

# Anisotropic Orlicz-Sobolev spaces of vector valued functions and Lagrange equations

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## Abstract

In this paper we study some properties of anisotropic Orlicz and Orlicz-Sobolev spaces of vector valued functions for a special class of G-functions. We introduce a variational setting for a class of Lagrangian Systems. We give conditions which ensure that the principal part of variational functional is finitely defined and continuously differentiable on Orlicz-Sobolev space.

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## 1. Introduction

In this paper we make some preliminary steps for variational analysis in anisotropic Orlicz-Sobolev spaces of vector valued functions. We consider the Euler-Lagrange equation

$$\frac{d}{dt}L_v(t, u(t), \dot{u}(t)) = L_x(t, u(t), \dot{u}(t)), \quad t \in (a, b) \quad (1)$$

where Lagrangian is of the form  $L(t, x, v) = F(t, x, v) + V(t, x)$ .

If  $F(v) = \frac{1}{2}|v|^2$  then the equation (1) reduces to  $\ddot{u}(t) + \nabla V(t, u(t)) = 0$ . One can consider more general case  $F(v) = \phi(|v|)$ , where  $\phi$  is convex and nonnegative. In the above cases  $F$  does not depend on  $v$  directly but rather on its norm  $|v|$  and the growth of  $F$  is the same in all directions, i.e.  $F$  has isotropic growth. Equation (1) with Lagrangian  $L(t, x, v) = \frac{1}{p}|v|^p + V(t, x)$  has been studied by

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many authors under different conditions. The classical reference is [1]. The isotropic Orlicz-Sobolev space setting was considered in [2].

We are interested in anisotropic case. This means that  $F$  depends on all components of  $v$  not only on  $|v|$  and has different growth in different directions. A simple example of such function is  $F(v) = \sum_{i=1}^N |v_i|^{p_i}$  or  $F(v) = \sum_{i=1}^N \phi_i(|v_i|)$ , where  $\phi_i$  are N-functions. We wish to consider more general situation. We assume that  $F: [a, b] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies

$$(F_1) \quad F \in C^1,$$

$$(F_2) \quad |F(t, x, v)| \leq a(|x|)(b(t) + G(v)),$$

$$(F_3) \quad |F_x(t, x, v)| \leq a(|x|)(b(t) + G(v)),$$

$$(F_4) \quad G^*(F_v(t, x, v)) \leq a(|x|)(c(t) + G^*(\nabla G(v))),$$

where  $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ ,  $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$  and  $G: \mathbb{R}^N \rightarrow \mathbb{R}$  is a G-function. Conditions  $(F_1)$ – $(F_4)$  are direct generalization of standard growth conditions from [1] (see also [2]). We show (see Theorem 5.7) that under these conditions the functional  $\mathcal{I}: \mathbf{W}^1 \mathbf{L}^G \rightarrow \mathbb{R}$  given by

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u}) dt$$

is continuously differentiable.

We restrict our considerations to a special class of G-functions. Here  $G: \mathbb{R}^n \rightarrow [0, \infty)$  is convex,  $G(-x) = G(x)$ , supercoercive,  $G(0) = 0$  and satisfies  $\Delta_2$  and  $\nabla_2$  conditions. We define the anisotropic Orlicz space to be

$$\mathbf{L}^G(I, \mathbb{R}^N) = \{u: I \rightarrow \mathbb{R}^N: \int_I G(u) dt < \infty\}.$$

The Orlicz space  $\mathbf{L}^G$  equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0: \int_I G\left(\frac{u}{\alpha}\right) dt \leq 1 \right\}$$

is a reflexive Banach space. An important example of Orlicz space is classical Lebesgue  $\mathbf{L}^p$  space, defined by  $G(x) = \frac{1}{p}|x|^p$ . In this case, the Luxemburg norm and the standard  $\mathbf{L}^p$  norm are equivalent. Therefore, Orlicz spaces can be viewed as a straightforward generalization of  $\mathbf{L}^p$  spaces.

Properties of N-functions and Orlicz spaces of real-valued functions has been studied in great details in monographs [3, 4, 5] and [6]. The standard references for vector-valued case are [7, 8, 9] and [10, 11] for Banach-space valued functions. In [7, 8] author considers a class of G-functions together with a uniformity conditions

which, for example, excludes the function  $G(x) = \sum |x_i|^{p_i}$  unless  $1 < p_1 = \dots = p_N < \infty$ . Moreover  $G$  is not necessarily assumed to be an even function. As was pointed out in [11], if  $G$  is not even then  $\mathbf{L}^G$  is no longer a vector space (see also [10, Example 2.1]).

Our strong conditions on  $G$  allow us to work in Orlicz spaces without worry about some technical difficulties arising in general case. For example, it is well known that the set  $\mathbf{L}^G(I, \mathbb{R}^N)$  is a vector space if and only if  $G$  satisfies  $\Delta_2$  condition. Otherwise  $\mathbf{L}^G$  is only a convex set. Another difficulty is the convergence notion. In Lebesgue spaces  $\|u_n - u\|_{\mathbf{L}^p} \rightarrow 0$  means simply  $\int |u_n - u|^p \rightarrow 0$ . For arbitrary G-function  $G$ , convergence in Luxemburg norm is not equivalent to  $\int G(u_n - u) dt \rightarrow 0$  unless  $G$  satisfies  $\Delta_2$ . The  $\Delta_2$  condition is also crucial for separability and reflexivity of  $\mathbf{L}^G$ .

The main consequence of anisotropic nature of  $G$  is the lack of monotonicity of the norm. It is no longer true that  $|u| \leq |v|$  implies  $\|u\|_{\mathbf{L}^G} \leq \|v\|_{\mathbf{L}^G}$ . In anisotropic case, standard dominance condition  $|u_n| \leq f$  does not implies convergence in  $\mathbf{L}^G$  norm and must be replaced by  $G(u_n) \leq f$  (see Theorem 3.17).

For every  $G$  there exist  $p, q \in (1, \infty)$  such that  $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$ . If  $G(x) = \sum |x_i|^{p_i}$  then  $\mathbf{L}^G$  can be identified with the product of  $\mathbf{L}^{p_i}$  but in many cases an anisotropic Orlicz Space is not equal to the space  $\mathbf{L}^{p_1} \times \mathbf{L}^{p_2} \times \dots \times \mathbf{L}^{p_N}$  (see Example 3.7).

To give a proper variational setting for equation (1) we introduce a notion of an anisotropic Orlicz-Sobolev space  $\mathbf{W}^1 \mathbf{L}^G$  of vector-valued functions. It is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^N) = \{u \in \mathbf{L}^G(I, \mathbb{R}^N) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^N)\}$$

with the norm

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

To the authors best knowledge there is no reference for the case of anisotropic norm and vector-valued functions of one variable. The references for other cases are [2, 9, 12, 13, 14, 15, 16, 17, 18, 19].

In [9] and [18] the space  $H^0(G, \Omega)$ ,  $\Omega \subset \mathbb{R}^n$  is defined as a completion of  $C_0^1(\Omega, \mathbb{R}^n)$  under norm  $\|u\|_{H^0(G, \Omega)} = \|Du\|_{G, \Omega}$ . It is classical result due to Trudinger  $H^0(G, \Omega) \hookrightarrow L_A(\Omega)$ , where  $A$  is some N-function (see also Cianchi [14]).

In [17] and [19] the anisotropic Orlicz-Sobolev space  $W^1 L_G$  is defined for G-function  $G : \mathbb{R}^{n+1} \rightarrow [0, \infty]$  as a space of weakly differentiable functions  $u : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}$  such that  $(u, D_1 u, D_2 u, \dots, D_n u)$  belongs to the Orlicz space generated by  $G$ . A norm for  $W^1 L_G$  is given by

$$\|u\|_{1, G, \Omega} = \|(u, Du)\|_{G, \Omega}.$$

In [12] we can find definition of isotropic Orlicz-Sobolev space of real valued functions

$$W_A^1(\Omega) = \{u \in \Omega \rightarrow \mathbb{R} \text{ measurable} : u, |\nabla u| \in L_A\},$$

where  $L_A$  is Orlicz Space and  $A$  is an N-function.

In [2] the isotropic Orlicz-Sobolev space of vector-valued functions is defined to be a space of absolutely continuous functions  $u : [0, T] \rightarrow \mathbb{R}^d$  such that  $u$  and  $\dot{u}$  belongs to Orlicz space generated by an N-function. Similar treatment can be found in [20].

## 2. G-functions

Let  $\langle \cdot, \cdot \rangle$  denote the standard inner product on  $\mathbb{R}^N$  and  $|\cdot|$  is the induced norm. We assume that  $G : \mathbb{R}^N \rightarrow [0, \infty)$  satisfies the following conditions:

$$(G_1) \quad G(0) = 0,$$

$$(G_2) \quad G \text{ is convex,}$$

$$(G_3) \quad G \text{ is even,}$$

$$(G_4) \quad G \text{ is supercoercive:}$$

$$\lim_{|x| \rightarrow \infty} \frac{G(x)}{|x|} = \infty,$$

$$(G_5) \quad G \text{ satisfies the } \Delta_2 \text{ condition:}$$

$$\exists_{K_1 \geq 2} \exists_{M_1 > 0} \forall_{|x| \geq M_1} G(2x) \leq K_1 G(x), \quad (\Delta_2)$$

$$(G_6) \quad G \text{ satisfies the } \nabla_2 \text{ condition:}$$

$$\exists_{K_2 \geq 1} \exists_{M_2 > 0} \forall_{|x| \geq M_2} G(x) \leq \frac{1}{2K_2} G(K_2 x). \quad (\nabla_2)$$

A function  $G$  is a G-function in the sense of Trudinger [9]. In general, G-function can be unbounded on bounded sets and need not satisfy conditions  $(G_4)$ – $(G_6)$  but only  $\lim_{x \rightarrow \infty} G(x) = \infty$ . A G-function of one variable is called N-function. Some typical examples of  $G$  are:

1.  $G_p(x) = \frac{1}{p}|x|^p, 1 < p < \infty,$
2.  $G(x) = \sum_{i=1}^N G_{p_i}(x_i), 1 < p_i < \infty,$
3.  $G(x_1, x_2) = (x_1 - x_2)^2 + x_2^4.$

A function  $G$  can be equal to zero in some neighborhood of 0. So that a function

$$G(x) = \begin{cases} 0 & |x| \leq 1 \\ |x|^2 - 1 & |x| > 1 \end{cases}$$

is also admissible. Condition  $\Delta_2$  implies that  $G$  is of polynomial growth (see Lemma 2.4 below and [3]). A function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$   $f(x) = e^{|x|} - |x| - 1$  does not satisfy  $\Delta_2$ .

Since  $G$  is convex and finite on  $\mathbb{R}^n$ ,  $G$  is locally Lipschitz and therefore continuous. Note that for every  $x \in \mathbb{R}^N$

$$\begin{aligned} G(\alpha x) &\leq \alpha G(x), \text{ if } 0 \leq \alpha \leq 1, \\ \alpha G(x) &\leq G(\alpha x), \text{ if } 1 \leq \alpha. \end{aligned}$$

We get immediately that  $G$  is non-decreasing along any half-line through the origin i.e. for every  $x \in \mathbb{R}^N$

$$0 < \alpha \leq \beta \implies G(\alpha x) \leq G(\beta x). \quad (2)$$

Our assumptions on  $G$  imply that for every  $x_0 \in \mathbb{R}^N$  there exists  $a \in \mathbb{R}^N$  and  $b \in \mathbb{R}$  such that for all  $x \in \mathbb{R}^N$

$$\langle a, x_0 \rangle + b = G(x_0) \text{ and } \langle a, x \rangle + b \leq G(x).$$

From this, we can easily obtain the Jensen integral inequality. Let  $I \subset \mathbb{R}$  be a finite interval and let  $u \in \mathbf{L}^1(I, \mathbb{R}^N)$ . Then

$$G\left(\frac{1}{\mu(I)} \int_I u \, dt\right) \leq \frac{1}{\mu(I)} \int_I G(u) \, dt.$$

We will often make use of the following simple observation.

**Proposition 2.1.** For all  $\alpha \in \mathbb{R}$  there exists  $K_1(\alpha) > 0$  such that

$$G(\alpha x) \leq K_1(\alpha)G(x)$$

for all  $|x| \geq M_1$ .

In fact, the above proposition provides a characterization of  $\Delta_2$  (see [7, 11]). It follows that for every  $\alpha \in \mathbb{R}$  there exists  $C_\alpha > 0$  such that for  $x \in \mathbb{R}^N$

$$G(\alpha x) \leq C_\alpha + K_1(\alpha)G(x).$$

We recall a notion of Fenchel conjugate. Define  $G^* : \mathbb{R}^N \rightarrow [0, \infty)$  by

$$G^*(y) := \sup_{x \in \mathbb{R}^N} \{\langle x, y \rangle - G(x)\}.$$

A function  $G^*$  is called Fenchel conjugate of  $G$ . As an immediate consequence of definition we have the so called Fenchel inequality:

$$\forall_{x,y \in \mathbb{R}^N} \langle x, y \rangle \leq G(x) + G^*(y).$$

Consider arbitrary  $f: \mathbb{R}^N \rightarrow [0, \infty)$ . It is obvious that the conjugate function  $f^*$  is always convex. But in general  $f^*$  need not be continuous, finite or coercive, even if  $f$  is. From the other hand, it is well known that if  $f$  is convex and l.s.c. then  $f^* \neq \infty$  and  $(f^*)^* = f$ .

**Example 2.2.**

1. If

$$g(x) = \begin{cases} 0 & |x| \leq 1 \\ \infty & |x| > 1 \end{cases}$$

then  $g^*(x) = |x|$ . Note that  $g$  and  $g^*$  are G-functions but do not satisfy our assumptions.

2. If  $G_p(x) = \frac{1}{p}|x|^p$ , then  $G_p^*(x) = \frac{1}{q}|x|^q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

3. If  $G(x) = \sum_{i=1}^N G_{p_i}(x_i)$ , then  $G^*(x) = \sum_{i=1}^N G_{p_i}^*(x_i)$ ,

4. If  $G(x, y) = (x - y)^2 + y^4$ , then

$$G^*(x, y) = \frac{1}{4}x^2 + \frac{3}{4}(x + y) \left( \frac{x + y}{4} \right)^{\frac{1}{3}}.$$

More information on general theory of conjugate functions can be found in standard books on convex analysis, see for instance [21, 22].

If a function  $G: \mathbb{R}^n \rightarrow [0, \infty)$  satisfies conditions  $(G_1)$ – $(G_6)$  then the same is true for its conjugate  $G^*$ . This is main reason we want to restrict class of considered functions.

**Theorem 2.3.** If  $G$  satisfies conditions  $(G_1)$ – $(G_6)$  then  $G^*$  also satisfies  $(G_1)$ – $(G_6)$  and  $(G^*)^* = G$ .

*Proof.* It is evident that  $G^*$  satisfies  $(G_1)$ ,  $(G_2)$  and  $(G_3)$ . It is well known that, under our conditions,  $G^*$  is finite (proposition 1.3.8, [21]),  $G^*$  is supercoercive (proposition 1.3.9, [21]) and  $G^*$  satisfies  $(G_5)$  and  $(G_6)$  (remark 2.3, [10]). Corollary [21, cor. 1.3.6] gives  $(G^*)^* = G$ .  $\square$

In order to compare growth rate of G-functions we define two relations. Let  $G_1$  and  $G_2$  be G-functions. Define

$$G_1 \prec G_2 \iff \exists_{M \geq 0} \exists_{K > 0} \forall_{|x| \geq M} G_1(x) \leq G_2(Kx)$$

and

$$G_1 \prec\prec G_2 \iff \forall \alpha > 0 \lim_{|x| \rightarrow \infty} \frac{G_2(\alpha x)}{G_1(x)} = \infty.$$

For conjugate functions we have (see [3, thm. 3.1])

$$G_1 \prec G_2 \Rightarrow G_2^* \prec G_1^*.$$

Obviously  $G_1 \prec\prec G_2$  implies  $G_1 \prec G_2$ . Assumption  $(G_4)$  implies  $|x| \prec\prec G$ . It is true that  $|x| \prec G$  holds under weaker assumption:  $G(x) \rightarrow \infty$ . Note that, if  $p > 1$  then  $|x| \prec\prec |x|^p$ . Hence, if  $|x|^p \prec G$  then  $|x| \prec\prec G$ . Since  $G$  satisfies  $(G_5)$  and  $(G_6)$  we have the following bounds for the growth of  $G$ .

**Lemma 2.4** ([10, Lemma 2.4]). There exist  $p, q \in (1, \infty)$  such that

$$|x|^p \prec G \prec |x|^q.$$

The exponents  $p$  and  $q$  depend on the constants in the  $\nabla_2$  and  $\Delta_2$  conditions respectively. Immediately from the above we get  $|x|^{\frac{q}{q-1}} \prec G^* \prec |x|^{\frac{p}{p-1}}$ .

### 3. Orlicz spaces

Let  $I \subset \mathbb{R}$  be a finite interval. The Orlicz space  $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^n)$  is defined to be

$$\mathbf{L}^G(I, \mathbb{R}^n) = \left\{ u: I \rightarrow \mathbb{R}^n: u \text{ - measurable, } \int_I G(u) dt < \infty \right\}.$$

As usual, we identify functions equal a.e. For  $u \in \mathbf{L}^G$  define:

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0: \int_I G\left(\frac{u}{\alpha}\right) dt \leq 1 \right\}.$$

The function  $\|\cdot\|_{\mathbf{L}^G}$  is called the Luxemburg norm. It is easy to see that

$$\int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1,$$

since  $G$  satisfies  $\Delta_2$ . Moreover

$$\int_I G\left(\frac{u}{k}\right) dt \leq 1 \iff \|u\|_{\mathbf{L}^G} \leq k.$$

Using Fenchel's inequality we obtain the Hölder inequality

$$\int_I \langle u, v \rangle dt \leq 2\|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^*}}, \quad u \in \mathbf{L}^G \text{ and } v \in \mathbf{L}^{G^*}$$



Similarly to [3] and [8] one can show that  $\mathbf{L}^G$  is a linear ([3, thm. 8.2]) and normed space ([8, thm. 2.3]). Completeness and separability of  $\mathbf{L}^G$  can be obtained in the same way as in [11, thm. 6.1, thm. 6.3, cor. 6.1]. Since  $\mathbf{L}^G \hookrightarrow \mathbf{L}^p \hookrightarrow \mathbf{L}^{p_0} \hookrightarrow \mathbf{L}^1$  (see propositions 3.3 and 3.4 below) and  $1 < p_0 < p$ , it follows that  $\mathbf{L}^G$  is reflexive space. The proof, in more general case, can be found in [11].

According to above remarks, we have the following theorem.

**Theorem 3.1.** If  $G: \mathbb{R}^n \rightarrow [0, \infty)$  satisfies  $(G_1)$ – $(G_6)$ , then  $(\mathbf{L}^G(I, \mathbb{R}^n), \|\cdot\|_{\mathbf{L}^G})$  is a separable, reflexive Banach space.

**Remark 3.2.**

1. All properties of  $\mathbf{L}^G$  remains true for  $\mathbf{L}^{G^*}$ , since  $G$  and  $G^*$  belongs to the same class of functions.
2. For an arbitrary G-function  $f: \mathbb{R}^n \rightarrow [0, \infty)$  which does not satisfies  $\Delta_2$  the set  $\mathbf{L}^f$  is not a linear space but only a convex set. In fact, it is well known that the set  $\mathbf{L}^f$  is linear space if and only if a G-function  $f$  satisfies  $\Delta_2$  condition.
3. It was pointed out by Schappacher [11, example 3.1] that if  $f$  is not bounded on bounded sets (i.e. we allow  $f(x) = +\infty$  for some  $x \in \mathbb{R}^n$ ) then  $\mathbf{L}^f$  need not be a linear space, even if  $f$  satisfies  $\Delta_2$  condition (see [3, 11]).
4. It is well known that if G-function does not satisfies  $\Delta_2$  condition then  $\mathbf{L}^G$  is not separable. One can define a subspace  $E^G$  as the closure of bounded functions under Luxemburg norm. In this case, the space  $E^G$  is a proper subset of  $\mathbf{L}^G$  and is always separable (see [3, 11]).
5. For every  $F \in (\mathbf{L}^G)^*$  there exists unique  $v \in \mathbf{L}^{G^*}$  such that for every  $u \in \mathbf{L}^G$

$$Fu = \int_I \langle u, v \rangle dt.$$

As a consequence we obtain that  $\mathbf{L}^{G^*} \simeq (\mathbf{L}^G)^*$ . Since  $G^{**} = G$ , we also get  $\mathbf{L}^G \simeq (\mathbf{L}^{G^*})^*$  (see [3, 8, 11]).

6. If G-function does not satisfies  $\Delta_2$  and  $\nabla_2$  conditions, then  $\mathbf{L}^G$  is not reflexive and  $(\mathbf{L}^G)^*$  is not isomorphic to  $\mathbf{L}^{G^*}$  (see [3, 11]).

An important example of Orlicz space is a classical Lebesgue space  $(\mathbf{L}^p, \|\cdot\|_{\mathbf{L}^p})$ ,  $p \in (1, \infty)$  defined by  $G(x) = \frac{1}{p}|x|^p$ . It is easy to check that in this case  $\mathbf{L}^G = \mathbf{L}^p$  and the Luxemburg norm and standard  $\mathbf{L}^p$  norm are equivalent. Two important examples of Lebesgue spaces are not covered in our setting, namely  $\mathbf{L}^1$  and  $\mathbf{L}^\infty$ .



The space  $\mathbf{L}^1$  is generated by  $f(x) = |x|$  and the space  $\mathbf{L}^\infty$  generated by  $f^*$ . We exclude these two spaces because we want to have only reflexive spaces in the class of Orlicz spaces we consider.

We will use the symbols  $\hookrightarrow$  and  $\hookleftrightarrow$  for, respectively, continuous and compact embeddings. Using the same methods as in [6, th. 8.12, 8.24] we obtain basic embedding theorems for anisotropic Orlicz spaces.

**Proposition 3.3.** Assume that  $F \prec G$ . Then  $L^G \hookrightarrow L^F$  and

$$\|u\|_{\mathbf{L}^F} \leq K(C\mu(I) + 1)\|u\|_{\mathbf{L}^G}$$

for some  $C > 0$ .

**Proposition 3.4.** If  $F \prec\prec G$  then  $\mathbf{L}^G \hookleftrightarrow \mathbf{L}^F$ .

Directly from Lemma 2.4 we obtain that Orlicz spaces can be viewed as a spaces between two Lebesgue spaces determined by constants in  $\Delta_2$  and  $\nabla_2$  conditions.

**Proposition 3.5.** For every  $G$  there exist  $p, q \in (1, \infty)$  such that

$$\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p.$$

In particular  $\mathbf{L}^\infty \hookrightarrow \mathbf{L}^G \hookleftrightarrow \mathbf{L}^1$ .

In some cases  $\mathbf{L}^G$  is simply a product of  $\mathbf{L}^{p_i}(I, \mathbb{R})$ , but there exist Orlicz spaces which are not in the form  $\mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R})$  (cf. [9, pp. 18-20]).

**Example 3.6.** Consider the Orlicz space  $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$  generated, by  $G(x) = |x_1|^{p_1} + |x_2|^{p_2}$ ,  $p_1, p_2 > 0$ . If  $u = (u_1, u_2) \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$ , then

$$\int_I G(u) dt = \int_I |u_1|^{p_1} dt + \int_I |u_2|^{p_2} dt < \infty.$$

Conversely, if  $u = (u_1, u_2) \in \mathbf{L}^G$  then

$$\int_I |u_1|^{p_1} dt \leq \int_I G(u) dt < \infty \text{ and } \int_I |u_2|^{p_2} dt \leq \int_I G(u) dt < \infty.$$

Hence  $u \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$ .

**Example 3.7.** Consider the Orlicz space  $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$  generated, by  $G(x) = (x_1 - x_2)^4 + x_2^2$ . From Lemma 2.4 and Proposition 3.5 we obtain that  $\mathbf{L}^4(I, \mathbb{R}^2) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}^2)$ . Let  $u_1$  be a function in  $\mathbf{L}^2(I, \mathbb{R})$  such that  $u_1 \notin \mathbf{L}^p(I, \mathbb{R})$ , for  $p > 2$ . Set  $u = (u_1, u_1)$ , then

$$\int_I G(u) dt = \int_I |u_1|^2 dt < \infty$$



but

$$\int_I |u|^p dt = \infty.$$

Therefore for every  $p > 2$  there exists  $u \in \mathbf{L}^G$  such that  $u \notin \mathbf{L}^p(I, \mathbb{R}^2)$ . Moreover,  $u \notin \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$  for any  $p > 2$ . From the other hand if  $u = (u_1, u_2) \in \mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R})$  then  $u \in \mathbf{L}^G$ . Therefore

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$$

but  $\mathbf{L}^G$  cannot be identified with any

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R}) \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R}).$$

### 3.1. Convergence

Now we investigate relations between Luxemburg norm and the integral

$$R_G(u) := \int_I G(u) dt.$$

A functional  $R_G$  is called modular. Theory of modulars is well known and is developed in more general setting than ours. More information can be found in [23, 5].

For Lebesgue spaces a notions of modular and norm are indistinguishable because modular  $\int_I |u|^p dt$  is equal to  $\|u\|_{\mathbf{L}^p}^p$ . But in Orlicz spaces relation between  $R_G$  and  $\|\cdot\|_{\mathbf{L}^G}$  is more complex.

There is remarkable difference between isotropic and anisotropic spaces. It is clear that if  $u, v \in \mathbf{L}^p$  (or more generally in isotropic Orlicz space) then  $|u(t)| \leq |v(t)|$  a.e. implies  $\|u\|_{\mathbf{L}^p} \leq \|v\|_{\mathbf{L}^p}$ . In anisotropic case it is no longer true, even if  $G(u(t)) < G(v(t))$ . Next two examples illustrates this point.

**Example 3.8.** Let  $G(x, y) = (x - y)^2 + y^4$ ,  $I = [0, 1]$ ,  $u(t) = (2, 0)$  and  $v(t) = (2, 3/2)$ . Then  $|u(t)| < |v(t)|$ ,  $G(u(t)) < G(v(t))$  and  $R_G(u) \leq R_G(v)$ , but  $2 = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.6$ .

**Example 3.9.** Let  $G(x, y) = x^2 + y^4$ ,  $u(t) = (1, 0)$  and  $v(t) = \frac{11}{10}(\cos t, \sqrt{\sin t})$ . In  $\mathbf{L}^G([0, \pi], \mathbb{R}^2)$  we have  $\sqrt{\pi} = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.7$ , but  $|u(t)| < |v(t)|$ ,  $G(u(t)) < G(v(t))$  for all  $t \in [0, \pi]$  and  $R_G(u) < R_G(v)$ .

**Definition 3.10.** We say that a subset  $K \subset \mathbf{L}^G$  is modular bounded if there exists  $C > 0$  such that

$$R_G(u) \leq C, \text{ for all } u \in K.$$

Modular boundedness is sometimes called mean boundedness. It is evident that  $R_G(u) \leq \|u\|_{\mathbf{L}^G}$  if  $\|u\|_{\mathbf{L}^G} \leq 1$  and  $R_G(u) > \|u\|_{\mathbf{L}^G}$  if  $\|u\|_{\mathbf{L}^G} > 1$ .

**Lemma 3.11.** Let  $u \in \mathbf{L}^G$ .

1. If  $R_G(u) \leq C$  then  $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$ .
2. If  $\|u\|_{\mathbf{L}^G} \leq C$  then  $R_G(u) \leq \mu(I)\tilde{C} + K_1(C)$  for some  $\tilde{C} > 0$ .

Moreover, a set  $K \subset \mathbf{L}^G$  is modular bounded if and only if is norm bounded.

*Proof.* Assume that  $R_G(u) \leq C$ . If  $C \leq 1$  then  $\|u\|_{\mathbf{L}^G} \leq 1$ . If  $C > 1$  then

$$\int_I G\left(\frac{u}{C}\right) dt \leq \frac{1}{C} \int_I G(u) dt \leq 1.$$

This implies  $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$ . For the second statement, assume  $\|u\|_{\mathbf{L}^G} \leq C$ . Then

$$R_G(u) = \int_{I_1} G(u) dt + \int_{I \setminus I_1} G\left(C \frac{u}{C}\right) dt \leq \mu(I_1)\tilde{C} + K_1(C) \int_I G\left(\frac{u}{C}\right) dt,$$

where  $I_1 = \{t \in I: |u(t)| \leq M_1 C\}$  and  $\tilde{C} > 0$ . To finish the proof observe that

$$\int_I G\left(\frac{u}{C}\right) dt \leq \int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1.$$

□

**Definition 3.12.** We say that a sequence of functions  $u_k \in \mathbf{L}^G$  is modular convergent to  $u \in \mathbf{L}^G$  if  $R_G(u_k - u) \rightarrow 0$  as  $k \rightarrow \infty$ .

Modular convergence is sometimes called mean convergence. Norm convergence always implies modular convergence. Let  $\|u_k\|_{\mathbf{L}^G} \rightarrow 0$  as  $k \rightarrow \infty$ . We can assume that  $\forall_k \|u_k\|_{\mathbf{L}^G} \leq 1$ , then

$$\frac{1}{\|u_k\|_{\mathbf{L}^G}} R_G(u_k) \leq R_G\left(\frac{u_k}{\|u_k\|_{\mathbf{L}^G}}\right) = 1.$$

Hence  $0 \leq R_G(u_k) \leq \|u_k\|_{\mathbf{L}^G}$ . In general, converse is not true unless  $G$  satisfies  $\Delta_2$  condition (see [3, 11]).

**Theorem 3.13.** Norm convergence is equivalent to modular convergence.

*Proof.* We need only to prove that modular convergence implies norm convergence. Fix  $\varepsilon > 0$  and assume that  $\{u_k\}$  is modular convergent to 0. Define

$$I_{1,k} = \{t \in I: |u_k(t)| \leq M_1\}.$$

Since  $G$  satisfies  $\Delta_2$ , for all  $k > 0$  we have

$$\begin{aligned} \int_I G(u_k/\varepsilon) dt &\leq \mu(I_{1,k}) C_{1/\varepsilon} + K_1(1/\varepsilon) \int_{I \setminus I_{1,k}} G(u_k) dt \leq \\ &\leq \mu(I) C_{1/\varepsilon} + K_1(1/\varepsilon) \int_I G(u_k) dt. \end{aligned}$$

For sufficiently large  $k$  we have

$$\int_I G(u_k) dt \leq \frac{1}{K_1(1/\varepsilon)}$$

and

$$\int_I G(u_k/\varepsilon) dt \leq \mu(I) C_{1/\varepsilon} + 1 = C.$$

Finally, Lemma 3.11 shows that  $\|u_k\|_{\mathbf{L}^G} \leq C\varepsilon$  and hence  $\|u_k\|_{\mathbf{L}^G} \rightarrow 0$ .  $\square$

It is standard result due to Riesz that for  $f_n, f \in \mathbf{L}^p$

$$f_n \rightarrow f \text{ a.e.} \implies (\|f_n\|_{\mathbf{L}^p} \rightarrow \|f\|_{\mathbf{L}^p} \iff \|f_n - f\|_{\mathbf{L}^p} \rightarrow 0).$$

Following lemmas establish Orlicz space version of this fact.

**Lemma 3.14.** For every  $k > 1$  and  $0 < \varepsilon < \frac{1}{k}$  and  $x, y \in \mathbb{R}^n$

$$|G(x+y) - G(x)| \leq \varepsilon |G(kx) - kG(x)| + 2G(C_\varepsilon y)$$

where  $C_\varepsilon = \frac{1}{\varepsilon(k-1)}$

The proof can be found in [24] (see also [25]).

**Lemma 3.15.** If  $u_n \rightarrow u$  in  $\mathbf{L}^G$  then  $R_G(u_n) \rightarrow R_G(u)$ .

*Proof.* In Lemma 3.14 set  $x+y = u_n$ ,  $x = u$ ,  $k = 2$ . Then  $\varepsilon < 1/2$ ,  $C_\varepsilon = \frac{1}{\varepsilon}$  and

$$|G(u_n) - G(u)| \leq \varepsilon |G(2u) - 2G(u)| + 2G\left(\frac{u_n - u}{\varepsilon}\right).$$

Since  $u_n \rightarrow u$  in  $\mathbf{L}^G$ , there exists  $n_0$  such that for  $n > n_0$  we have  $\|u_n - u\|_{\mathbf{L}^G} < \varepsilon^2 \leq \varepsilon < 1$ . Thus

$$\int_I G\left(\frac{u_n - u}{\varepsilon}\right) dt \leq \frac{1}{\varepsilon} \|u_n - u\|_{\mathbf{L}^G} < \varepsilon.$$

From this and inequality above we obtain

$$|R_G(u_n) - R_G(u)| \leq \varepsilon \int_I |G(2u) - 2G(u)| dt + 2\varepsilon.$$

Letting  $\varepsilon \rightarrow 0$  we have  $R_G(u_n) \rightarrow R_G(u)$ .  $\square$

According to the above lemma, if  $u_n \rightarrow u$  in  $\mathbf{L}^G$  then:

1. Since  $\mathbf{L}^G \hookrightarrow \mathbf{L}^1$  (see Lemma 3.5 below), we can extract a subsequence  $u_{n_k}$  such that

$$u_{n_k} \rightarrow u \text{ a.e and } |u_{n_k}| \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

2. Since  $R_G(u_n - u) \rightarrow 0$ ,  $G(u_n - u) \rightarrow 0$  in  $\mathbf{L}^1$ . Thus we can extract a subsequence  $\{u_{n_k}\}$  such that

$$G(u_{n_k} - u) \rightarrow 0 \text{ a.e and } G(u_{n_k} - u) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

3. Since  $R_G(u_n) \rightarrow R_G(u)$ ,  $G(u_n) \rightarrow G(u)$  in  $\mathbf{L}^1$ . Hence there exists a subsequence  $\{u_{n_k}\}$  such that

$$G(u_{n_k}) \rightarrow G(u) \text{ a.e and } G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

**Lemma 3.16.** Let  $\{u_n\} \subset \mathbf{L}^G$  and  $u \in \mathbf{L}^G$ . Suppose that

1.  $u_n \rightarrow u$  a.e.,
2.  $R_G(u_n) \rightarrow R_G(u)$ .

Then  $u_n \rightarrow u$  in  $\mathbf{L}^G$ .

*Proof.* This lemma was proved in [4, p. 83] for N-functions. Since  $G$  is convex, we get  $\frac{1}{2}(G(u_n(t)) + G(u(t))) - G\left(\frac{u_n(t) - u(t)}{2}\right) \geq 0$ . Continuity of  $G$  and  $u_n \rightarrow u$  a.e. implies

$$\frac{1}{2}(G(u_n(t)) + G(u(t))) - G\left(\frac{u_n(t) - u(t)}{2}\right) \rightarrow G(u) \text{ a.e.}$$

So that by the Fatou Lemma, we have

$$\begin{aligned} \int_I G(u) dt &\leq \liminf_{n \rightarrow \infty} \int_I \frac{1}{2}(G(u_n) + G(u)) dt - G\left(\frac{u_n - u}{2}\right) dt \leq \\ &\leq \lim_{n \rightarrow \infty} \int_I \frac{1}{2}(G(u_n) + G(u)) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt = \\ &= \int_I G(u) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt. \end{aligned}$$

This implies that

$$\int_I G\left(\frac{u_k(t) - u(t)}{2}\right) dt \rightarrow 0$$

and  $\|u_k - u\|_{\mathbf{L}^G} \rightarrow 0$  by Theorem 3.13. □

As a consequence we obtain dominated convergence theorem for anisotropic Orlicz spaces:

**Theorem 3.17.** Suppose that  $\{u_n\} \subset \mathbf{L}^G$  and

1.  $u_n \rightarrow u$  a.e.
2. there exists  $h \in \mathbf{L}^1(I, \mathbb{R})$  such that  $G(u_n) \leq h$  a.e.

Then  $u \in \mathbf{L}^G$  and  $u_n \rightarrow u$  in  $\mathbf{L}^G$ .

*Proof.* Since  $G$  is continuous and  $u_n \rightarrow u$  a.e.,  $G(u_n) \rightarrow G(u)$  a.e. It follows that  $G(u) \leq h$  a.e. Thus  $G(u) \in \mathbf{L}^1(I, \mathbb{R})$  and hence  $u \in \mathbf{L}^G$ . In a standard way we get  $R_G(u_n) \rightarrow R_G(u)$ . Hence  $u_n \rightarrow u$  in  $\mathbf{L}^G$ , by the Lemma 3.16.  $\square$

In the above theorem, assumption  $G(u_n) \leq h$  can be replaced by  $G(u_n) \leq G(h)$ ,  $h \in \mathbf{L}^G$ . Consider a sequence  $\{u_n\} \subset \mathbf{L}^G$  convergent pointwise to measurable function  $u$ . Under standard dominance condition (i.e.  $|u_n| \leq |g|$ ,  $g \in \mathbf{L}^G$ ) it is not true in general that  $u_n \rightarrow u \in \mathbf{L}^G$ .

**Example 3.18.** Let  $G(x, y) = x^2 + y^4$ ,  $I = (0, 1)$ ,  $u(t) = (0, t^{-1/4})$  and  $h(t) = (t^{-3/8}, 0)$ . Define

$$u_n(t) = \begin{cases} u(t) & |u(t)| \leq n \\ 0 & |u(t)| > n \end{cases}$$

Then  $u_n \rightarrow u$  a.e.,  $u_n, h \in \mathbf{L}^G$  and  $|u_n| \leq |h|$  for every  $t$ . But  $G(u(t)) = t^{-1} \notin \mathbf{L}^1(I, \mathbb{R})$ . Hence  $u \notin \mathbf{L}^G$ .

**Remark 3.19.** Modular  $R_G$  is called monotone modular if  $|x| \leq |y|$  implies  $R_G(x) \leq R_G(y)$ . If  $R_G$  is monotone modular then  $u_k \rightarrow u$  a.e and  $|u_k| \leq |g|$ ,  $g \in \mathbf{L}^G$  implies  $u \in \mathbf{L}^G$  and  $\|u_k - u\|_{\mathbf{L}^G} \rightarrow 0$ . We refer the reader to [25] for more details.

#### 4. Orlicz-Sobolev spaces

The Orlicz-Sobolev space  $\mathbf{W}^1 \mathbf{L}^G = \mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n)$  is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) := \{u \in \mathbf{L}^G(I, \mathbb{R}^n) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^n)\}.$$

For  $u \in \mathbf{W}^1 \mathbf{L}^G$  we define

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} := \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

Define  $\mathbf{W}_0^1 \mathbf{L}^G = \mathbf{W}_0^1 \mathbf{L}^G(I, \mathbb{R}^n)$  as the closure of  $C_0^1(I, \mathbb{R}^n)$  in  $\mathbf{W}^1 \mathbf{L}^G$  with respect to the  $\|\cdot\|_{\mathbf{W}^1 \mathbf{L}^G}$ .



**Theorem 4.1.** The space  $(\mathbf{W}^1 \mathbf{L}^G, \|\cdot\|_{\mathbf{W}^1 \mathbf{L}^G})$  is a separable reflexive Banach space.

Proof is standard and will be omitted, see for instance [26]. If  $G(x) = \frac{1}{p}|x|^p$ , then the Orlicz-Sobolev space  $\mathbf{W}^1 \mathbf{L}^G$  coincides with the Sobolev space  $\mathbf{W}^{1,p}(I, \mathbb{R}^n)$ . Observe that  $u_n \rightarrow u$  in  $\mathbf{W}^1 \mathbf{L}^G$  is equivalent to  $R_G(u_n - u) \rightarrow 0$  and  $R_G(\dot{u}_n - \dot{u}) \rightarrow 0$ .

Since there exist  $p, q \in (1, \infty)$  such that  $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$ , the following continuous embeddings exist

$$\mathbf{W}^{1,q} \hookrightarrow \mathbf{W}^1 \mathbf{L}^G \hookrightarrow \mathbf{W}^{1,p}$$

Using standard results from the theory of Sobolev spaces we get

1.  $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow \mathbf{W}^{1,1}$ ,
2.  $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow \mathbf{L}^q$ , for all  $1 \leq q \leq \infty$ ,
3.  $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow C(\bar{I})$ .

As a consequence we have

**Theorem 4.2.** A function  $u \in \mathbf{W}^1 \mathbf{L}^G$  is absolutely continuous. Precisely, there exists absolutely continuous representative of  $u$  such that for all  $a, b \in I$

$$u(b) - u(a) = \int_a^b \dot{u}(t) dt.$$

Directly from definition of  $\mathbf{W}_0^1 \mathbf{L}^G$  we obtain important property of functions in  $\mathbf{W}_0^1 \mathbf{L}^G$ .

**Theorem 4.3.** If  $u \in \mathbf{W}_0^1 \mathbf{L}^G$ , then  $u = 0$  on  $\partial I$ .

Using embeddings mentioned above we have for every  $u \in \mathbf{W}^1 \mathbf{L}^G$

$$\|u\|_{\mathbf{L}^\infty} \leq C \|u\|_{\mathbf{W}^1 \mathbf{L}^G} \quad (3)$$

**Theorem 4.4** (Sobolev inequality). For every function  $u \in \mathbf{W}^1 \mathbf{L}^G$

$$\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I) \|\dot{u}\|_{\mathbf{L}^G}$$

where  $u_I = \frac{1}{\mu(I)} \int_I u$ .

*Proof.* Since  $u$  is absolutely continuous, there exists  $t_0 \in I$  such that  $u(t_0) = \frac{1}{\mu(I)} \int_I u$  and for every  $t \in I$  we have

$$u(t) - u(t_0) = \int_{t_0}^t \dot{u} dt.$$

By Jensen's inequality,

$$\begin{aligned} G\left(\frac{u(t) - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) &= G\left(\frac{1}{|t - t_0|} \int_{t_0}^t \frac{\dot{u}}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}} dt\right) \leq \\ &\leq \frac{1}{|t - t_0|} \int_{t_0}^t G\left(\frac{\dot{u}}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)} \int_I G\left(\frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)}. \end{aligned}$$

Integrating both sides over  $I$  we get

$$\int_I G\left(\frac{u - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq 1.$$

Thus  $\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$ . □

In similar way we get

**Theorem 4.5** (Poincaré inequality). For every  $u \in \mathbf{W}_0^1 \mathbf{L}^G$

$$\|u\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$$

It follows that one can introduce equivalent norm in  $\mathbf{W}_0^1 \mathbf{L}^G$ :

$$\|u\|_{\mathbf{W}_0^1 \mathbf{L}^G} = \|\dot{u}\|_{\mathbf{L}^G}.$$

Every linear functional  $F$  on  $\mathbf{W}_0^1 \mathbf{L}^G$  can be represented in the form

$$F(u) = \int_I \langle u, v_0 \rangle + \langle \dot{u}, v_1 \rangle dt,$$

where  $v_0, v_1 \in \mathbf{L}^{G^*}$ . Moreover,  $\|F\| = \max\{\|v_0\|_{\mathbf{L}^{G^*}}, \|v_1\|_{\mathbf{L}^{G^*}}\}$ . In the case of Sobolev space  $\mathbf{W}^{1,p}$  the proof is given in [26, proposition 8.14], but it remains the same for Orlicz-Sobolev spaces. As was pointed out in [26], the first assertion of the above proposition holds for every linear functional on  $\mathbf{W}^1 \mathbf{L}^G$ .



## 5. Variational setting

In this section we examine the principal part

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u}) dt \quad (4)$$

of the variational functional associated with Euler-Lagrange equation

$$\frac{d}{dt} F_v(t, u, \dot{u}) = F_x(t, u, \dot{u}) + \nabla V(t, u), \quad t \in I$$

where  $u: I \rightarrow \mathbb{R}^N$  and the Lagrangian  $L: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  is given by  $L(t, x, v) = F(t, x, v) + V(t, x)$ .

In definition of the Orlicz space we need not to assume that  $G$  is differentiable, but when we consider the functional  $\mathcal{I}$  we need it to show that  $\mathcal{I} \in C^1$ . Throughout this section we will assume, in addition to  $(G_1)$ – $(G_6)$ , that  $G$  satisfies  $(G_7)$   $G$  is of a class  $C^1$ .

**Remark 5.1.** Differentiability of  $f$  is not sufficient to differentiability of  $f^*$ . But if  $f$  is finite, strictly convex, 1-coercive and differentiable then so is  $f^*$ . This result is in close relation with Legendre duality (see [21, p. 239] and [1] for more details).

It is well known that if  $G$  is continuously differentiable then for all  $x, y \in \mathbb{R}^n$

$$G(x) - G(x - y) \leq \langle \nabla G(x), y \rangle \leq G(x + y) - G(x) \quad (5)$$

and

$$\langle x, \nabla G(x) \rangle = G(x) + G^*(\nabla G(x)).$$

Let  $y = x$  in (5). Then  $\langle \nabla G(x), x \rangle \leq G(2x) - G(x)$ . Therefore, for all  $x \in \mathbb{R}^n$

$$G^*(\nabla G(x)) \leq G(2x).$$

Directly from the above we get

**Proposition 5.2.** If  $u \in \mathbf{L}^G$  then  $\nabla G(u) \in \mathbf{L}^{G^*}$ .

**Lemma 5.3** (cf. [16, lemma A.5]). If  $u_n \rightarrow u$  in  $\mathbf{L}^G$  then  $R_{G^*}(\nabla G(u_n)) \rightarrow R_{G^*}(\nabla G(u))$ .

*Proof.* There exists a subsequence  $\{u_{n_k}\}$  such that  $u_{n_k} \rightarrow u$  a.e.,  $G(u_{n_k}) \rightarrow G(u)$  a.e. and  $G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R})$ . By continuity of  $\nabla G$  and  $G^*$  we have  $\nabla G(u_{n_k}) \rightarrow \nabla G(u)$  a.e. and

$$G^*(\nabla G(u_{n_k})) \rightarrow G^*(\nabla G(u)) \text{ a.e.}$$

Since  $G^*(\nabla G(x)) \leq G(2x)$ ,

$$G^*(\nabla G(u_{n_k})) \leq G(2u_{n_k}) \leq C + K_1 G(u_{n_k}) \leq C + K_1 h.$$

By dominated convergence theorem  $R_{G^*}(\nabla G(u_{n_k})) \rightarrow R_{G^*}(\nabla G(u))$ . Since this holds for any subsequence of  $\{u_n\}$  we have that

$$R_{G^*}(\nabla G(u_n)) \rightarrow R_{G^*}(\nabla G(u)).$$

□

As a direct consequence of the above lemma and Lemma 3.16 we obtain

**Proposition 5.4.**

$$\|u_n - u\|_{\mathbf{L}^G} \rightarrow 0 \implies \|\nabla G(u_n) - \nabla G(u)\|_{\mathbf{L}^{G^*}} \rightarrow 0.$$

5.1. Case I

We shall first examine a special case  $F(t, x, v) = G(v)$ , now functional (4) takes the form

$$\mathcal{I}(u) = \int_I G(\dot{u}) dt.$$

**Theorem 5.5.**  $\mathcal{I} \in C^1(\mathbf{W}^1 \mathbf{L}^G, \mathbb{R})$ . Moreover

$$\mathcal{I}'(u)\varphi = \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt. \quad (6)$$

*Proof.* The proof follows similar lines as [2, th. 3.2] (see also [1, thm 1.4]). First, note that  $\dot{u} \in \mathbf{L}^G$  implies

$$0 \leq \mathcal{I}(u) < \infty.$$

It suffices to show that  $\mathcal{I}$  has at every point  $u$  directional derivative  $\mathcal{I}'(u) \in (\mathbf{W}^1 \mathbf{L}^G)^*$  given by (6) and that the mapping  $\mathcal{I}' : \mathbf{W}^1 \mathbf{L}^G \rightarrow (\mathbf{W}^1 \mathbf{L}^G)^*$  is continuous. Let  $u \in \mathbf{W}^1 \mathbf{L}^G$ ,  $\varphi \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$ ,  $t \in I$ ,  $s \in [-1, 1]$ . Define

$$H(s, t) := G(\dot{u}(t) + s\dot{\varphi}(t)).$$

By (5) we obtain

$$\int_I |H_s(s, t)| dt = \int_I |\langle \nabla G(\dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt \leq \int_I G(\dot{u} + (s+1)\dot{\varphi}) + \int_I G(\dot{u} + s\dot{\varphi}) dt < \infty.$$

Consequently,  $\mathcal{I}$  has a directional derivative and

$$\mathcal{I}'(u)\varphi = \frac{d}{ds} \mathcal{I}(u + s\varphi) \Big|_{s=0} = \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt.$$

By Proposition 5.2 and the Hölder inequality

$$|\mathcal{I}'(u)\varphi| = \left| \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|\nabla G(\dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \leq C \|\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

To finish the proof it suffices to show that if  $u_n \rightarrow u$  in  $\mathbf{W}^1 \mathbf{L}^G$ , then  $\mathcal{I}'(u_n) \rightarrow \mathcal{I}'(u)$  in  $(\mathbf{W}^1 \mathbf{L}^G)^*$ . Using the Hölder inequality and Proposition 5.4 we obtain

$$\begin{aligned} |\mathcal{I}'(u_n)\varphi - \mathcal{I}'(u)\varphi| &= \left| \int_I \langle \nabla G(\dot{u}_n) - \nabla G(\dot{u}), \dot{\varphi} \rangle dt \right| \leq \\ &\leq 2 \|\nabla G(\dot{u}_n) - \nabla G(\dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \rightarrow 0. \end{aligned}$$

□

### 5.2. Case II

We turn to general case. Suppose that  $F: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies

$$(F_1) \quad F \in C^1,$$

$$(F_2) \quad |F(t, x, v)| \leq a(|x|)(b(t) + G(v)),$$

$$(F_3) \quad |F_x(t, x, v)| \leq a(|x|)(b(t) + G(v)),$$

$$(F_4) \quad G^*(F_v(t, x, v)) \leq a(|x|)(c(t) + G^*(\nabla G(v))).$$

where  $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ ,  $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$ .

If  $G(v) = |v|^p$  then conditions  $(F_2)$ ,  $(F_3)$  and  $(F_4)$  take the standard form (Theorem 1.4 from [1]). In [2] there are similar conditions with  $G(v) = \Phi(|v|)$ , where  $\Phi$  is an N-function. In this case, condition  $(F_4)$  takes the form  $|F_v(t, x, v)| \leq \tilde{a}(|x|)(\tilde{c}(t) + \Phi'(|u|))$ . In anisotropic case we need to use  $G^*$ , because vector valued G-function is not necessarily monotone with respect to  $|\cdot|$ .

**Lemma 5.6.** If  $u \in \mathbf{W}^1 \mathbf{L}^G$ , then  $F_x(\cdot, u, \dot{u}) \in \mathbf{L}^1$  and  $F_v(\cdot, u, \dot{u}) \in \mathbf{L}^{G^*}$ .

*Proof.* Define non decreasing function

$$\alpha(s) = \sup_{\tau \in [0, s]} a(\tau).$$

Then, for  $u \in \mathbf{W}^1 \mathbf{L}^G$  we have

$$a(|u(t)|) \leq \alpha(\|u\|_{\mathbf{L}^\infty}) \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}). \quad (7)$$

Let  $u \in \mathbf{W}^1 \mathbf{L}^G$ . By (7) and  $(F_3)$

$$\begin{aligned} \int_I |F_x(t, u, \dot{u})| dt &\leq \int_I a(|u(t)|)(b(t) + G(\dot{u})) dt \leq \\ &\leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u})) dt < \infty. \end{aligned}$$

Moreover, by  $(F_4)$  and Proposition 5.2

$$\int_I G^*(F_v(t, u, \dot{u})) dt \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (c(t) + G^*(\nabla G(\dot{u}))) dt < \infty.$$

□

**Theorem 5.7.**  $\mathcal{I} \in C^1(\mathbf{W}^1 \mathbf{L}^G, \mathbb{R})$ . Moreover

$$\mathcal{I}'(u)\varphi = \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt + \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt. \quad (8)$$

*Proof.* By  $(F_2)$

$$|\mathcal{I}(u)| \leq \int_I a(|u|)(b(t) + G(\dot{u})) dt \leq \alpha(\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u})) dt < \infty.$$

It suffices to show that directional derivative  $\mathcal{I}'(u) \in (\mathbf{W}^1 \mathbf{L}^G)^*$  exists, is given by (8) and that the mapping  $\mathcal{I}' : \mathbf{W}^1 \mathbf{L}^G \rightarrow (\mathbf{W}^1 \mathbf{L}^G)^*$  is continuous.

Let  $u \in \mathbf{W}^1 \mathbf{L}^G$ ,  $\varphi \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$ ,  $t \in I$ ,  $s \in [-1, 1]$ . Define

$$H(s, t) := F(t, u + s\varphi, \dot{u} + s\dot{\varphi}).$$

By  $(F_3)$ , continuity of  $\varphi$ , (7) and the fact that  $u + s\varphi \in \mathbf{W}^1 \mathbf{L}^G$  we obtain

$$\begin{aligned} \int_I |\langle F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle| dt &\leq \int_I |F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi})| |\varphi| dt \leq \\ &\leq \int_I a(|u + s\varphi|)(b(t) + G(\dot{u} + s\dot{\varphi})) |\varphi| dt \leq \\ &\leq \alpha(\|u + s\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u} + s\dot{\varphi})) |\varphi| dt < \infty. \end{aligned}$$

By the Fenchel inequality,  $(F_4)$  and Lemma 5.6 we obtain

$$\int_I |\langle F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt \leq \int_I [G^*(F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi})) + G(\dot{\varphi})] dt < \infty.$$

It follows that

$$\int_I |H_s(s, t)| dt = \int_I |\langle F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle + \langle F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt < \infty.$$

Consequently,  $\mathcal{I}$  has a directional derivative and

$$\mathcal{I}'(u)\varphi = \left. \frac{d}{ds} \mathcal{I}(u + s\varphi) \right|_{s=0} = \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt + \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$



By Lemma 5.6, the Hölder inequality and (3) we get

$$|\mathcal{I}'(u)\varphi| \leq \|F_x(\cdot, u, \dot{u})\|_{\mathbf{L}^1} \|\varphi\|_{\mathbf{L}^\infty} + \|F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \leq C \|\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

To finish the proof it suffices to show that  $\mathcal{I}'$  is continuous. Since  $u_n \rightarrow u$  in  $\mathbf{W}^1 \mathbf{L}^G$ , it follows that  $u_n \rightarrow u$  in  $\mathbf{L}^G$ ,  $\dot{u}_n \rightarrow \dot{u}$  in  $\mathbf{L}^G$  and there exists  $M > 0$  such that  $\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G} < M$ .

By Lemma 3.15 we have  $G(\dot{u}_n) \rightarrow G(\dot{u})$  in  $\mathbf{L}^1(I, \mathbb{R})$ . Hence there exists a subsequence  $\{u_{n_k}\}$  and  $h \in \mathbf{L}^1(I, \mathbb{R})$  such that

$$G(\dot{u}_{n_k}) \rightarrow G(\dot{u}) \text{ a.e and } G(\dot{u}_{n_k}) \leq h.$$

By (F<sub>3</sub>) and since  $\{u_{n_k}\}$  is bounded, we obtain

$$|F_x(t, u_{n_k}, \dot{u}_{n_k})| \leq \alpha(\|u_{n_k}\|_{\mathbf{W}^1 \mathbf{L}^G})(b(t) + G(\dot{u}_{n_k})) \leq \alpha(M)(b(t) + h(t)).$$

By (F<sub>1</sub>) we have

$$F_x(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow F_x(t, u(t), \dot{u}(t))$$

for a.e  $t \in I$ . Applying dominated convergence theorem we obtain

$$\int_I \langle F_x(t, u_{n_k}, \dot{u}_{n_k}), \varphi \rangle dt \rightarrow \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt.$$

Since this holds for any subsequence of  $\{u_n\}$  we have that

$$\int_I \langle F_x(t, u_n, \dot{u}_n), \varphi \rangle dt \rightarrow \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt.$$

By (F<sub>4</sub>) and Lemma 5.6

$$G^*(F_v(t, u_{n_k}(t), \dot{u}_{n_k}(t))) \leq \alpha(M)(c(t) + G^*(\nabla G(\dot{u}_{n_k}(t)))).$$

In the same way as in the proof of Lemma 5.3 we obtain

$$G^*(F_v(t, u_{n_k}(t), \dot{u}_{n_k}(t))) \leq \alpha(M)(c(t) + C + K_1 h(t)).$$

By continuity of  $F_v$  we obtain

$$G^*(F_v(t, u_{n_k}(t), \dot{u}_{n_k}(t))) \rightarrow G^*(F_v(t, u(t), \dot{u}(t)))$$

for a.e  $t \in I$  and consequently

$$\int_I G^*(F_v(t, u_{n_k}, \dot{u}_{n_k})) dt \rightarrow \int_I G^*(F_v(t, u, \dot{u})) dt.$$

It follows that

$$\int_I G^*(F_v(t, u_n, \dot{u}_n)) dt \rightarrow \int_I G^*(F_v(t, u, \dot{u})) dt.$$

Application of Lemma 3.16 to  $R_{G^*}$  yields  $\|F_v(\cdot, u_n, \dot{u}_n) - F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \rightarrow 0$ . By the Hölder inequality

$$\left| \int_I \langle F_v(t, u_n, \dot{u}_n) - F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|F_v(\cdot, u_n, \dot{u}_n) - F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \rightarrow 0.$$

Finally,

$$\int_I \langle F_v(t, u_n, \dot{u}_n), \dot{\varphi} \rangle dt \rightarrow \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$

□

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