .

**Proposition 2.9.** [18] If  $G$  is a connected graph of order  $n$ , then  $\gamma(G) \leq \gamma_c(G) \leq \gamma_{\text{wcon}}(G)$ .

From above Propositions and Corollary 2.3 we obtain the next result.

**Proposition 2.10.** For any connected graph  $G$  of order  $n \geq 3$  with  $g(G) = 5$ ,

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \gamma_{\text{wcon}}(G).$$

*Proof.* By Propositions 2.8 and 2.9, we obtain  $\gamma_{\text{wcon}}(G) \geq \delta(G)$  and the result follows from Corollary 2.3.  $\square$

### 3. Graphs with Small Weakly Convex Domination Subdivision Number

In this section, we consider graphs with small weakly convex domination subdivision number. We make use of the following results in this section.

**Proposition 3.1.** [10] If a connected graph  $G$  of order  $n \geq 3$  satisfies one of the following properties

- (i)  $\gamma_c(G) = 1$ ;
- (ii)  $\gamma_c(G) = 2$  and  $G$  contains a  $\gamma_c(G)$ -set  $\{a, b\}$  such that  $N(a) \cap N(b) = \emptyset$ ,

then  $\text{sd}_{\gamma_c}(G) = 1$ .

**Proposition 3.2.** If  $\gamma_c(G) = \gamma_{\text{wcon}}(G)$ , then  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \text{sd}_{\gamma_c}(G)$ .

*Proof.* After subdividing  $\text{sd}_{\gamma_c}(G)$  edges of  $G$ , the resulting graph  $G'$  satisfies  $\gamma_c(G') > \gamma_c(G) = \gamma_{\text{wcon}}(G)$ . Hence  $\gamma_{\text{wcon}}(G') \geq \gamma_c(G') > \gamma_{\text{wcon}}(G)$  and  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \text{sd}_{\gamma_c}(G)$ .  $\square$

Now we present sufficient conditions for a graph to have weakly convex domination subdivision number equal to 1.

**Proposition 3.3.** Let  $G$  be a connected graph of order  $n \geq 3$ . If  $G$  satisfies one of the following properties:

- (i)  $\gamma_{\text{wcon}}(G) = 1$ ;
- (ii)  $\gamma_{\text{wcon}}(G) = 2$  and  $G$  contains a  $\gamma_{\text{wcon}}(G)$ -set  $\{a, b\}$  such that  $N(a) \cap N(b) = \emptyset$ ;
- (iii)  $G$  contains two adjacent vertices of degree 2;
- (iv)  $g(G) \geq 6$ ;
- (v)  $G$  has an edge  $e$  such that if  $e$  is subdivided with a vertex  $w$ , then  $G_e$  has a  $\gamma_{\text{wcon}}(G_e)$ -set containing  $w$ ,

then  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ .

*Proof.* (i, ii) Clearly  $\gamma_c(G) = \gamma_{\text{wcon}}(G)$  and the result follows from Proposition 3.2 and Proposition 3.1.

(iii) Let  $x_1$  and  $y_1$  be two adjacent vertices of degree 2 in  $G$  and let  $G'$  be obtained from  $G$  by subdividing the edge  $x_1y_1$  with a vertex  $z$ . Then  $G'$  contains an induced path  $x, x_1, z, y_1, y$  (possibly a cycle if  $x$  and  $y$  are adjacent or if  $x = y$ ). Let  $D$  be a  $\gamma_{\text{wcon}}(G')$ -set. If  $z \in D$ , then  $D - \{z\}$  is a weakly convex dominating set of  $G$ . Let  $z \notin D$ . To dominate  $z$ , without loss of generality we can suppose that  $x_1 \in D$ . Since  $D$  is weakly convex,  $\{x, x_1\}$  is a subset of  $D$ . Since  $x_1$  was in  $D$  only to dominate the vertex  $z$ , the set  $D - \{x_1\}$  is a weakly convex dominating set of  $G$ . Therefore  $\gamma_{\text{wcon}}(G) < \gamma_{\text{wcon}}(G')$ . In particular for paths and cycles,  $\text{sd}_{\gamma_{\text{wcon}}}(P_n) = \text{sd}_{\gamma_{\text{wcon}}}(C_n) = 1$ .

(iv) Let  $e = u_1u_2$  be an arbitrary edge of  $G$ . If  $e$  is a cut edge, then clearly  $\gamma_{\text{wcon}}(G_e) > \gamma_{\text{wcon}}(G)$ . Let  $C = (u_1, u_2, \dots, u_k)$  be a cycle containing  $e$ . Assume  $G_e$  is obtained from  $G$  by subdividing the edge  $e$  with subdivision vertex  $w$  and  $D$  is a  $\gamma_{\text{wcon}}(G_e)$ -set. We show that  $w \in D$  which implies  $D - \{w\}$  is a weakly convex dominating set of  $G$ , as desired. Assume to the contrary that  $w \notin D$ . It follows that  $\{u_1, u_2\} \not\subseteq D$ . Assume without loss of generality that  $u_2 \notin D$ . Then to dominate  $w$  and  $u_2$ , we must have  $u_1 \in D$  and  $D \cap N_G(u_2) \neq \emptyset$ . Let  $v \in D \cap N_G(u_2)$ . Since  $g(G) \geq 6$ , then  $u_1, w, u_2, v$  is the unique  $u_1v$ -path in  $G'$  that implies  $u_1, w, u_2, v \in D$ , a contradiction. Therefore,  $w \in D$  and  $\gamma_{\text{wcon}}(G_e) > \gamma_{\text{wcon}}(G)$ . Thus  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ .

(v) Let  $e = ab$  be an edge of  $G$  such that if  $e$  is subdivided with a vertex  $w$  then  $G_e$  has a  $\gamma_{\text{wcon}}(G_e)$ -set  $D'$  containing  $w$ . Since  $D'$  is a weakly convex dominating set,  $D' \cap \{a, b\} \neq \emptyset$ . If  $a, b \in D'$ , then  $w \in D'$  and  $D' - \{w\}$  is obviously a weakly convex dominating set of  $G$  implying that  $\gamma_{\text{wcon}}(G) \leq |D'| - 1 < \gamma_{\text{wcon}}(G_e)$ . Let  $\{a, b\} \not\subseteq D'$ . Assume without loss of generality that  $a \in D'$  and  $b \notin D'$ . If  $D' = \{a, w\}$ , then obviously  $D' - \{w\}$  is a weakly convex dominating set of  $G$  and hence  $\gamma_{\text{wcon}}(G) < \gamma_{\text{wcon}}(G_e)$  again. Let  $\{a, w\} \not\subseteq D'$ . Since  $D'$  is a weakly convex dominating set, we deduce that for any  $x \in D' - \{a, w\}$  there exists a  $xw$ -geodesic  $P_{xw}$  in  $G'$  such that  $V(P_{xw}) \subseteq D'$ . Obviously  $P_{xw} - \{w\}$  is a  $xa$ -geodesic in  $G$  for each  $x \in D'$ . It follows that  $D' - \{w\}$  is a weakly convex dominating set of  $G$  implying that  $\gamma_{\text{wcon}}(G) < \gamma_{\text{wcon}}(G_e)$ . Thus  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1$ .  $\square$



Note that the case (i) includes the complete graphs and the case (ii) includes the complete bipartite graph  $K_{p,q}$  with  $p, q \geq 2$ , and the graph obtained from  $K_4$  by subdividing one edge once.

Now we give upper bounds for weakly convex domination subdivision number of graphs with weakly convex domination number 2 or 3.

**Proposition 3.4.** Let  $G$  be a connected graph of order  $n \geq 3$  with  $\gamma_{\text{wcon}}(G) = 2$ . Then

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq 2.$$

*Proof.* Let  $G$  be a connected graph of order  $n \geq 3$  with  $\gamma_{\text{wcon}}(G) = 2$ . Then  $\Delta(G) \leq n - 2$ . Let  $S = \{u, v\}$  be a  $\gamma_{\text{wcon}}(G)$ -set,  $u'$  a private neighbor of  $u$  with respect to  $S$  and  $v'$  a private neighbor of  $v$  with respect to  $S$ . Let  $G'$  be the graph obtained from  $G$  by subdividing the edges  $uu', vv'$  with subdivision vertices  $x$  and  $y$ , respectively, and let  $D$  be a  $\gamma_{\text{wcon}}(G')$ -set. We show that  $|D| \geq 3$  that implies  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq 2$ . Suppose to the contrary that  $|D| \leq 2$ . To dominate  $x, y$ , we must have  $|D \cap \{u, u'\}| \geq 1$  and  $|D \cap \{v, v'\}| \geq 1$ . Since  $|D| \leq 2$ , we have  $|D \cap \{u, u'\}| = 1$  and  $|D \cap \{v, v'\}| = 1$ . Since  $G[D]$  is connected,  $uv' \notin E(G)$  and  $vu' \notin E(G)$ , we deduce that either  $D = \{u, v\}$  or  $D = \{u', v'\}$ . In each case,  $D$  is not a dominating set of  $G'$  which is a contradiction.  $\square$

**Proposition 3.5.** Let  $k \geq 2$  be an integer. For the complete  $k$ -partite graph  $G = K_{p_1, p_2, \dots, p_k}$  with  $2 \leq p_1 \leq p_2 \leq \dots \leq p_k$ ,

$$\text{sd}_{\gamma_{\text{wcon}}}(G) = \begin{cases} 1 & \text{if } k = 2 \\ 2 & \text{otherwise.} \end{cases}$$

*Proof.* It is clear that any two adjacent vertices form a minimum weakly convex dominating set of  $G$  which implies  $\gamma_{\text{wcon}}(G) = 2$ . If  $k = 2$ , the result follows from Proposition 3.3 (ii). Let  $k \geq 3$  and let  $V_1, V_2, \dots, V_k$  be the partite sets of  $V(G)$ . By Proposition 3.4,  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq 2$ . Now we show that  $\text{sd}_{\gamma_{\text{wcon}}}(G) \geq 2$ . Let  $e = ab$  be an edge of  $G$ . Hence  $a \in V_i, b \in V_j$ , where  $i \neq j$ . In this case the set  $\{a, v\}$ , where  $v$  is a vertex belonging to  $V_k$  and  $k \notin \{i, j\}$ , forms a minimum weakly convex dominating set of  $G$ . Thus  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 2$  and the proof is completed.  $\square$

Proposition 3.5 shows that the bound in Proposition 3.4 is sharp.

In [10] the following Proposition was presented.

**Proposition 3.6.** If  $G$  is a connected graph of order  $n \geq 3$  and  $\gamma_c(G) = 3$ , then  $1 \leq \text{sd}_{\gamma_c}(G) \leq 3$ .

We use above result when we consider graphs with weakly convex domination subdivision number equal to 3.

**Proposition 3.7.** For every connected graph  $G$  of order  $n \geq 3$ , if  $\gamma_{\text{wcon}}(G) = 3$ , then  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq 3$ .

*Proof.* If  $\gamma_{\text{wcon}}(G) = 3$ , then the  $\gamma_c(G)$ -sets and the  $\gamma_{\text{wcon}}(G)$ -sets are the same and so  $\gamma_c(G) = \gamma_{\text{wcon}}(G)$ . By Proposition 3.2,  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \text{sd}_{\gamma_c}(G)$  and the result follows from Proposition 3.6.  $\square$

Using Propositions 3.3, 3.4 and 3.7, we obtain other two general upper bounds for weakly convex domination subdivision number.

**Theorem 3.8.** For any connected graph  $G$  of order  $n \geq 3$ ,

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

*Proof.* The result is immediate for  $n = 3$ . Let  $n \geq 4$ . If  $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$ , the result is true by Corollary 2.3. Let  $\delta(G) > \lfloor \frac{n}{2} \rfloor$ . If  $n = 4$ , then  $\gamma_{\text{wcon}}(G) = 1$  and it follows from Proposition 3.3 that  $\text{sd}_{\gamma_{\text{wcon}}}(G) = 1 < \lfloor \frac{n}{2} \rfloor$ . If  $n = 5$ , then clearly  $\gamma_{\text{wcon}}(G) \leq 2$  and by Propositions 3.3 and 3.4 we have  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq 2 = \lfloor \frac{n}{2} \rfloor$ . Let  $n \geq 6$ . We deduce from Theorem 1.2 that  $\gamma_{\text{wcon}}(G) \leq \max\{3, 2\lfloor \frac{n}{2} \rfloor - \delta(G)\}$ . Now the result follows from Proposition 3.7 and Theorem 2.7.  $\square$

**Corollary 3.9.** For any connected graph  $G$  of order  $n \geq 3$ ,

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \alpha'(G).$$

*Proof.* By Corollary 2.3 and Theorem 3.8, we have  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$ . On the other hand, it is known from [6] that the matching number of every graph is at least  $\min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$ . Thus  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \alpha'(G)$ .  $\square$

Next result gives a bound for the weakly convex domination subdivision number of a triangle-free graph  $G$  with  $\gamma_{\text{wcon}}(G) = 4$ .

**Theorem 3.10.** Let  $G$  be a connected triangle-free graph  $G$  with  $\gamma_{\text{wcon}}(G) = 4$ . Then  $\text{sd}_{\gamma_{\text{wcon}}}(G) \leq 4$ .

*Proof.* Let  $D = \{u_1, u_2, u_3, u_4\}$  be a  $\gamma_{\text{wcon}}(G)$ -set such that the size of  $G[D]$  is as large as possible. Since the induced subgraph  $G[D]$  is connected, we consider three cases.

**Case 1.**  $G[D] = C_4$  such that  $u_1u_2 \in E(C_4)$ ,  $u_2u_3 \in E(C_4)$ ,  $u_3u_4 \in E(C_4)$ ,  $u_4u_1 \in E(C_4)$ .

If  $u_i$  has no private neighbor with respect to  $D$  for some  $i$ , then clearly  $D - \{u_i\}$  is a weakly convex dominating set of  $G$  which is a contradiction. Let  $v_i$  be a private neighbor of  $u_i$  with respect to  $D$  for each  $i$ . Assume  $G'$  is obtained from  $G$  by subdividing the edges  $u_i v_i$  with vertices  $x_i$  for each  $i$ . Suppose  $S_1$  is a  $\gamma_{\text{wcon}}(G')$ -set. We show that  $|S_1| \geq 5$ . Assume to the contrary that  $|S_1| \leq 4$ . To dominate  $x_i$ , we must have  $S_1 \cap \{u_i, v_i\} \neq \emptyset$ . Since  $|S_1| \leq 4$ ,  $|S_1 \cap \{u_i, v_i\}| = 1$ . Since  $v_i$  is a private neighbor of  $u_i$  with respect to  $D$  for each  $i$ ,  $S_1 \cap \{u_i \mid 1 \leq i \leq 4\} \neq \emptyset$  and  $S_1 \cap \{v_i \mid 1 \leq i \leq 4\} \neq \emptyset$ . Let  $u_i \in S_1, v_j \in S_1$  for some  $i \neq j$ . Then clearly every  $u_i v_j$ -path contains a vertex not in  $\{u_i, v_i \mid 1 \leq i \leq 4\}$  which leads to a contradiction.

**Case 2.**  $G[D] = P_4$  such that  $u_1u_2 \in E(P_4)$ ,  $u_2u_3 \in E(P_4)$ ,  $u_3u_4 \in E(P_4)$ .

Obviously  $u_1$  and  $u_4$  have no common neighbor. If  $u_1$  has no private neighbor with respect to  $D$ , then clearly  $D - \{u_1\}$  is a weakly convex dominating set of  $G$ , a contradiction. Let  $v_1$  be a private neighbor of  $u_1$  with respect to  $D$ . Similarly,  $u_4$  has a private neighbor with respect to  $D$ , say  $v_4$ . Assume  $G'$  is obtained from  $G$  by subdividing the edges  $u_1v_1, u_1u_2, u_2u_3, u_4v_4$  with vertices  $x_1, x_2, x_3, x_4$ , respectively. Assume  $S_2$  is a  $\gamma_{\text{wcon}}(G')$ -set. Now we show that  $|S_2| \geq 5$ . Let  $|S_2| \leq 4$ . Clearly,  $S_2 \cap \{u_1, v_1\} \neq \emptyset, S_2 \cap \{u_4, v_4\} \neq \emptyset$  and  $S_2 \cap \{u_2, u_3\} \neq \emptyset$ . If  $\{u_1, v_1\} \subseteq S_2$  then  $x_1 \in S_2$  implying that  $|S_2| \geq 5$  which is a contradiction. Therefore  $|S_2 \cap \{u_1, v_1\}| = 1$ . Similarly,  $|S_2 \cap \{u_4, v_4\}| = 1$ . First let  $u_1 \notin S_2$ . Then we must have  $u_2, v_1 \in S_2$ . Since  $G$  is triangle-free and  $|S_2| \leq 4$ , we have  $2 \leq d_G(v_1, u_2) \leq 3$ . If  $d_G(v_1, u_2) = 3$ , then let  $v_1, w_1, w_2, u_2$  is a geodesic path such that  $S_2 = \{v_1, u_2, w_1, w_2\}$ . Since  $G$  is triangle-free,  $u_1w_1 \notin E(G)$  and  $u_1w_2 \notin E(G)$  and so  $S_2$  does not dominate  $u_1$ , a contradiction. Therefore  $d_G(v_1, u_2) = 2$  and so  $u_2$  and  $v_1$  have a common neighbor  $w$  not in  $\{u_3, u_4, v_4\}$ . Hence  $S_2 = \{v_1, w, u_2, w'\}$  where  $w' \in \{u_4, v_4\}$ . But then, to dominate  $u_1$ , we must have  $wu_1 \in E(G)$  which is a contradiction because  $G$  is triangle-free. Assume now  $u_1 \in S_2$ . If  $u_2 \in S_2$ , then  $x_2 \in S_2$  and clearly  $G'[S_2]$  will be not connected which is a contradiction. Let  $u_2 \notin S_2$  that yields  $u_3 \in S_2$ . Since  $G$  is triangle-free and  $|S_2| \leq 4$ ,  $u_3, u_1$  have a common neighbor  $w$  which belongs to  $S_2$ . If  $w' \in S_2 \cap \{u_4, v_4\}$ , then  $S_2 = \{w, w', u_1, u_3\}$ . It is easy to see that  $S_2$  does not dominate  $u_2$ , a contradiction.

**Subcase 2.3.**  $G[D] = K_{1,3}$ .

Assume  $u = u_4$  is the center of  $G[D] = K_{1,3}$  and  $u_1, u_2, u_3$  are leaves adjacent to  $u$ . As above, we can see that  $u_i$  has a private neighbor with respect to  $D$ , say  $v_i$ , for each  $i$ . Let  $G'$  be the graph obtained from  $G$  by subdividing the edges  $u_1v_1, u_2v_2, u_3v_3$  with vertices  $x_1, x_2, x_3$ , respectively, and let  $S_3$  be a  $\gamma_{\text{wcon}}(G')$ -set. We show that  $|S_3| \geq 5$ . Assume to the contrary that  $|S_3| \leq 4$ . To dominate  $x_i$ , we must have  $S_3 \cap \{u_i, v_i\} \neq \emptyset$  for each  $i$ . If  $\{u_i, v_i\} \subseteq S_3$  for some  $i$ , then  $x_i \in S_3$  implying that  $|S_3| \geq 5$ , a contradiction. If  $u_i, v_j \in S_3$ , then  $u_i, v_j$  must have a common neighbor  $w \in S_3$  that dominates  $v_i$  or is adjacent to  $u_k, v_k$  ( $k \notin \{i, j\}$ ) which is a contradiction because  $G$  is triangle-free. If  $u_i, v_j \in S_3$  for some  $i \neq j$ , then  $u_i, v_j$  must have a common neighbor  $w$  that dominates  $u_j$ , a contradiction again. So we assume  $\{v_1, v_2, v_3\} \subseteq S_3$ . If  $S_3 = \{v_1, v_2, v_3, w\}$ , then  $w$  must be adjacent to  $u_i$  for each  $i$  which leads to a contradiction because  $G$  is triangle-free. This completes the proof.  $\square$

Next result is an immediate consequence of Propositions 2.10, 3.3 (part (iv)), 3.4, 3.7 and Theorem 3.10.

**Corollary 3.11.** For any connected graph  $G$  of order  $n \geq 3$  with  $g(G) \geq \gamma_{\text{wcon}}(G)$ ,

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \gamma_{\text{wcon}}(G).$$



#### 4. Graphs with Large Weakly Convex Domination Subdivision Number

In the previous sections, we essentially presented bounds on the weakly convex domination subdivision number in graphs. Our goal in this section is to show that the weakly convex domination subdivision number of a graph can be arbitrarily large. The following graph was introduced by Haynes et al. in [14] to prove a similar result for  $\text{sd}_{\gamma_i}(G)$ .

Let  $X = \{1, 2, \dots, 3(k-1)\}$  and let  $\mathcal{Y} = \{Y \subset X \mid |Y| = k\}$ . Thus,  $\mathcal{Y}$  consists of all  $k$ -subsets of  $X$ , and so  $|\mathcal{Y}| = \binom{3(k-1)}{k}$ . Let  $G$  be the graph with vertex set  $X \cup \mathcal{Y}$  and with edge set constructed as follows: add an edge joining every two distinct vertices of  $X$  and for each  $x \in X$  and  $Y \in \mathcal{Y}$ , add an edge joining  $x$  and  $Y$  if and only if  $x \in Y$ . Then,  $G_k$  is a connected graph of order  $n = \binom{3(k-1)}{k} + 3(k-1)$ . The set  $X$  induces a clique in  $G_k$ , while the set  $\mathcal{Y}$  is an independent set and each vertex of  $\mathcal{Y}$  has degree  $k$  in  $G_k$ . Therefore  $\delta(G) = k$ . Favaron et al. [10] proved that  $\gamma_c(G_k) = 2(k-1)$  and  $\text{sd}_{\gamma_c}(G_k) = k$ .

**Proposition 4.1.** For any integer  $k \geq 2$ ,  $\gamma_{\text{wcon}}(G_k) = 2(k-1)$ .

*Proof.* By Proposition 3.6,  $\gamma_{\text{wcon}}(G_k) \geq \gamma_c(G_k) = 2(k-1)$ . On the other hand, any subset of  $X$  of cardinality  $2(k-1)$  is a weakly convex dominating set of  $G$ , and so  $\gamma_{\text{wcon}}(G_k) \leq 2(k-1)$ . Consequently,  $\gamma_{\text{wcon}}(G_k) = \gamma_c(G_k) = 2(k-1)$ .  $\square$

**Theorem 4.2.** For any integer  $k \geq 2$ ,  $\text{sd}_{\gamma_{\text{wcon}}}(G_{2k}) \geq k+1$ .

*Proof.* Assume  $F = \{e_1, \dots, e_k\}$  is an arbitrary subset of  $k$  edges of  $G_{2k}$  and let  $G'_{2k}$  be the graph obtained from  $G_{2k}$  by subdividing all edges in  $F$ . We show that  $\gamma_{\text{wcon}}(G'_{2k}) \leq \gamma_{\text{wcon}}(G_{2k}) = 2(2k-1)$ . Assume  $e_i = u_i v_i$  for each  $i$  and let  $S = X \cap \{u_i, v_i \mid 1 \leq i \leq k\}$ . Clearly  $|X - S| \geq 4k - 3$  and each vertex in  $X - S$  is adjacent to all vertices in  $S$ . Since every edge of  $G_{2k}$  is incident with at least one vertex of  $X$ , we may assume that  $u_i \in X$  for each  $i$ . If  $v_i \in \mathcal{Y}$ , then since  $d_{G_{2k}}(v_i) = 2k$  and  $|F| = k$ ,  $v_i$  is adjacent to a vertex of  $X - S$ , say  $w_i$ , such that  $v_i w_i \notin F$ . If  $v_i \in X$ , then let  $w_i$  be any vertex of  $X - S$ . Assume  $D_F = \{u_i, w_i \mid 1 \leq i \leq k\}$ . Then  $|D_F| \leq 2k$ . Now extend  $D_F$  to a set  $D$  of size  $2(2k-1)$  by adding  $4k-2-|D_F|$  vertices of  $X - S$ . Obviously  $D$  is a weakly convex dominating set of  $G'_{2k}$  and so  $\gamma_{\text{wcon}}(G'_{2k}) \leq 2(2k-1) = \gamma_{\text{wcon}}(G_{2k})$ . This implies that  $\text{sd}_{\gamma_{\text{wcon}}}(G_{2k}) \geq k+1$ .  $\square$

We conclude this paper with an open problem.

**Problem 4.3.** Prove or disprove: For any connected graph  $G$  of order  $n \geq 3$ ,

$$\text{sd}_{\gamma_{\text{wcon}}}(G) \leq \gamma_{\text{wcon}}(G).$$

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