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# Mountain pass type periodic solutions for Euler-Lagrange equations in anisotropic Orlicz-Sobolev space

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**Abstract**

Using the Mountain Pass Theorem, we establish the existence of periodic solution for Euler-Lagrange equation. Lagrangian consists of kinetic part (an anisotropic G-function), potential part $K-W$ and a forcing term. We consider two situations: $G$ satisfying $\Delta_2 \cap \nabla 2$ at infinity and globally. We give conditions on the growth of the potential near zero for both situations.

*Keywords:* anisotropic Orlicz-Sobolev space, Euler-Lagrange equations, variational functional, Mountain Pass Theorem, Palais-Smale condition
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## 1. Introduction

We consider the second order system

$$\begin{cases} \frac{d}{dt} L_v(t,u(t),\dot{u}(t)) = L_x(t,u(t),\dot{u}(t)) & \text{for a.e. } t \in (-T,T) \\ u(-T) = u(T) \end{cases} \tag{ELT}$$

where $L\colon [-T,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$L(t,x,v) = G(v) + V(t,x) + \langle f(t), x\rangle.$$

We assume that $G$ is a differentiable G-function (in the sense of Trudinger [1]) and $V$ satisfies suitable growth conditions. If $G(v) = \frac{1}{p}|v|^p$ then the equation (ELT) reduces to p-laplacian. More general case is $G(v) = \phi(|v|)$, where $\phi$ is convex and nonnegative. In the above cases, $G$ depends on norm $|v|$ and its

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growth is the same in all directions (isotropic). In this paper we consider the situation when the growth of $G$ is different in different directions (anisotropic) e.g. $G(x,y) = |x|^p + |y|^q$.

Existence of periodic solutions for the problem (ELT) was investigated in many papers, e.g.: [2] (anisotropic case), [3] (isotropic case), [4] (($p,q$)-laplacian), [5, 6] ($p$-laplacian), [7] (laplacian) and many others.

This paper is motivated by [8, 9, 10], where the existence of homoclinic solution of $\frac{d}{dt}L_v(t,u(t),\dot{u}(t)) = L_x(t,u(t),\dot{u}(t))$ is investigated (see also [11, 12]). In all these papers an intermediate step is to show, using the Mountain Pass Theorem, that corresponding periodic problem has a solution.

We want to adapt methods from [8] to anisotropic Orlicz-Sobolev space setting. It turns out, that the mountain pass geometry of action functional is strongly depended on Simonenko indices $p_G$ and $q_G$ (see section 2). To show that the action functional satisfies the Palais-Smale condition we need index $q_G^\infty$. Similar observation can be found in [13, 14, 15, 16] where the existence of solutions to elliptic systems via the Mountain Pass Theorem is considered. In [14] authors deal with an anisotropic problem. The isotropic case is considered in [13, 15, 16].

We assume that:

($A_1$) $G\colon \mathbb{R}^N \to [0,\infty)$ is a continuously differentiable G-function (i.e. G is convex, even, $G(0)=0$ and $G(x)/|x| \to \infty$, as $|x| \to \infty$) satisfying $\Delta_2$ and $\nabla_2$ condition,

($A_2$) $V(t,x) = K(t,x) - W(t,x)$, where $K, W \in \mathbf{C}^1([-T,T]\times\mathbb{R}^N,\mathbb{R})$,

($A_3$) there exist $a \in \mathbf{L}^1([-T,T],\mathbb{R})$, $b>1$ and $\rho_0 > 0$ such that

$$V(t,x) \geq b\,G(x) - a(t) \quad \text{ for } |x| \leq \rho_0,\ t\in[-T,T],$$

($A_4$) there exist $b_1 > 0$ and $p>1$ satisfying $|\cdot|^p \prec G$, such that

$$\liminf_{|x|\to\infty} \frac{K(t,x)}{|x|^p} \geq b_1 \text{ uniformly in } t \in [-T,T]$$

and

$$\liminf_{|x|\to\infty} \frac{W(t,x)}{\max\{K(t,x), G(x)\}} > 3 \text{ uniformly in } t\in[-T,T],$$

($A_5$) there exist $\nu \in \mathbb{R}$, $\mu > q_G^\infty + \nu$ and $\kappa \in \mathbf{L}^1([-T,T],[0,\infty))$ such that

$$\langle V_x(t,x), x\rangle \leq (q_G^\infty + \nu)K(t,x) - \mu W(t,x) + \kappa(t) \quad \text{ for } (t,x)\in[-T,T]\times\mathbb{R}^N,$$

($A_6$) $\int_{-T}^{T} V(t,0)\,dt = 0$,

$(A_7)$ $f \in \mathbf{L}^{G^*}([-T,T],\mathbb{R}^N)$.

Assumptions $(A_3)$, $(A_4)$ and $(A_5)$ are essential for the Mountain Pass Theorem. We need $(A_3)$ to show that there exists $\alpha > 0$ such that functional

$$\mathcal{J}(u) = \int_{-T}^{T} G(\dot{u}) + V(t,u) + \langle f, u\rangle \, dt \tag{$\mathcal{J}$}$$

is greater than $\alpha$ on the boundary of some ball (see lemma 3.4). To do this we need to control behavior of V near zero.

Condition $(A_4)$ allows us to control the growth of $V$ at infinity. The first condition, together with $(A_5)$, is used to show that the Palais-Smale condition is satisfied. The latter condition is used to show that functional $\mathcal{J}$ is negative far from zero. Assumption $(A_5)$ is a modification of the well known Ambrosetti-Rabinowitz condition.

Let us denote by $C_{\infty,\mathbf{W}^1\mathbf{L}^G}$ an embedding constant for $\mathbf{W}^1\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ and define

$$\rho := \frac{\rho_0}{C_{\infty,\mathbf{W}^1\mathbf{L}^G}}. \tag{$\rho$}$$

Now we can formulate our main theorems.

**Theorem 1.1.** *Let* $L\colon [-T,T]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ *satisfies* $(A_1)$*–*$(A_7)$*. Assume that* $G$ *satisfies* $\Delta_2$ *and* $\nabla_2$ *globally, and*

$$\int_{-T}^{T} G^*(f(t)) + a(t)\, dt < \min\{1, b-1\} \begin{cases} (\rho/2)^{q_G}, & \rho \le 2 \\ (\rho/2)^{p_G} & \rho > 2 \end{cases} \tag{1}$$

*Then* (ELT) *possesses a periodic solution.*

The assumption that $G$ satisfies $\Delta_2$ and $\nabla_2$ globally can be relaxed if we assume that $\rho \ge 2$. In this case we need a stronger assumption on $f$ and $a$.

**Theorem 1.2.** *Let* $L\colon [-T,T]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ *satisfies* $(A_1)$*–*$(A_7)$*. Assume that* $\rho \ge 2$ *and*

$$\int_{-T}^{T} G^*(f(t)) + a(t)\, dt < \min\{1, b-1\}(\rho/2) \tag{2}$$

*Then* (ELT) *possesses a periodic solution.*

Theorem 1.1 generalizes Lemma 3.1 from [8]. Actually, assumption (1) has the same form as $(H_5)$ in [8], since in p-laplacian case $p_G = q_G = q_G^\infty = p$. Note that p-laplacian satisfies $\Delta_2$ and $\nabla_2$ globally. To the best of the authors knowledge there is no analogue of Theorem 1.2 in the literature.

Now we give two examples of potentials suitable for our setting.

**Example 1.3.** *Consider the functions*

$$G(x,y) = x^2 + (x-y)^4, \quad K(t,x,y) = (2+\sin t)G(x,y) + |x^2+y^2|^2 \cos^2 t$$

$$W(t,x,y) = \frac{|x^2+y^2|^{5/2}(e^{t^2(x^2+y^2-1)}-1)}{t^2+1} + \sin t.$$

*$G$ is differentiable G-function satisfying $\Delta_2$ and $\nabla_2$ globally. Here $V = K - W$ satisfies $(A_2)$-$(A_6)$, where $p_G = 2$, $q_G^\infty = q_G = 4$, $\mu = 5$, $a(t) = \sin t$, $b = 2$, $\kappa(t) \geq 5\sin t$. On the other hand $K$ does not satisfy assumption $(H_1)$ and $W$ does not satisfy assumption $(H_2)$ from [8].*

The next example shows that our results generalize Lemma 7 from [9].

**Example 1.4.** *Set*

$$V(t,x) = c(t)G(x) - \lambda d(t)F(x),$$

*where $F$ is convex function satisfying $\Delta_2$ globally, $G \prec\prec F$, the functions $c(t)$, $d(t)$ are continuously differentiable, even on $\mathbb{R}$, $0 < c \leq c(t) \leq C$, $0 < d \leq d(t) \leq D$, $tc'(t) > 0$ for $t \neq 0$ and $tc'(t) < 0$ for $t \neq 0$. Then $V$ satisfies conditions $(A_2)$-$(A_7)$.*

*Theorems 1.1 and 1.2 assert the existence of periodic solutions for*

$$\begin{aligned} &\frac{d}{dt}\nabla G(\dot{u}) - c(t)\nabla G(u) + \lambda d(t)\nabla F(u) = f(t), \\ &u(-T) = u(T) = 0 \end{aligned}$$

*which is a generalization of the problem (2) from [9].*

## 2. Some facts about G-functions and Orlicz-Sobolev spaces

Assume that $G\colon \mathbb{R}^N \to [0,\infty)$ satisfies assumption $(A_1)$. We say that

- $G$ satisfies the $\Delta_2$ condition if

$$\exists_{K_1>2}\, \exists_{M_1\geq 0}\, \forall_{|x|\geq M_1}\; G(2x) \leq K_1 G(x), \tag{$\Delta_2$}$$

- $G$ satisfies the $\nabla_2$ condition if

$$\exists_{K_2>1}\, \exists_{M_2\geq 0}\, \forall_{|x|\geq M_2}\; G(x) \leq \frac{1}{2K_2} G(K_2 x). \tag{$\nabla_2$}$$

- $G$ satisfies $\Delta_2$ (resp. $\nabla_2$) globally if $M_1 = 0$ (resp. $M_2 = 0$).

Functions $G_1(x) = |x|^p$, $G_2(x) = |x|^{p_1} + |x|^{p_2}$ satisfy $\Delta_2$ and $\nabla_2$ globally. If $G$ does not satisfy $\Delta_2$ globally, then it could decrease very fast near zero. For example,

$$G(x) = \begin{cases} |x|^2 e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies $\Delta_2$ but does not satisfy $\Delta_2$ globally. For more details about $\Delta_2$ condition in case of N-function we refer the reader to [17].

Since $G$ is differentiable and convex,

$$G(x) - G(x-y) \leq \langle \nabla G(x), y \rangle \leq G(x+y) - G(x) \quad \text{for all } x, y \in \mathbb{R}^N. \tag{3}$$

A function $G^*(y) = \sup_{x\in\mathbb{R}^N}\{\langle x, y\rangle - G(x)\}$ is called the Fenchel conjugate of $G$. As an immediate consequence of the definition we have the Fenchel inequality:

$$\forall_{x,y\in\mathbb{R}^N} \ \langle x, y\rangle \leq G(x) + G^*(y).$$

Now we briefly recall the notion of anisotropic Orlicz space. For more details we refer the reader to [18] and [19]. The Orlicz space associated with $G$ is defined to be

$$\mathbf{L}^G = \{u\colon [-T,T] \to \mathbb{R}^N \colon \int_{-T}^{T} G(u)\,dt < \infty\}.$$

The space $\mathbf{L}^G$ equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf\left\{\lambda > 0\colon \int_{-T}^{T} G\left(\frac{u}{\lambda}\right)\,dt \leq 1\right\}$$

is a reflexive Banach space. We have the Hölder inequality

$$\int_I \langle u, v\rangle\,dt \leq 2\|u\|_{\mathbf{L}^G}\|v\|_{\mathbf{L}^{G^*}} \qquad \text{for every } u \in \mathbf{L}^G \text{ and } v \in \mathbf{L}^{G^*}.$$

Let us denote by

$$\mathbf{W}^1\mathbf{L}^G := \left\{u \in \mathbf{L}^G : \dot{u} \in \mathbf{L}^G\right\}$$

an anisotropic Orlicz-Sobolev space of vector valued functions with norm

$$\|u\|_{\mathbf{W}^1\mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}.$$

We introduce the following subset of $\mathbf{W}^1\mathbf{L}^G$

$$\mathbf{W}^1_T\mathbf{L}^G := \left\{u \in \mathbf{W}^1\mathbf{L}^G : u(-T) = u(T)\right\}.$$

We will also consider an equivalent norm given by

$$\|u\|_{1,\mathbf{W}^1\mathbf{L}^G} = \inf\left\{\lambda > 0\colon \int_{-T}^{T} G\left(\frac{u}{\lambda}\right) + G\left(\frac{\dot{u}}{\lambda}\right)\,dt \leq 1\right\}.$$

The following proposition will be crucial to Lemma 3.4.

**Proposition 2.1.**

$$\frac{1}{2}\|u\|_{\mathbf{W}^1\mathbf{L}^G} \leq \|u\|_{1,\mathbf{W}^1\mathbf{L}^G} \leq 2\|u\|_{\mathbf{W}^1\mathbf{L}^G}$$

The proof for isotropic case can be found in [20, Proposition 9, p.177]. It remains the same for anisotropic case.

Functional $R_G(u) := \int_{-T}^{T} G(u)\,dt$ is called modular.

**Proposition 2.2.** *[21, Proposition 2.7] $R_G(u)$ is coercive on $\mathbf{L}^G$ in the following sense:*

$$\lim_{\|u\|_{\mathbf{L}^G}\to\infty} \frac{R_G(u)}{\|u\|_{\mathbf{L}^G}} = \infty.$$

Define the Simonenko indices for G-function

$$p_G = \inf_{|x|>0} \frac{\langle x, \nabla G(x)\rangle}{G(x)}, \quad q_G = \sup_{|x|>0} \frac{\langle x, \nabla G(x)\rangle}{G(x)}, \quad q_G^\infty = \limsup_{|x|\to\infty} \frac{\langle x, \nabla G(x)\rangle}{G(x)}.$$

It is obvious that $p_G \leq q_G^\infty \leq q_G$. Moreover, if $G$ satisfies $\Delta_2$ and $\nabla_2$ globally, then $1 < p_G$ and $q_G < \infty$. The following results is crucial to Lemma 3.4.

**Proposition 2.3.** *Let $G$ satisfies $\Delta_2$ and $\nabla_2$ globally.*

1. *If $\|u\|_{\mathbf{L}^G} \leq 1$, then $\|u\|_{\mathbf{L}^G}^{q_G} \leq R_G(u)$.*
2. *If $\|u\|_{\mathbf{L}^G} > 1$, then $\|u\|_{\mathbf{L}^G}^{p_G} \leq R_G(u)$.*

The proof can be found in appendix. More information about indices for isotropic case can be found in [22], [23] and [13]. For relations between Luxemburg norm and modular for anisotropic spaces we refer the reader to [19, Examples 3.8 and 3.9].

For, respectively, continuous and compact embeddings we will use the symbols $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$.

Let $G_1$ and $G_2$ be G-functions. Define

$$G_1 \prec G_2 \iff \exists_{M\geq 0}\, \exists_{K>0}\, \forall_{|x|\geq M}\; G_1(x) \leq G_2(K\,x). \tag{4}$$

The relation $\prec$ allows to compare growth rate of functions $G_1$ and $G_2$.

It is well known that if $G_1 \prec G_2$, then $\mathbf{L}^{G_2} \hookrightarrow \mathbf{L}^{G_1}$. Let $u \in \mathbf{W}^1\mathbf{L}^G$, $A_G : \mathbb{R}^N \to [0,\infty)$ be the greatest convex radial minorant of $G$ (see [2]). Then

$$\|u\|_{\mathbf{L}^\infty} \leq C_{\infty,\mathbf{W}^1\mathbf{L}^G}\|u\|_{\mathbf{W}^1\mathbf{L}^G},$$

where $C_{\infty,\mathbf{W}^1\mathbf{L}^G} = A_G^{-1}\left(\frac{1}{2T}\right)\max\{1, 2T\}$.

The following proposition will be used in the proof of Lemma 3.2.

**Proposition 2.4.** *(cf. [8]) For any $1 < p \leq q < \infty$, such that $|\cdot|^p \prec G(\cdot) \prec |\cdot|^q$,*

$$\int_{-T}^{T} |u|^p \, dt \geq C_{\infty,\mathbf{W}^1\mathbf{L}^G}^{p-q} C_{G,q}^{-q} \|u\|_{\mathbf{W}^1\mathbf{L}^G}^{p-q} \|u\|_{\mathbf{L}^G}^{q}$$

*for $u \in \mathbf{W}^1\mathbf{L}^G \setminus \{0\}$, where $C_{G,q}$ is an embedding constant from $\mathbf{L}^q \hookrightarrow \mathbf{L}^G$.*

*Proof.* Let $u \in \mathbf{W}^1\mathbf{L}^G \setminus \{0\}$. Since $G \prec |\cdot|^q$,

$$\int_{-T}^{T} |u|^q \, dt = \|u\|_{\mathbf{L}^q}^q \geq C_{G,q}^{-q} \|u\|_{\mathbf{L}^G}^q.$$

From Hölder's inequality and embedding $\mathbf{W}^1\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ we obtain

$$\int_{-T}^{T} |u|^q \, dt = \int_{-T}^{T} |u|^p |u|^{q-p}, dt \leq$$

$$\leq \|u\|_{\mathbf{L}^\infty}^{q-p} \int_{-T}^{T} |u|^p \, dt \leq (C_{\infty,\mathbf{W}^1\mathbf{L}^G} \|u\|_{\mathbf{W}^1\mathbf{L}^G})^{q-p} \int_{-T}^{T} |u|^p \, dt.$$

□

## 3. Proof of the main results

Let $\mathcal{J} : \mathbf{W}_T^1\mathbf{L}^G \to \mathbb{R}$ be given by

$$\mathcal{J}(u) = \int_{-T}^{T} G(\dot{u}) + K(t,u) - W(t,u) + \langle f, u \rangle \, dt. \tag{$\mathcal{J}$}$$

From $(A_1)$, $(A_2)$ and [19, Thm. 5.5] we have $\mathcal{J} \in C^1$ and

$$\mathcal{J}'(u)\varphi = \int_{-T}^{T} \langle \nabla G(\dot{u}), \dot{\varphi} \rangle \, dt + \int_{-T}^{T} \langle V_x(t,u) + f(t), \varphi \rangle \, dt. \tag{$\mathcal{J}'$}$$

It is standard to prove that critical points of $\mathcal{J}$ are solutions of (ELT).

Our proof is based on the well-known Mountain Pass Theorem (see [24]).

**Theorem 3.1.** *Let $X$ be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies the following conditions:*

1. *$I(0) = 0$,*
2. *$I$ satisfies Palais-Smale condition,*
3. *there exist $\rho > 0$, $e \in X$ such that $\|e\|_X > \rho$ and $I(e) < 0$,*

*4. there exists $\alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$.*

*Then $I$ possesses a critical value $c \geq \alpha$ given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

*where* $\Gamma = \{g \in C([0,1], X);\quad g(0) = 0,\ g(1) = e\}$.

We divide the proof into sequence of lemmas.

**Lemma 3.2.** *$\mathcal{J}$ satisfies the Palais-Smale condition, i.e. every sequence $\{u_n\} \subset \mathbf{W}^1_T\mathbf{L}^G$ such that $\{\mathcal{J}(u_n)\}$ is bounded and $\mathcal{J}'(u_n) \to 0$ as $n \to \infty$ contains a convergent subsequence.*

*Proof.* From $(A_5)$ and $(\mathcal{J}')$ we get

$$\int_{-T}^{T} \mu W(t,u) - (q_G^\infty + \nu)K(t,u)\,dt \leq$$
$$\leq -\mathcal{J}'(u)u + \int_{-T}^{T} \langle \nabla G(\dot u), \dot u\rangle\,dt + \int_{-T}^{T} \langle f(t), u\rangle + \kappa(t)\,dt. \quad (5)$$

From the definition of the functional we obtain

$$\mu \int_{-T}^{T} G(\dot u)\,dt + (\mu - q_G^\infty - \nu)\int_{-T}^{T} K(t,u)\,dt =$$
$$= \mu\,\mathcal{J}(u) + \int_{-T}^{T} \mu W(t,u) - (q_G^\infty + \nu)K(t,u)\,dt - \mu\int_{-T}^{T} \langle f(t), u\rangle\,dt.$$

Applying (5), the Hölder inequality and $(A_7)$ we have

$$\mu \int_{-T}^{T} G(\dot u)\,dt + (\mu - q_G^\infty - \nu)\int_{-T}^{T} K(t,u)\,dt \leq$$
$$\leq \mu\,\mathcal{J}(u) - \mathcal{J}'(u)u + \int_{-T}^{T} \langle \nabla G(\dot u), \dot u\rangle\,dt + C_\kappa + (1-\mu)C_f\|u\|_{\mathbf{W}^1\mathbf{L}^G}, \quad (6)$$

where $C_\kappa = \int_{-T}^{T} \kappa(t)\,dt$ and $C_f = 2\|f\|_{\mathbf{L}^{G^*}}$. By the definition of $q_G^\infty$, there exists $M > 0$ such that

$$\langle x, \nabla G(x)\rangle \leq (q_G^\infty + \nu)\,G(x),\ \text{ for } |x| > M.$$

Hence

$$\int_{-T}^{T} \langle \nabla G(\dot u), \dot u\rangle\,dt = \int_{\{|\dot u|>M\}} \langle \nabla G(\dot u), \dot u\rangle\,dt + \int_{\{|\dot u|\leq M\}} \langle \nabla G(\dot u), \dot u\rangle\,dt \leq$$
$$\leq (q_G^\infty + \nu)\int_{\{|\dot u|>M\}} G(\dot u)\,dt + C_{\nabla G} \leq (q_G^\infty + \nu)\int_{-T}^{T} G(\dot u)\,dt + C_{\nabla G}, \quad (7)$$

where $C_{\nabla_G} = \max_{|x|\leq M} 2\,T\,M\,\nabla G(x)$. Applying (7) we can rewrite (6) as

$$(\mu - q_G^\infty - \nu)\int_{-T}^{T} G(\dot{u}) + K(t,u)\,dt \leq$$
$$\leq \mu\,\mathcal{J}(u) - \mathcal{J}'(u)u + C_\kappa + (1-\mu)C_f\|u\|_{\mathbf{W}^1\,\mathbf{L}^G} + C_{\nabla G}. \quad (8)$$

From $(A_4)$, given any $0 < \varepsilon_1 < b_1$, there exists $\delta_1 \geq 0$ such that

$$K(t,x) \geq (b_1 - \varepsilon_1)|x|^p - \delta_1, \qquad \text{for } x \in \mathbb{R}^N \text{ and } t \in [-T,T].$$

By Proposition 2.4 we obtain

$$\int_{-T}^{T} K(t,u)\,dt \geq \int_{-T}^{T} (b_1 - \varepsilon_1)|u|^p\,dt - 2T\delta_1 \geq$$
$$\geq (b_1 - \varepsilon_1)C_{\infty,\mathbf{W}^1\,\mathbf{L}^G}^{p-q} C_{G,q}^{-q} \frac{\|u\|_{\mathbf{L}^G}^q}{\|u\|_{\mathbf{W}^1\,\mathbf{L}^G}^{q-p}} - 2T\delta_1, \quad (9)$$

for any $q$ such that $G \prec |\cdot|^q$. Finally, applying (9) to (8) we obtain

$$(\mu - q_G^\infty - \nu)\left(\int_{-T}^{T} G(\dot{u})\,dt + (b_1 - \varepsilon_1)C_{\infty,\mathbf{W}^1\,\mathbf{L}^G}^{p-q} C_{G,q}^{-q} \frac{\|u\|_{\mathbf{L}^G}^q}{\|u\|_{\mathbf{W}^1\,\mathbf{L}^G}^{q-p}}\right) +$$
$$- (\mu - 1)C_f\|u\|_{\mathbf{W}^1\,\mathbf{L}^G} + \mathcal{J}'(u)u \leq \mu\,\mathcal{J}(u) + C_\kappa + C_{\nabla_G} + C_{\delta_1}, \quad (10)$$

where $C_{\delta_1} = 2T\delta_1(\mu - q_G^\infty - \nu)$.

Let $\{u_n\} \subset \mathbf{W}_T^1\,\mathbf{L}^G$ will be a Palais-Smale sequence for $\mathcal{J}$. There exist $C_J$, $C_{J'} > 0$ such that

$$\mathcal{J}(u_n) \leq C_J, \quad \mathcal{J}'(u_n)u_n \geq -C_{J'}\|u_n\|_{\mathbf{W}^1\,\mathbf{L}^G}$$

Without loss of generality we can assume, that $\|u_n\|_{\mathbf{W}^1\,\mathbf{L}^G} > 0$. Substituting $u_n$ into (10) we obtain

$$\|u_n\|_{\mathbf{W}^1\,\mathbf{L}^G}\left(\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1\,\mathbf{L}^G}} + \frac{\|u_n\|_{\mathbf{L}^G}^q}{\|u_n\|_{\mathbf{W}^1\,\mathbf{L}^G}^{1+q-p}} - C'\right) \leq C'', \quad (11)$$

where $C', C'' > 0$ are suitable constants independent of $n$.

We show that $\{u_n\}$ is bounded. On the contrary, suppose that there exists a subsequence of $u_n$ (still denoted $u_n$) such that $\|u_n\|_{\mathbf{W}^1\,\mathbf{L}^G} \to \infty$. Consider three cases.

1. Let $\|u_n\|_{\mathbf{L}^G} \to \infty$ and $\|\dot{u}_n\|_{\mathbf{L}^G} \to \infty$ (again, w.l.o.g. $\|\dot{u}_n\|_{\mathbf{L}^G} > 0$). From Proposition 2.2 we have that

$$\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1\mathbf{L}^G}} + \frac{\|u_n\|^q_{\mathbf{L}^G}}{\|u_n\|^{1+q-p}_{\mathbf{W}^1\mathbf{L}^G}} = \\ = \frac{R_G(\dot{u}_n)}{\|\dot{u}_n\|_{\mathbf{L}^G}}\frac{\|\dot{u}_n\|_{\mathbf{L}^G}}{\|u_n\|_{\mathbf{W}^1\mathbf{L}^G}} + \left(\frac{\|u_n\|_{\mathbf{L}^G}}{\|u_n\|_{\mathbf{W}^1\mathbf{L}^G}}\right)^{1+q-p} \|u_n\|^{p-1}_{\mathbf{L}^G} \to \infty.$$

2. Let $\|\dot{u}_n\|_{\mathbf{L}^G} \to \infty$ and $\|u_n\|_{\mathbf{L}^G}$ is bounded. Then

$$\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1\mathbf{L}^G}} = \frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{L}^G} + \|\dot{u}_n\|_{\mathbf{L}^G}} = \frac{\frac{R_G(\dot{u}_n)}{\|\dot{u}_n\|_{\mathbf{L}^G}}}{\frac{\|u_n\|_{\mathbf{L}^G}}{\|\dot{u}_n\|_{\mathbf{L}^G}} + 1} \to \infty \text{ as } \|\dot{u}_n\|_{\mathbf{L}^G} \to \infty.$$

3. Let $\|u_n\|_{\mathbf{L}^G} \to \infty$ and $\|\dot{u}_n\|_{\mathbf{L}^G}$ is bounded. Since $p > 1$, we have

$$\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1\mathbf{L}^G}} + \frac{\|u_n\|^q_{\mathbf{L}^G}}{\|u_n\|^{1+q-p}_{\mathbf{W}^1\mathbf{L}^G}} \geq \frac{\|u_n\|^q_{\mathbf{L}^G}}{(\|u_n\|_{\mathbf{L}^G} + \|\dot{u}_n\|_{\mathbf{L}^G})^{1+q-p}} \to \infty.$$

Therefore, in view of (11), $\{u_n\}$ is bounded in $\mathbf{W}^1_T\mathbf{L}^G$.

It follows from reflexivity of $\mathbf{W}^1_T\mathbf{L}^G$ and embeddings $\mathbf{W}^1_T\mathbf{L}^G \hookrightarrow\hookrightarrow \mathbf{L}^G$, $\mathbf{W}^1_T\mathbf{L}^G \hookrightarrow\hookrightarrow \mathbf{W}^{1,1}$ that there exists $u \in \mathbf{W}^1_T\mathbf{L}^G$ and a subsequence of $u_n$ (still denoted $u_n$) such that $u_n \to u$ in $\mathbf{L}^G$. Moreover, $\dot{u}_n \to \dot{u}$ in $\mathbf{L}^1$ and hence pointwise a.e.

Since $\{u_n\}$ is a Palais-Smale sequence, we have

$$0 \leftarrow \mathcal{J}'(u_n)(u_n - u) = \int_{-T}^{T} \langle \nabla G(\dot{u}_n), \dot{u}_n - \dot{u}\rangle\, dt + \int_{-T}^{T} \langle V_x(t, u_n) + f(t), u_n - u\rangle\, dt.$$

Since $\int_{-T}^{T}\langle V_x(t,u_n) + f(t), u_n - u\rangle\, dt \to 0$ we can deduce that

$$\int_{-T}^{T} \langle \nabla G(\dot{u}_n), \dot{u}_n - \dot{u}\rangle\, dt \to 0.$$

From (3) we obtain

$$\int_{-T}^{T} G(\dot{u}_n)\, dt \leq \int_{-T}^{T} G(\dot{u})\, dt + \int_{-T}^{T} \langle \nabla G(\dot{u}_n), (\dot{u}_n - \dot{u})\rangle\, dt$$

Hence

$$\limsup_{n\to+\infty} \int_{-T}^{T} G(\dot{u}_n)\, dt \leq \int_{-T}^{T} G(\dot{u})\, dt.$$

On the other hand, by Fatou's Theorem we have

$$\liminf_{n\to+\infty}\int_{-T}^{T} G(\dot{u}_n)\,dt \ge \int_{-T}^{T} G(\dot{u})\,dt.$$

Combining these inequalities we get that

$$\int_{-T}^{T} G(\dot{u}_n)\,dt \to \int_{-T}^{T} G(\dot{u})\,dt.$$

Therefore $\dot{u}_n \to \dot{u}$ in $\mathbf{L}^G$ by [19, Lemma 3.16]. □

We next prove that $\mathcal{J}$ is negative for some point outside $B_\rho(0)$.

**Lemma 3.3.** *There exist $e \in \mathbf{W}^1_T\mathbf{L}^G$ such that $\|e\|_{\mathbf{W}^1\mathbf{L}^G} > \rho$ and $\mathcal{J}(e) < 0$.*

*Proof.* By assumption $(A_4)$, there exist $\varepsilon_0, r > 0$ such that

$$W(t,x) \ge (3+\varepsilon_0)\max\{K(t,x), G(x)\} \quad \text{for } |x| > r.$$

This gives

$$K(t,x) - W(t,x) \le -(2+\varepsilon_0)G(x) \quad \text{for } |x| > r. \tag{12}$$

Fix $v \in \mathbb{R}^N$. For $\xi > T+1$ define $e\colon [-T,T] \to \mathbb{R}^N$ by

$$e(t) = \xi\left(1 - \frac{|t|}{T+1}\right)v.$$

Direct computation shows

$$\dot{e}(t) = \begin{cases} -\frac{\xi}{T+1}\,v, & t \in (0,T], \\ \frac{\xi}{T+1}\,v, & t \in [-T,0). \end{cases}$$

Since $\|e\|_{\mathbf{L}^\infty} = \xi > T+1$ and $\|\dot{e}\|_{\mathbf{L}^\infty} = \xi/(T+1) > 1$, we can choose $\xi$ such that both (12) and $\|e\|_{\mathbf{W}^1\mathbf{L}^G} \ge \rho$ hold. From $(\mathcal{J})$, the Fenchel inequality and (12) we have

$$\mathcal{J}(e) \le \int_{-T}^{T} G(\dot{e}) + K(t,e) - W(t,e) + G(e) + G^*(f)\,dt \le$$
$$\le \int_{-T}^{T} G(\dot{e}) - G(e) - \varepsilon_0 G(e) + G^*(f)\,dt.$$

Since $1 - \frac{|t|}{T+1} \ge \frac{1}{T+1}$ for $t \in [-T,T]$, we have

$$\int_{-T}^{T} G(\dot{e}) - G(e)\,dt = \int_{-T}^{T} G\left(\frac{\xi}{T+1}v\right) - G\left(\xi\left(1 - \frac{|t|}{T+1}\right)v\right)\,dt \le 0.$$

Choosing $\xi$ large enough we get

$$\mathcal{J}(e) \le \int_{-T}^{T} -\varepsilon_0\, G(e) + G^*(f)\, dt < 0.$$

□

In order to show that $\mathcal{J}$ satisfies the fourth assumption of the Mountain Pass Theorem, we first provide some estimates for $R_G(\dot u) + R_G(u)$ on $\partial B_\rho(0)$.

If $\Delta_2$ and $\nabla_2$ are satisfied globally then we can use Proposition 2.3 to estimate $R_G(\dot u) + R_G(u)$ from below by $(\rho/2)^r$, $r > 1$, for any $\rho > 0$. If $G$ does not satisfies $\Delta_2$ and $\nabla_2$ globally then we cannot use Proposition 2.3 (for explanation see Remark A.2). In this case we use equivalent norm and Proposition 2.1 but we obtain only that $R_G(\dot u) + R_G(u) \ge \rho/2$. Moreover, we are forced to assume $\rho > 2$.

Let $u \in \mathbf{W}^1_T\mathbf{L}^G$ be such that $\|u\|_{\mathbf{W}^1\mathbf{L}^G} = \rho$. Set $\rho_1 = \|u\|_{\mathbf{L}^G}$, $\rho_2 = \|\dot u\|_{\mathbf{L}^G}$, $\rho_1 + \rho_2 = \rho$. Assuming that $G$ satisfies $\Delta_2$ an $\nabla_2$ globally we get, by Proposition 2.3, the following estimates:

1. If $\rho_1, \rho_2 \le 1$ then $R_G(\dot u) + R_G(u) \ge \|\dot u\|^{q_G}_{\mathbf{L}^G} + \|u\|^{q_G}_{\mathbf{L}^G}$. Hence
$$R_G(\dot u) + R_G(u) \ge 2^{1-q_G}(\|\dot u\|_{\mathbf{L}^G} + \|u\|_{\mathbf{L}^G})^{q_G} \ge (\rho/2)^{q_G}\,, \tag{13}$$
since ${\rho_1}^{q_G} + \rho_2^{q_G} \ge 2^{1-q_G}(\rho_1+\rho_2)^{q_G}$.

2. If $\rho_1 \le 1$, $\rho_2 \ge 1$ then $(\rho_1+\rho_2)^{p_G} \le (2\rho_2)^{p_G} \le 2^{p_G}\left(\rho_1^{q_G} + \rho_2^{p_G}\right)$. Hence
$$R_G(\dot u) + R_G(u) \ge \|\dot u\|^{p_G}_{\mathbf{L}^G} + \|u\|^{q_G}_{\mathbf{L}^G} \ge (\rho/2)^{p_G}. \tag{14}$$

3. If $\rho_1 \ge 1$, $\rho_2 \le 1$ then $(\rho_1+\rho_2)^{p_G} \le (2\rho_1)^{p_G} \le 2^{p_G}\left(\rho_1^{p_G} + \rho_2^{q_G}\right)$. Thus
$$R_G(\dot u) + R_G(u) \ge \|\dot u\|^{q_G}_{\mathbf{L}^G} + \|u\|^{p_G}_{\mathbf{L}^G} \ge (\rho/2)^{p_G}. \tag{15}$$

4. If $\rho_1, \rho_2 \ge 1$ then
$$R_G(\dot u) + R_G(u) \ge \|\dot u\|^{p_G}_{\mathbf{L}^G} + \|u\|^{p_G}_{\mathbf{L}^G} \ge (\rho/2)^{p_G}. \tag{16}$$

From the other hand, Proposition 2.1 implies

$$\inf\left\{\lambda > 0\colon \int_{-T}^{T} G\left(\frac{u}{\lambda}\right) + G\left(\frac{\dot u}{\lambda}\right)\, dt \le 1\right\} \ge \frac{1}{2}\rho.$$

Therefore

$$\int_{-T}^{T} G\left(\frac{2u}{\rho}\right) + G\left(\frac{2\dot u}{\rho}\right)\, dt \ge 1$$

and consequently,

$$R_G(u) + R_G(\dot u) \ge \frac{\rho}{2}, \tag{17}$$

provided $\rho > 2$.

**Lemma 3.4.** *Assume that either* (1) *or* (2) *holds. There exists positive constant* $\alpha$ *such that* $\mathcal{J}|_{\partial B_\rho(0)} \geq \alpha$.

*Proof.* From the definition of $\rho$ and embedding $\mathbf{W}^1_T\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ we have

$$|u(t)| \leq \|u\|_{\mathbf{L}^\infty} \leq C_{\infty,\mathbf{W}^1\mathbf{L}^G}\|u\|_{\mathbf{W}^1\mathbf{L}^G} = C_{\infty,\mathbf{W}^1\mathbf{L}^G}\rho = \rho_0 \quad \text{for } t \in [-T,T].$$

From $(A_3)$ and the Fenchel inequality we obtain

$$\mathcal{J}(u) \geq \int_{-T}^{T} G(\dot{u}) + bG(u) - a(t) + \langle f, u\rangle \, dt \geq$$
$$\geq \min\{1, b-1\}(R_G(\dot{u}) + R_G(u)) - R_{G^*}(f) - \int_{-T}^{T} a(t)\, dt.$$

Assume that (1) holds. If $\rho \leq 2$ then (13), (14) and (15) yields

$$\mathcal{J}(u) \geq \min\{1, b-1\}(\rho/2)^{q_G} - R_{G^*}(f) - \int_{-T}^{T} a(t)\, dt =: \alpha.$$

If $\rho > 2$, then by (14), (15) and (16) we get

$$\mathcal{J}(u) \geq \min\{1, b-1\}(\rho/2)^{p_G} - R_{G^*}(f) - \int_{-T}^{T} a(t)\, dt > 0 =: \alpha.$$

From (1) it follows that in both cases $\alpha > 0$.

Assume that (2) holds. From (17) we obtain

$$\mathcal{J}(u) \geq \min\{1, b-1\}(R_G(\dot{u}) + R_G(u)) - R_{G^*}(f) - \int_{-T}^{T} a(t) \geq$$
$$\geq \min\{1, b-1\}(\rho/2) - R_{G^*}(f) - \int_{-T}^{T} a(t)\, dt =: \alpha.$$

From (2) we have $\alpha > 0$. $\square$

Now we are in position to prove our main theorems. Note that by $(A_6)$ and $G(0) = 0$ we have $\mathcal{J}(0) = 0$. From Lemmas 3.2, 3.3 and 3.4 we have that $\mathcal{J}$ satisfies all assumptions of the Mountain Pass Theorem. Hence there exists a critical point $u \in \mathbf{W}^1_T\mathbf{L}^G$ of $\mathcal{J}$ and (ELT) have periodic solution.

Actually, we can show that any solution to (ELT) is more regular (cf. Corollary 16.16 in [25]).

**Proposition 3.5.** *If* $u \in \mathbf{W}^1_T\mathbf{L}^G$ *is a solution of* (ELT)*, then* $u \in \mathbf{W}^1\mathbf{L}^\infty$.

*Proof.* Let $u \in \mathbf{W}^1\mathbf{L}^G$ be a solution of (ELT). Then

$$\nabla G(\dot{u}(t)) = \int_{-T}^{t} \nabla V(t, u(t))\, dt + C$$

and there exists $M > 0$ such that $|\nabla G(\dot{u}(t))| \leq M < \infty$. From the other hand

$$G(\dot{u}(t)) \leq \langle \nabla G(\dot{u}(t)), \dot{u}(t)\rangle \leq M|\dot{u}(t)|.$$

Since $\frac{G(v)}{|v|} \to \infty$ as $|v| \to \infty$, we obtain $|\dot{u}(t)|$ is bounded. $\square$

**Remark 3.6.** *If $G$ is strictly convex then one can show that if $u \in \mathbf{W}^1_T\mathbf{L}^G$ is a solution of* (ELT)*, then $u \in \mathbf{C}^1$.*

**Remark 3.7.** *Theorem 1.1 remains true if we change assumption* (1) *to*

$$\int_{-T}^{T} G^*(f(t)) + a(t)\, dt < \min\{1, b-1\} \begin{cases} 2(\rho/2)^{q_G}, & \rho \leq 2^{1-1/(q_G-p_G)} \\ (\rho/2)^{p_G} & \rho > 2^{1-1/(q_G-p_G)}. \end{cases} \tag{18}$$

*Estimate in the first case is better than* (1) *but it is taken on smaller set. In the second case estimate is the same as in* (1) *but can be taken on bigger set.*

**Remark 3.8.** *In the proof of Lemma 3.4 we can use the Hölder inequality instead of the Fenchel inequality to estimate $\int_{-T}^{T}\langle f, u\rangle\, dt$. It allows us to take $b > 0$ if $\rho \leq 1$.*

## Appendix A.

Assume that $G$ satisfies $\nabla_2$ globally. It is easy to show that $G^*$ satisfies $\Delta_2$ globally with $K_1^* = 2K_2$.

Since $G \in \mathbf{C}^1$ and is convex, we have

$$K_1 G(x) \geq G(2x) \geq G(2x) - G(x) \geq \langle x, \nabla G(x)\rangle \quad \text{for all } x \in \mathbb{R}^N.$$

Let $y \in \mathbb{R}^N$ and $s \in \partial G^*(y)$, where $\partial G^*$ denotes the subdifferential of $G^*$. Since $G^*$ satisfies $\Delta_2$ globally, we have

$$K_1^* G^*(y) \geq G^*(2y) \geq G^*(2y) - G^*(y) \geq \langle s, y\rangle \quad \text{for all } y \in \mathbb{R}^N$$

Let $x \in \mathbb{R}^N$. Then $x \in \partial G^*(\nabla G(x))$ and $G(x) + G^*(\nabla G(x)) = \langle x, \nabla G(x)\rangle$. It follows that

$$G(x) = \langle x, \nabla G(x)\rangle - G^*(\nabla G(x)) \leq \left(1 - \frac{1}{K_1^*}\right)\langle x, \nabla G(x)\rangle.$$

Finally

$$\frac{2K_2}{2K_2-1} \leq \frac{\langle x, \nabla G(x)\rangle}{G(x)} \leq K_1 \quad \text{for all } x \in \mathbb{R}^N. \tag{A.1}$$

Since $K_2 > 1$, we have $\frac{2K_2}{2K_2-1} > 1$ and from (A.1) we obtain

$$p_G > 1 \quad \text{and} \quad q_G < \infty.$$

For any $x \in \mathbb{R}^N$ and $\lambda \geq 1$ we have

$$\log G(\lambda x) - \log G(x) = \int_1^\lambda \frac{\langle \nabla G(\lambda x), x\rangle}{G(\lambda x)}\, d\lambda \leq \int_1^\lambda \frac{q_G}{\lambda}\, d\lambda = \log \lambda^{q_G}.$$

Thus

$$G(\lambda x) \leq \lambda^{q_G} G(x) \quad \text{for all } x \in \mathbb{R}^N,\ \lambda \geq 1. \tag{A.2}$$

Similarly, we get

$$G(\lambda x) \geq \lambda^{p_G} G(x) \quad \text{for all } x \in \mathbb{R}^N,\ \lambda \geq 1. \tag{A.3}$$

**Lemma A.1.** *Let $u \in \mathbf{L}^G$.*

1. *If $\|u\|_{\mathbf{L}^G} < 1$ then $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{q_G}$.*
2. *If $\|u\|_{\mathbf{L}^G} > 1$ then $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{p_G}$.*

*Proof.* For any $0 < \beta < \|u\|_{\mathbf{L}^G} < 1$ we have $R_G\left(\frac{u}{\beta}\right) \geq 1$. From (A.2) we obtain that $G\left(\frac{x}{\beta}\right) \leq \left(\frac{1}{\beta}\right)^{q_G} G(x)$ for all $x \in \mathbb{R}^N$. Hence

$$R_G(u) \geq \beta^{q_G} R_G\left(\frac{u}{\beta}\right) \geq \beta^{q_G}.$$

Letting $\beta \uparrow \|u\|_{\mathbf{L}^G}$ gives $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{q_G}$.

For any $1 < \beta < \|u\|_{\mathbf{L}^G}$ we have $R_G\left(\frac{u}{\beta}\right) > 1$. Then from (A.3) we obtain that $G(x) \geq \beta^{p_G} G\left(\frac{x}{\beta}\right)$ for all $x \in \mathbb{R}^N$. Hence

$$R_G(u) \geq \beta^{p_G} R_G\left(\frac{u}{\beta}\right) \geq \beta^{p_G}.$$

Letting $\beta \uparrow \|u\|_{\mathbf{L}^G}$ gives $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{p_G}$. □

**Remark A.2.** *If $G$ satisfies $\Delta_2$ and $\nabla_2$ (not globally), estimates similar to* (A.2) *and* (A.3) *can be obtained for sufficiently large $|x|$. However, even if $\|u\|_{\mathbf{L}^G}$ is large it does not necessarily mean that $|u(t)|$ is large. Hence we cannot use these estimates to obtain result similar to Lemma A.1.*

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