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E-COHOMOLOGICAL CONLEY INDEX

PhD Thesis

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Abstract

In this thesis we continue with developing the E-cohomological Conley index which was introduced by A.Abbondandolo. In particular, we generalize the index to non-gradient flows, we show that it an possesses additional multiplicative structure and we prove the continuation principle. Then, using continuation principle, we show how the computation of the E-cohomological Conley index can be reduced to the computation of the classical Conley index for a gradient flow on a finite-dimensional space. We conclude that the index is isomorphic to the local Morse cohomology on a Hilbert space. Finally, we apply above results to give a short proof of the Arnold conjecture on $2n$-dimensional torus in both degenerate and non-degenerate case.

Streszczenie

W niniejszej rozprawie rozwijamy teorie E-kohomologicznego indeksu Conleya wprowadzonego przez A.Abbondandolo. W szczególności, uogólniamy indeks na potoki niegradientowe; pokazujemy, że indeks posiada dodatkową strukturę multiplikatywną i dowodzimy homotopijnej niezmienniczonśći indeksu. Następnie, korzystając z homotopijnej niezmienniczości, pokazujemy jak można zredukować problem liczenia E-kohomologicznego indeksu Conleya do policzenia klasycznego indeksu Conleya dla gradientowego potoku na skończenie wymiarowej przestrzeni. Jako wniosek otrzymujemy fakt mówiący, że E-kohomologiczny indeks jest izomorficzny do kohomologii Morse'a na przestrzeni Hilberta. Pokazujemy również jak można zastosować powyższe wyniki aby otrzymać krótki dowód hipotezy Arnolda na $2n$-wymiarowym torusie zarówno w wersji zdegenerowanej jak i niezdegenerowanej.

Contents

Chapter 1

Introduction

In 1999, A.Abbondandolo defined the E-cohomological Conley index for a certain class of gradient flows in Hilbert spaces (see [Abb99]). The idea was to combine the classical Conley index ([Con78]) and the E-cohomology theory. The construction used in the definition of E-cohomology theory goes back to A.Granas and K.Gęba ([GG73]). Suppose that $\mathbb{H}$ is a separable Hilbert space with a splitting into closed infinite-dimensional subspaces E^+ and E^-. The E-cohomology is a generalized cohomology theory which detects non-triviality of a sphere in E^-, i.e. a sphere which is infinite-dimensional and infinite-codimensional. This property allows for effective use of the E-cohomological Conley index in detecting critical points of infinite Morse index and coindex i.e. to strongly indefinite problems.

In this thesis we will continue with further developing of the E-cohomological Conley index. In particular we

1. generalize the index to non-gradient flows;

2. show that the index possesses a module structure and introduce the cup-length;

3. apply the index and its cup-length to get a simple proof of the Arnold conjecture on T^{2n}.

Let us be more precise about the contents

In Chapter 2 we recall the definition and some properties of E-cohomology groups. Most of them are already stated by A.Abbondandolo in [Abb97]. However, we make a remark (Remark 2.2.4) that the original class of morphisms can be slightly extended. Namely it can be extended in such a way that it includes certain flow deformations. Additionally, we show that the E-cohomology groups have some additional module structure and we define a cup-length.

Chapter 3 is devoted to the definition and properties of the E-cohomological Conley index. We start by introducing a compactness property for flows. Let η be a flow.

(C) If a set $\bigcup_{n\in\mathbb{N}} \eta(x_n, [-n, n])$ is bounded for a sequence $\{x_n\}$, then $\{x_n\}$ has a convergent subsequence.

Stronger condition, with the interval $[-n, n]$ replaced with $[0, n]$, was already used by K.P.Rybakowski and E.Zehnder in [Ryb83]. However, the flows that we are interested in, satisfy only the weaker assumption. It turns out, that the above condition is very useful in the forthcoming proofs. It also implies the local Palais-Smale condition if the flow is a gradient flow (see Proposition 4.4.5). We show that the class of $\mathcal{LS}$-flows satisfy condition (C) . For this class we define an E-cohomological Conley index and prove some usual properties like non-triviality. The proof of homotopy invariance, i.e. the continuation principle, is contained in a separate section. We find it important since the proof is not a straightforward generalization of any proof for the classical Conley index. Moreover, the continuation principle is crucial for applications as shown in Chapter 4 and 5.

In Chapter 4 we show that the any $\mathcal{LS}$-flow is related by continuation to a product flow for which all the interesting dynamics is contained in a finite-dimensional subspaces (Corollary 4.1.5). This allows us to compare the E-cohomological Conley index with other invariants. In particular we show that it is isomorphic to the local Morse cohomology (Theorem 4.4.4) and to the cohomological $\mathcal{LS}$-index defined by M.Izydorek (Proposition 4.2.3). As mentioned, the E-cohomological Conley index has a module structure. Since this index is isomorphic to the local Morse cohomology there should be a module structure on the side of Morse cohomology. However, it is not clear how to interpret the later in terms of connecting orbits. As a consequence of the finite-dimensional reduction one also gets a statement that any $\mathcal{LS}$-flow is related by continuation to a gradient flow (Theorem 4.3.1).

Finally in Chapter 5 we apply the theory to the Arnold conjecture on the $2n$-dimensional torus. Computation of E-cohomological Conley index for a Rabinowitz functional for arbitrary Hamiltonian on T^{2n} is reduced by continuation principle to computation for the trivial one. The most degenerate case turns out to be the easiest to compute. Then the degenerate Arnold conjecture follows from cup-length of the index (Proposition 3.4.4) while the non-degenerate version from the isomorphism of the index with the local Morse cohomology (Theorem 4.4.4).

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Chapter 2

E-cohomology - preliminaries

To avoid some technical difficulties with orientation issues, we assume that the coefficient ring of the cohomology groups is equal to $\mathbb{Z}_2$.
Let $\mathbb{H}$ be a real separable Hilbert space with an orthogonal splitting $\mathbb{H} = E^+ \oplus E^-$ such that each of E^+ and E^- is either infinite-dimensional or trivial. Endow E^+, E^- and $\mathbb{H}$ with a weak, strong and product topology respectively.

2.1 Special cases

We start by considering the case of either E^+ or E^- is trivial. Suppose that $E^- = 0$. Then the E-cohomology groups for a pair of locally compact subsets of $\mathbb{H}$ are defined as Alexander-Spanier cohomology groups with compact supports. The later will be denoted by H_c^*.

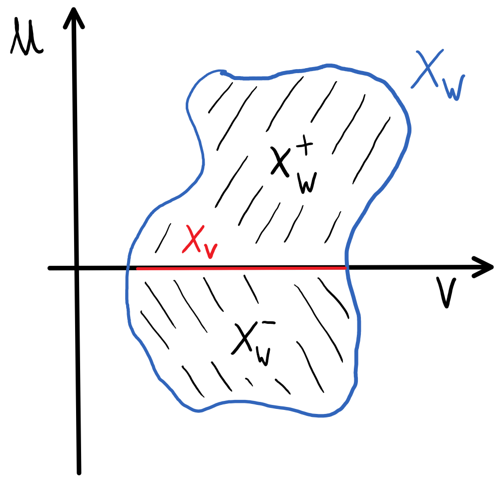
Remark. The reason why we use Alexander-Spanier cohomology is that it satisfies the strong excision axiom. Any other cohomology theory with that property would work as well.

The second case is more complicated. Suppose now that E^+ is trivial and E^- is infinite-dimensional. In that case, the definition of E-cohomology groups coincides with the definition of finite-codimensional cohomology groups introduced by K.Gęba and A.Granas in [GG73]. In fact, this definition lies at the heart of the most general case so we take a closer look on that.
Let V, U, W be finite-dimensional subspaces of E^- such that $W = V \oplus U$. Let us

start with $\dim U = 1$. Choose an orientation on U by picking a vector $u \in U$. Put

$$W^+ = \{w \in W | \langle w, u \rangle \geq 0\}$$
$$W^- = \{w \in W | \langle w, u \rangle \leq 0\}.$$

We define $X_W = X \cap W$, $X_W^+ = X \cap W^+$, $X_W^- = X \cap W^-$.



Note that $X_W = X_W^+ \cup X_W^-$ and $X_V = X_W^+ \cap X_W^-$ and therefore the Mayer-Vietoris sequence for the triad (X_W, X_W^+, X_W^-) reads

$$\ldots \longrightarrow H^k(X_W^+) \oplus H^k(X_W^-) \longrightarrow H^k(X_V) \xrightarrow{\Delta^k_{V,W}} H^{k+1}(X_W) \longrightarrow H^{k+1}(X_W^+) \oplus H^{k+1}(X_W^-) \longrightarrow \ldots \tag{2.1}$$

If $\dim U = n > 0$ then we split U into one dimensional subspaces and repeat the above construction n times to get $\Delta^k_{V,W}$. One can show that $\Delta^k_{V,W}$ does not depend on the factorization of U.

Definition 2.1.1. The *finite-codimensional cohomology groups* of a closed and bounded set X are defined by

$$H^{\infty - q}(X) = \varinjlim_{V \subset E^-, \dim V < \infty} \{H_c^{\dim V - q}(X_V); \Delta^q_{V,W}(X)\}$$

Example 2.1.2. Let T be a subspace of $E = E^-$ and let $X = S(T)$ be a sphere in T. If $\operatorname{codim} T = p < \infty$ then it is easy to see that

$$H^{\infty-q}(X) = \begin{cases} \mathbb{Z}_2 & \text{if q = p+1;} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, it is easily seen that if T is of infinite codimension then all *finite-codimensional cohomology groups* are trivial.

The extension of $H^{\infty-q}$ to the category of pairs is straightforward.

Finite-codimensional cohomology groups, as introduced above, are well described in [GD13, VI,18]. However, K.Gęba and A.Granas in their paper (see [GG73]) developed a more general case. They showed that one can take any generalized cohomology theory instead of the Alexander-Spanier cohomology. The analogous direct limit leads to the generalized cohomology theory on the Leray-Schauder category. Taking, for example, the cohomotopy groups in the definition leads to the *stable* cohomotopy theory. Morphisms in the Leray-Schauder category are compact fields, i.e. maps of the form $\mathbb{I} + K$, where K is compact. The set of morphisms seems to be too restricted. For example, translations are not compact fields but one expects that the resulting cohomology groups are translation-invariant. For applications to differential equations, one would also like to use flow deformations as morphisms.
The E-cohomology theory deals only with the ordinary cohomology theory rather than a generalized one. On the other hand, it generalize finite-codimensional cohomology groups in two directions. Firstly, both E^+ and E^- can be infinite-dimensional. As will be seen later, this allows us to apply the E-cohomology to the strongly indefinite problems. Secondly, A.Abbondandolo in [Abb97] introduces rather large class of E-morphisms with respect to which the E-cohomology groups are invariant. In particular, the class includes flow deformations.

2.2 Definition of E-cohomology

Let us now move to the general case, i.e. the case where both E^+ and E^- are infinite dimensional. Let $V' \in \mathcal{V} = \{V \subset E^- : \dim V < \infty\}$ and set $V := E^+ \oplus V'$. For a pair (X, A) of closed and bounded subsets of $\mathbb{H}$ define $\Delta^k_{V,W}(X, A)$ by the Mayer-Vietoris sequence as in the previous subsection (compare also [Abb97, p.336]).

Definition 2.2.1. The *E-cohomology groups* for a pair (X, A) are defined as

$$H^q_E(X, A) = \varinjlim_{V' \subset E^-, \dim V' < \infty} \{H_c^{q+\dim V'}(X_V, A_V); \Delta^q_{V,W}(X, A)\}$$

Example 2.2.2. Let T be a subspace of $\mathbb{H}$ such that both $\dim(T \cap E^+)$ and $\operatorname{codim}(T + E^+)$ are finite. Then the the E-dimension is defined as

$$\text{E-dim}\, T = \dim(T \cap E^+) - \operatorname{codim}(T + E^+).$$

Let X be the unit sphere in T. Then it is easily seen that

$$H^q_E(X) = \begin{cases} \mathbb{Z}_2 & \text{if } q = \text{E-dim}\, T - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let us make two remarks about the above example. Firstly, the groups H^q_E can be non-trivial for both positive and negative q. Secondly, the E-cohomology theory detects non-triviality of spheres which are neither finite-dimensional nor finite-codimensional. For that reason, one can think of the E-cohomology theory as of a version of a *middle-dimensional* cohomology. Note that a similar phenomenon occurs in Floer theory.

We would like to emphasize that the E-cohomology groups satisfy the Eilenberg-Steenrod axioms restricted set of morphisms. Let us be more precise about that statement.

Definition 2.2.3. A continuous map $\Psi : (X, A) \to (Y, B)$ is an *E-morphism* if:

1. it is of the form
$$\Psi(x) = Mx + K(x),$$
where M is a linear automorphism of $\mathbb{H}$ such that $ME^+ = E^+$ and K maps bounded sets into precompact sets.

2. $\Psi^{-1}(U)$ is bounded for every bounded set U.

Remark 2.2.4. In fact, the above class of morphisms can be extended to the class of continuous maps Ψ $(X, A) \to (Y, B)$ of the form $\Psi(x) = M(x)x + K(x)$, where K is as above, $M : X \to GL(\mathbb{H})$ has compact image and $M(x)E^+ = E^+$.

Having defined the E-morphisms, the E-homotopy is simply a homotopy in the category. To be more precise, the E-homotopy is a continuous map $\Psi : (X, A) \times [0.1] \to (Y, B)$ such that $\Psi(\cdot, t)$ is an E-morphism for every $t \in [0, 1]$. If Ψ is an E-homotopy, then we say that $\Psi(\cdot, 0)$ and $\Psi(\cdot, 1)$ are E-homotopic.
Moreover, H^*_E is a functor and therefore an E-morphism Ψ induces the map $H^*_E(\Psi) : H^*_E(Y, B) \to H^*_E(X, A)$. The following properties are satisfied (see [Abb97, Thm.0.2 and Thm.6.4]):

- (Homotopy invariance) If two E - morphisms Ψ and Ψ' are E - homotopic then $H^*_E(\Psi) = H^*_E(\Psi')$;
- (Strong excision) If X and Y are closed and bounded subsets of $\mathbb{H}$ and $i\colon (X, X \cap Y) \to (X \cup Y, Y)$ is the inclusion map then $H^*_E(i)$ is an isomorphism ;
- (Long exact sequence) For a triple $X \subset Y \subset Z$ of closed and bounded subsets of $\mathbb{H}$ the following sequence is exact:

$$\ldots \to H^k_E(Z, X) \to H^k_E(Y, X) \xrightarrow{\delta} H^{k+1}_E(Z, Y) \to H^{k+1}_E(Z, X) \to \ldots$$

Note that in [Abb97] the existence of a long exact sequence for a pair, rather than for a triple, is proven. However, the long exact sequence for a triple follows from the one for a pair by purely algebraic arguments (see [Hu66, pp.31-38]).

Another property of the E-cohomology, that we will need later on, is the continuity property. Let $\{(U^m, V^m)\}$ be a sequence of closed and bounded pairs in $\mathbb{H}$ such that:

1. $U^m \subset U^n$ and $V^m \subset U^n$ if $n \leq m$;
2. $\bigcap_{n\in\mathbb{N}} U^m = X$ and $\bigcap_{m\in\mathbb{N}} V^m = A$.

Proposition 2.2.5 (Prop.4.1 in [Abb97])**.** *The direct limit*

$$\varinjlim_{m\in\mathbb{N}} H^*_E(i^m) : \varinjlim_{m\in\mathbb{N}} \{H^*_E(U^m, V^m); H^*_E(j^{m,n})\} \to H^*_E(X, A)$$

is an isomorphism.

Finally, we observe that the deformation retract in the weak sense induces an isomorphism.

Lemma 2.2.6. *Let X be closed and bounded in $\mathbb{H}$ and let $j : Y \to X$ be the inclusion. Moreover, suppose that $H : [0,1] \times X \to X$ is such that*

1. $H(0, \cdot) = \mathbb{1}_X$;
2. $H(\lambda, \cdot)$ *is an E-morphism;*
3. $H(\lambda, Y) \subset Y$ *for all* $\lambda \in [0,1]$;
4. $H(1, X) \subset Y$.

Then $[H(1,\cdot)]^* : H_E^*(Y) \to H_E^*(X)$ *is an isomorphism and* $[H(1,\cdot)]^* = [j^*]^{-1}$.

Proof of the above lemma follows directly from the homotopy invariance of H_E^*.

2.3 Module structure

In this section we show that H_E^* possesses an additional module structure. Let X be a closed and bounded subset of $\mathbb{H}$ and let $V' \subset W'$ be finite-dimensional subspaces of E^-. Put $V = E^+ + V'$ and $W = E^+ + W'$. Denote by $i_{V,W}(X)$ the inclusion $X_V \hookrightarrow X_W$. Define the groups $H_0^p(X)$ by the inverse limit

$$H_0^p(X) = \varprojlim_{V' \subset E^-, \dim V' < \infty} \{H_c^p(X_V); i_{V,W}^*(X)\}.$$

Note that $H_0^*(X)$ is a ring.

Proposition 2.3.1. $H_E^*(X, A)$ *is a right module over* $H_0^*(X)$

Proof. Let $[\alpha_V]_0$, $[\beta_V]$ be the classes in $H_0^r(X)$ and $H^q(X, A)$ respectively. We will show that the formula

$$[\beta_V]_E \star [\alpha_V]_0 := [\beta_V \cup \alpha_V]_E$$

gives a well-defined multiplication. In other words, we have to check that

$$[\beta_V \cup \alpha_V]_E = [\beta_W \cup \alpha_W]_E$$

By putting $\beta_W = \Delta_{V,W}\beta_V$ and $\alpha_V = i_{V,W}^*\alpha_W$ we get

$$\Delta_{V,W}(\beta_V \cup i_{V,W}^*\alpha_W) = \Delta_{V,W}\beta_V \cup \alpha_W.$$

This equality follows from the naturality of the cup product and a compatibility with the suspension (compare [tD08, 17.2.1]). □

Suppose that Ω is closed and bounded and that $j : X \hookrightarrow \Omega$ is the inclusion.

Lemma 2.3.2. *j induces a ring homomorphism $j^* : H_0^*(\Omega) \to H_0^*(X)$.*

Proof. Put $j^*[\alpha_V]_0 := [j_V^*\alpha_V]_0$ where $j_V : X_V \hookrightarrow \Omega_V$ is the inclusion, and j_V^* is the induced homomorphism on the Alexander-Spanier cohomology. We will show that j^* is well defined, i.e. that $[j_V^*\alpha_V]_0 = [j_W^*\alpha_W]_0$. This is equivalent to $i_{V,W}^* j_W^* \alpha_W = j_V^*\alpha_V$. But $\alpha_V = k_{V,W}^*\alpha_W$, where $k_{V,W} : \Omega_V \hookrightarrow \Omega_W$ is the inclusion. Therefore we have $j_W \circ i_{V,W} = k_{V,W} \circ j_V$.
On the other hand,

$$j^*([\alpha_V]_0 \cup [\alpha'_V]_0) = j^*[\alpha_V \cup \alpha'_V]_0 = [j_V^*(\alpha_V \cup \alpha'_V)]_0 = \\ = [j_V^*\alpha_V \cup j_V^*\alpha'_V]_0 = [j_V^*\alpha_V]_0 \cup [j_v^*\alpha'_V]_0 = j^*[\alpha_V]_0 \cup j^*[\alpha'_V]_0 \quad (2.2)$$

and thus j^* is a ring homomorphism. $\square$

Corollary 2.3.3. *$H_E^*(X, A)$ is a right $H_0^*(\Omega)$-module.*

Definition 2.3.4. A *relative cup-length* $\mathrm{CL}(\Omega; X, A)$ is defined by

- $\mathrm{CL}(\Omega; X, A) := 0$ if $H_E^*(X, A) = 0$
- $\mathrm{CL}(\Omega; X, A) := 1$ if $H_E^*(X, A) \neq 0$ and $\alpha \star j^*\beta = 0$ for every $\alpha \in H_E^*(X, A)$ and $\beta \in H_0^{>0}(\Omega)$
- $\mathrm{CL}(\Omega; X, A) := k \geq 2$ if there exist $\alpha_0 \in H_E^*(X, A)$,$\beta_0 \in H_0^{>0}(\Omega)$ such that $\alpha_0 \star j^*\beta_0^{k-1} \neq 0$ and $\alpha \star j^*\beta^k = 0$ for all $\alpha \in H_E^*(X, A)$, $\beta \in H_0^{>0}(\Omega)$

Proofs of the following Lemmas are identical as for the finite-dimensional version of CL (see [DGU11, Lemma 2.2 and 2.3]).

Lemma 2.3.5. *If $B \subset A \subset X \subset Y$, then*

$$\mathrm{CL}(Y; X, B) \leq \mathrm{CL}(Y; X, A) + \mathrm{CL}(Y; A, B)$$

Lemma 2.3.6. *If $A \subset X \subset Y_1 \subset Y_2$, then*

$$\mathrm{CL}(Y_2; X, A) \leq \mathrm{CL}(Y_1; X, A)$$

Chapter 3

E-cohomological Conley index

3.1 Compactness conditions

The lack of compactness of $\mathbb{H}$ is compensated by a certain assumption on a flow. This assumptions guarantees that the Conley index is well defined and that it has expected properties.

Let $\phi : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ be a flow. If no confusion can arise, we write shortly $p \cdot t$ for $\phi(p, t)$. Let $x = \{x_n\}$ be a sequence in $\mathbb{H}$ and put

$$\delta(x) = \bigcup_n x_n \cdot [-n, n]; \quad \delta^+(x) = \bigcup_n x_n \cdot [0, n]; \delta^-(x) = \bigcup_n x_n \cdot [-n, 0];$$

Definition 3.1.1. We say that a sequence x is δ-bounded (δ^+-bounded, δ^--bounded) if $\delta(x)$ ($\delta^+(x)$,$\delta^-(x)$) is bounded in $\mathbb{H}$.

Definition 3.1.2. We say that a flow on a metric space satisfies the *compactness condition* (C) if any δ-bounded sequence contains a convergent subsequence.

Remark 3.1.3. Note that the analogous condition (C^+) "*any δ^+-bounded sequence contains a convergent subsequence*" is stronger than condition (C) . Such a condition was already used by K.Rybakowski and E.Zehnder in [RZ85]. However, condition (C) is satisfied for flows defined by strongly indefinite functionals (both E^+ and E^- infinite-dimensional), while the condition (C^+) might be not.

We will show that the condition (C) is satisfied for a large class of flows, namely those which are generated by compact perturbations of self-adjoint Fredholm operators. Let $L : E \to E$ be a bounded self-adjoint Fredholm operator.

Lemma 3.1.4. *(see [Wat15, Lemma 2.2]) Either* 0 *belongs to the resolvent of* L *or it is an isolated eigenvalue of finite multiplicity.*

Definition 3.1.5. A vector field $F = L + K : \mathbb{H} \to \mathbb{H}$ is called an $\mathcal{LS}$-vector field if L is a bounded self-adjoint Fredholm operator, K maps bounded sets onto precompact sets and is locally Lipschitz. Moreover we assume that there exists a basis of orthogonal eigenvectors of L. In particular, there is a sequence of finite-dimensional subspaces $\{E_n\}$ such that

1. $E_0 = \ker L$
2. $L(E_n) = E_n$ for all $n > 0$
3. $E_n \subset E_{n+1}$
4. $\overline{\bigcup_{n\in\mathbb{N}} E_n} = \mathbb{H}$

Such a sequence will be called an approximating scheme. Moreover, we say that a flow is an $\mathcal{LS}$-flow if it is uniformly continuous flow generated by an $\mathcal{LS}$-vector field.

Remark 3.1.6. Note that this is a slight abuse of notation. In [GIP99] authors give a more general definition of an $\mathcal{LS}$-vector field. Namely, L may not be self-adjoint, but then the assumption on the spectrum has to be made. We have chosen to work with a slightly smaller class of vector fields because the formulations of our theorems are simpler and the resulting category of flows is large enough for our applications.

The following Lemma is crucial to most of the proofs in that section.

Lemma 3.1.7. *An* $\mathcal{LS}$*-flow satisfies the compactness condition (C) .*

Proof. Write an $\mathcal{LS}$-flow in the form $\phi(x,t) = e^{tL}x + U(x,t)$. It is enough to show that sets $\{P^+x_n\}$ and $\{P^-x_n\}$ are compact. Suppose that $\{x_n^+\} = \{P^+x_n\}$ does not contain a convergent subsequence. Then, after going to a subsequence, we can find an $\epsilon > 0$ such that $|x_k^+ - x_l^+| > \epsilon$ whenever $k \neq l$. Set $X = \bigcup_{n\in\mathbb{N}} x_n \cdot [-n,n]$ is bounded, i.e. there is an $R > 0$ such that $X \subset B(R)$. Set $\delta = \frac{3R}{\epsilon}$. By Lemma 3.1.4 there exists $T > 0$ such that

$$|e^{TL}x| \geq \delta|x^+|$$

for every $x^+ \in E^+$. Then

$$3R \leq |e^{TL}(x_k - x_l)| \leq |x_k \cdot T| + |x_l \cdot T| + |U(x_k, T) - U(x_l, T)| \leq 2R + |U(x_k, T) - U(x_l, T)|$$

i.e.

$$R \leq |U(x_k, T) - U(x_l, T)|$$

but U is compact and we arrived at a contradiction. Analogously, we show that $\{x_n^-\}$ is compact. □

3.2 E-cohomological Conley index

Let U be a closed and bounded subset of $\mathbb{H}$ and let ϕ be a flow on $\mathbb{H}$. Denote by

$$\mathrm{Inv}\,(U, \phi) = \{x \in U : \phi(x, \mathbb{R}) \subset U\}$$

Definition 3.2.1. We say that a closed and bounded set U is an *isolating neighbourhood* for S if $S = \mathrm{Inv}\,(U, \phi) \subset \mathrm{Int}\, U$. On the other hand, set S is called an *isolated invariant set* if there exists an isolating neighbourhood for S.

Below fact follows directly from the definitions.

Fact 3.2.2. If a flow satisfies (C) then any isolated invariant set is compact.

Let S be an isolated invariant set.

Definition 3.2.3. A pair (N, L) of closed and bounded sets is an index pair for S if

1. $\overline{N \setminus L}$ is an isolating neighbourhood for S,
2. L is positively invariant with respect to N,
3. L is an exit set of N, i.e., if $x \in N$ and there exists $T > 0$ such that $x \cdot T \notin N$ then there exists $T' \in [0, T]$ such that $x \cdot [0, T'] \subset N$ and $x \cdot T' \in L$.

3.2.1 Existence of regular index pairs

Let us recall the existence of the index pairs in our setting. Following V.Benci (see [Ben91]), we define sets $G^T(U)$, $\Gamma(U, Y)$ by the formulas

$$G^T(U) = \{x \in U | x \cdot [-T, T] \subset U\}, \quad \Gamma^T(U, Y) = \{x \in G^T(U) | x \cdot [0, T] \cap Y \neq \emptyset\}$$

and we put

$$\Gamma^T(U) = \Gamma^T(U, \partial U).$$

Remark. Note that V.Benci uses only $G^T(U)$ and $\Gamma^T(U)$. We decided to introduce also relative Γ^T because it is more convenient while working with regular index pairs.

Lemma 3.2.4. *Let U be an isolating neighbourhood. If the flow satisfies (C) then there exists $T > 0$ such that $G^T(U) \subset \operatorname{Int} U$.*

Proof. Suppose the contrary. Then there exists a sequence $\{x_n\}$ such that $x_n \in G^n(U) \cap \partial U$. By (C) sequence $\{x_n\}$, converges up to subsequence, to $x_0 \in \partial U \cap S$. A contradiction. □

Theorem 3.2.5 (V.Benci)**.** *Let U be an isolating neighbourhood. If $G^T(U) \subset \operatorname{Int} U$ then $(G^T(U), \Gamma^T(U))$ is an index pair.*

Together with Lemma 3.2.4 we have:

Corollary 3.2.6. *If a flow satisfies (C) and U is an isolating neighbourhood then for T sufficiently large a pair $(G^T(U), \Gamma^T(U))$ is an index pair.*

It is often more convenient to work with a regular index pair rather than with an arbitrary one.

Definition 3.2.7. We say that an index pair (N, L) is *regular* if the function $\tau_{N,L} : N \to [0, \infty]$

$$\tau_{N,L}(x) = \sup\{t \in \mathbb{R}^{\geq 0} : x \cdot [0, t] \subset N \setminus L\}$$

is continuous. We drop the subscript when confusion is unlikely.

Any index pair (N, L) can be regularized in the sense that there exists a regular index pair (N, L') of the same isolated invariant set. The idea of regularizing index pairs goes back to Ch.Conley and the detailed proof can be found in Salamon's paper ([Sal85, pp.20-22]). We sketch the construction because it allows us to make further conclusions while dealing with $\mathcal{LS}$-flows .

Lemma 3.2.8. *Let (N, L) be an index pair for the isolated invariant set S for a local flow in a metric space. Then there exists a continuous Lyapunov function $g : N \to [0, 1]$ such that*

1. $g(x) = 1 \iff x \cdot \mathbb{R}^+ \subset N$ *and* $\omega(x) \subset S$;
2. $g(x) = 0 \iff x \in L$;
3. $t > 0, 0 < g(x) < 1, x \cdot [0, t] \subset N \Rightarrow g(x \cdot t) < g(x)$.

We briefly show how a Lyapunov function is constructed. Define the function $l : N \to [0, 1]$ by

$$l(x) = \frac{d(x, L)}{d(x, L) + d(x, S)}.$$

Clearly l is continuous and satisfies $L = l^{-1}(0)$ and $S = l^{-1}(1)$. Now, using l, we define the second function $k : N \to [0, 1]$ which takes into account the flow lines:

$$k(x) = \sup\{l(x \cdot t) | t \geq 0, x \cdot [0, t] \subset N\}.$$

k already satisfies (1),(2) and

$$t > 0, \quad 0 < g(x) < 1, \quad x \cdot [0, t] \subset N \Rightarrow g(x \cdot t) \leq g(x).$$

For that reason we have to do last modification to get a strict inequality in the last condition. This is done by averaging i.e. define $g : N \to [0, 1]$ by

$$g(x) = \int_0^{t(x)} e^{-\xi} k(x \cdot \xi)\, d\xi,$$

where $t(x) = \sup\{t \geq 0 | x \cdot [0, t] \subset N \setminus L\}$. For the proof of the continuity of g we refer to [Sal85, pp.20-22].

Corollary 3.2.9. *For a given index pair (N, L), a Lyapunov function $g : N \to [0, 1]$ satisfying (1) - (3)) and $\epsilon \in (0, 1)$ define $L_\epsilon = \{x \in N | g(x) \leq \epsilon\}$. From (3) it is easily seen that (N, L_ϵ) is a regular index pair.*

For flows generated by bounded vector fields, in particular for $\mathcal{LS}$-flows, we can make further conclusions.

Proposition 3.2.10. *Let (N, L) be an index pair for the flow generated by a vector field F. Suppose that F is bounded on N. Then for every $\rho > 0$ there exists a regular index pair (N, L') such that $d(L', L) < \rho$.*

Proof. Functions l,k,$t(x)$ and g are defined in the sketch of construction of the Lyapunov function in Lemma 3.2.8. In fact, we will show that for a given $\rho > 0$ there exists $\epsilon > 0$ such that the pair (N, L_ϵ) (see Corollary 3.2.9) satisfies the claimed condition. Suppose the contrary, i.e. that we have a sequence $\{x_n\} \subset N$ such that $g(x_n) \le \frac{1}{n}$ and $d(x_n, L) \ge \rho$. Put $c = \sup_{x \in N} ||F(x)||$. Then

$$\rho \le d(x_n, L) \le \int_0^{t(x_n)} ||F(x_n \cdot \xi)||\, d\xi \le t(x_n) \cdot c$$

so $t(x_n) > 2c_1 = \frac{\rho}{c}$. On the other hand

$$\frac{1}{n} \ge g(x_n) = \int_0^{t(x_n)} e^{-\xi} k(x_n \cdot \xi)\, d\xi \ge \int_0^{c_1} e^{-\xi} k(x_n \cdot \xi)\, d\xi \ge$$
$$\ge e^{-c_1} c_1 k(x_n \cdot c_1) \ge e^{-c_1} c_1 l(x_n \cdot c_1) \ge e^{-c_1} c_1 c_2 d(x_n \cdot c_1, L) \quad (3.1)$$

where $\frac{1}{2c_2} = \sup_{x \in N} ||x||$. Moreover,

$$d(x_n \cdot c_1, L) \ge d(x_n, L) - \int_0^{c_1} ||F(x_n \cdot \xi)||\, d\xi \ge \frac{\rho}{2}.$$

and we arrived at a contradiction. $\square$

Note that $\mathcal{LS}$-vector fields are bounded on bounded sets. In particular, the statement of the above Proposition is true for $\mathcal{LS}$-flows.

Remark 3.2.11. This is exactly where the definition of relative Γ^T is useful. Having any regular index pair (N, L) we have a whole family of regular index pairs by $(G^T(N), \Gamma^T(N, L))$.

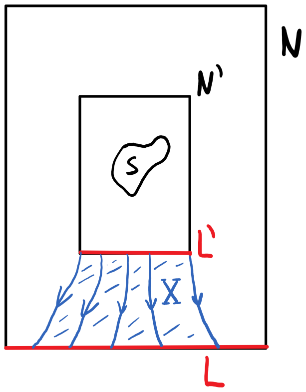
3.2.2 Definition of the Conley index

Lemma 3.2.12. *Let ϕ be an $\mathcal{LS}$-flow and let (N, L) and (N', L') be index pairs for the isolated invariant set S. Suppose that $N' \subset N \setminus L$ and that (N, L) is regular. Define a map $f_T : N' \to N$ by*

$$f_T(x) = \begin{cases} x \cdot T & \text{if } x \cdot [0, T] \subset N; \\ x \cdot \tau_{N,L}(x) & \text{otherwise} \end{cases}$$

If $f_{T'}(L') \subset L$ then there exists $T'' \ge T'$ such that $[f_T(\cdot)]^* : H^*_E(N, L) \to H^*_E(N', L')$ is an isomorphism for all $T > T''$.

Proof. Put $X = L' \cdot [0, \infty) \cap N$.
Let $i : (N', L') \hookrightarrow (N' \cup X, X)$ and $j : (N' \cup X \cup L, L) \hookrightarrow (N, L)$ be the inclusions.



Define $h_T : [0,1] \times (N' \cup X) \to (N' \cup X)$ by extending the formula for $f_{\lambda T}$, namely

$$h_T(\lambda, x) = \begin{cases} x \cdot \lambda T & \text{if } x \cdot [0,T] \subset N; \\ x \cdot \lambda \tau_{N,L}(x) & \text{otherwise} \end{cases}$$

for $T \geq T'$. Note that

$$f_T = j \circ h_T(1, \cdot) \circ i$$

so it is enough to show that i,j and $h_T(1, \cdot)$ induce isomorpisms. For i^* the conclusion follows from the strong excision axiom. To see that $h_T(1, \cdot)$ induces an ismorphism, observe that $h_T(\cdot, \cdot)$ restrcited to X is a deformation retract in the weak sense onto L and the statement follows from Lemma 3.2.4. Finally, define $\hat{h}_T : [0,1] \times N \to N$ by the same formula as h_T. Then there exists $T \geq T'$ such that $\hat{h}_T(1, N) \subset N' \cup X$. To see this suppose that there exists a sequence $\{x_n\}$ such that $x_n \cdot [0, 2n] \subset N \setminus (N' \cup X)$. Then $y_n := x_n \cdot n$ converges along a subsequence to $y_0 \in S$. But $S \subset \operatorname{Int} N'$ and we arrived at a contradiction. Therefore $\hat{h}_T(1, N) \subset N' \cup X$ and by the Lemma 3.2.4, $\hat{h}_T$ induces an isomorphisms with j^* as its inverse.

□

Proposition 3.2.13. *Let S be an isolated invariant set for an $\mathcal{LS}$-flow. If (N, L), (N', L') are two index pairs for S then*

$$H^*_E(N, L) \simeq H^*_E(N', L').$$

Proof. The proof will be divided into two steps.

Step (1) Suppose that both (N, L) and (N', L') are regular. Put $U = \overline{(N \setminus L) \cap (N' \setminus L')}$. Then by Lemma 3.2.4 and Theorem 3.2.5 pair $(G^T(U), \Gamma^T(U))$ is an index pair. Moreover, it is contained in both $N \setminus L$ and $N' \setminus L'$. By Lemma 3.2.12,

$$H_E^*(N, L) \simeq H_E^*(G^T, \Gamma^T) \simeq H_E^*(N', L').$$

Step (2) Suppose that (N, L) is not regular. Then by defining L_ϵ as in the Corollary 3.2.9 we get a sequence of regular index pairs $(N, L_{\frac{1}{n+1}})$. By Proposition 2.2.5

$$\varinjlim_{n \in \mathbb{N}} \{H_E^*(N, L_{\frac{1}{n+1}}); H_E^*(j^n)\} \to H_E^*(N, L)$$

is an isomorphism. By Step 1, $H_E^*(j^n)$ is an isomorphism for every n. Therefore $H_E^*(N, L) \simeq H_E^*(N, L_{\frac{1}{n+1}})$ i.e. $H_E^*(N, L)$ is isomorphic to E-cohomology of some regular index pair. We follow the same argument if (N', L') is not regular.

□

Definition 3.2.14. Let S be an isolated invariant set of an $\mathcal{LS}$-flow ϕ and let (N, L) be an index pair for S. The *E-cohomological Conley index* is defined by

$$\mathrm{ch_E}(S, \phi) = H_E^*(N, L).$$

We also write $\mathrm{ch_E}(U, \phi)$ where U is an isolating neighbourhood of S and drop ϕ from the notation if no confusion can arise.

Example 3.2.15. Suppose that $F = \nabla f$ is a gradient vector field and that x is a non-degenerate critical point. Let W_x be the negative eigenspace of $d^2 f(x)$. Define the Morse index of x relative to the splitting by

$$m_E(x) = \text{E-dim}\, W_x$$

Note that if both E^+ and E^- are infinite dimensional then the ordinary Morse index and Morse co-index are infinite. As in the finite dimensional Conley index theory, we have the following result.

$$\mathrm{ch_E}^k(\{x\}) = \tilde{H}^k(S^{m_E(x)}) = \begin{cases} \mathbb{Z}_2, & \text{if } k = m_E(x); \\ 0, & \text{otherwise.} \end{cases}$$

One of the crucial properties of the Conley index is the non-triviality. Here is a precise statement.

Proposition 3.2.16 (Non-triviality)**.** *Let U be an isolating neighbourhood and $S = \operatorname{Inv} U$. If $\mathrm{ch}_{\mathrm{E}}(S) \neq 0$ then $S \neq \emptyset$.*

Proof. Suppose that $S = \emptyset$ and suppose that $G^T(U)$ is non-empty for every $T > 0$ and take a sequence $\{x_n\}$ such that $x_n \in G^n(U)$. Then by compactness condition (C) and continuity of the flow x_n converges to a point $x_0 \in S$. Contradiction proves that G^T is empty for sufficiently large T. But then $\mathrm{ch}_{\mathrm{E}}(S) = H_E^*(\emptyset, \emptyset) = 0$. □

3.3 Continuation principle

Another important property of the Conley index is the homotopy invariance, usually referred to as the continuation principle. The proof of the homotopy invariance is rather technical and we decided to devote separate paragraph to that. For the classical Conley index there are two methodologically different proofs known to the author. First one was orginally given by Conley (see also D.Salamon ([Sal85]) or J.Smoller ([Smo12])). The second one was proposed by V.Benci ([Ben91]). Since Benci's proof works for locally non-compact spaces, it uses different techniques.
We follow Benci's ideas. However, there are some technical difficulties we have to overcome. The classical Conley index is a homotopy type of a quotient space. Therefore, to show that it is invariant under homotopy one has to find homotopy equivalence. In our setting we cannot work with quotient spaces, since they are not subsets of the underlying Hilbert space anymore. For that reason, the map introduced by Benci does not work in our setting, despite the fact it works for the classical Conley index on spaces which are not locally compact.

Lemma 3.3.1. *Suppose that ϕ satisfies compactness property (C) . Let U be an isolating neighbourhood of S. Then there exists $\rho > 0$ such that the tubular neighbourhood $U_\rho := N_\rho U$ is an isolating neighbourhood of S.*

Proof. Suppose, contrary to our claim, that there is a sequence

$$x_n \in (U_{\frac{1}{n}} \setminus U) \cap \operatorname{Inv} U_{\frac{1}{n}}.$$

By the compactness property (C) , x_n converges, up to a subsequence, to $x_0 \in \partial U \cap S = \emptyset$. A contradiction. □

Lemma 3.3.2. *Suppose that ϕ satisfies compactness property (C) . Let U and U_ρ be the isolating neighbourhoods as in Lemma 3.3.1. Take $T > 0$ such that $G^T_\phi(U_\rho) \subset \operatorname{Int} U$. If a flow ψ satisfies*

$$\forall_{t\in T, x\in U_\rho} ||\phi(t,x) - \psi(t,x)|| < \rho \tag{3.2}$$

then

1. *U is an isolating neighbourhood for ψ;*
2. *ψ satisfies property (C) on U.*

Proof. 1. Observe that by (3.2)

$$G^T_\psi(U) \subset G^T_\phi(U_\rho)$$

and therefore we have $\operatorname{Inv}(U,\psi) \subset G^T_\psi(U) \subset G^T_\phi(U_\rho) \subset \operatorname{Int} U$, which proves the assertion.

2. Take a sequence $\{x_n\}$ such that $x_n \in G^n_\psi(U)$. Since $G^n_\psi(U) \subset G^n_\phi(U_\rho)$ and ϕ satisfies (C) , $\{x_n\}$ contains a convergent subsequence.

□

Remark. The second statement of Lemma 3.3.2 will not be used in the subsequent arguments. However, we wanted to emphasise that the property (C) is open, i.e. if a flow satisfies (C) then all close (in a sense of the equation 3.2) flows also satisfy that property.

Theorem 3.3.3. *Let ϕ be an $\mathcal{LS}$-flow and let U be an isolating neighbourhood. Choose $\rho > 0$ as in Lemma 3.3.1 and $T > 0$ as in Lemma 3.3.2. If an $\mathcal{LS}$-flow ψ satisfies*

$$||\psi(t,x) - \phi(t,x)|| < \frac{\rho}{4} \tag{3.3}$$

for all $x \in U_\rho$ and $t \in [-T, T]$, then

$$\operatorname{ch}_{\mathrm{E}}(\psi, U) = \operatorname{ch}_{\mathrm{E}}(\phi, U).$$

Before giving a proof we state the corollary.

Corollary 3.3.4 (Continuation Principle). *Let $\{H_\lambda : \lambda \in [0,1]\}$ be a continuous family of $\mathcal{LS}$-flows such that for every $\lambda \in [0,1]$ set U is an isolating neighbourhood for H_λ. Then*

$$\mathrm{ch}_{\mathrm{E}}(H_0, U) = \mathrm{ch}_{\mathrm{E}}(H_1, U)$$

Proof of Theorem 3.3.3. Directly by definition of G^T and 3.3 we have the following inclusions

$$G^T_\psi(U_{\frac{\rho}{4}}) \subset G^T_\phi(U_{\frac{2\rho}{4}}) \subset G^T_\psi(U_{\frac{3\rho}{4}}) \subset G^T_\phi(U_\rho) \subset \operatorname{Int} U$$

By uniform continuity we can take ϵ such that if $d(x,y) < \epsilon$ then

$$d(\phi(t,x),\phi(t,y)) < \frac{\rho}{4} \text{ and } d(\psi(t,x),\psi(t,y)) < \frac{\rho}{4}$$

for all $t \in [-T,T]$. Denote by (N_1,L_1), (N_2,L_2), $(\tilde{N}_1,\tilde{L}_1)$, $(\tilde{N}_2,\tilde{L}_2)$ regularizations of index pairs $(G^T_\psi(U_{\frac{\rho}{4}}),\Gamma^T_\psi(U_{\frac{\rho}{4}}))$, $(G^T_\psi(U_{\frac{3\rho}{4}}),\Gamma^T_\psi(U_{\frac{3\rho}{4}}))$, $(G^T_\phi(U_{\frac{2\rho}{4}}),\Gamma^T_\phi(U_{\frac{2\rho}{4}}))$, $(G^T_\phi(U_\rho),\Gamma^T_\phi(U_\rho))$ such that the distance of L and the respective Γ is less than ϵ (see Proposition 3.2.10). We will now define maps f_1,f_2,ξ_λ, $\tilde{\xi}_\lambda$ and g as in the following diagram

$$\begin{tikzcd}
(N_1,L_1) \arrow[r, "f_1"] \arrow[dr, "\xi"'] & (\tilde{N}_1,\tilde{L}_1) \arrow[d, "g"] \arrow[dr, "\tilde{\xi}"] & \\
& (N_2,L_2) \arrow[r, "f_2"] & (\tilde{N}_2,\tilde{L}_2)
\end{tikzcd} \tag{3.4}$$

Put

$$\begin{aligned}
f_1(x) :=& \psi(\nu_{\tilde{N}_1,\tilde{L}_1}(x),x)\\
f_2(x) :=& \psi(\nu_{\tilde{N}_2,\tilde{L}_2}(x),x)\\
g(x) =:& \phi(\nu_{N_2,L_2}(x),x)\\
\xi(\lambda,x) :=& \phi(\nu_{N_2,L_2},(\psi(\lambda\nu_{\tilde{N}_1,\tilde{L}_1}(x),x))\\
\tilde{\xi}(\lambda,x) :=& \psi(\nu_{\tilde{N}_2,\tilde{L}_2},(\phi(\lambda\nu_{N_2,L_2}(x),x))
\end{aligned}$$

where $\nu_{N,L}(x) := \min\{T, \tau_{N,L}(x)\}$. We now prove that the above maps are indeed maps between pairs, as shown on the diagram above. If $x \in L_1$ then there exists $y \in \Gamma^T_\psi(U_{\frac{\rho}{4}})$ such that $d(x,y) < \epsilon$. Therefore $d(\psi(t,x),\partial U_{\frac{\rho}{4}}) \leq d(\psi(t,x),\psi(x,t) < \frac{\rho}{4}$ and $x \notin \tilde{N}_1 \subset \operatorname{Int} U$. Similar arguments work for other maps.

By Lemma 3.2.12, $\xi(0,\cdot)$ and $\tilde{\xi}(0,\cdot)$ induce isomorphisms on E-cohomology level and, by homotopy invariance, so do $\xi(\lambda,\cdot)$ and $\tilde{\xi}(\lambda,\cdot)$. On the other hand,

$\xi(1,\cdot) = g \circ f_1$ and $\xi(1,\cdot) = f_2 \circ g$. Map $g^* : H^*_E(N_2, L_2) \to H^*_E(\tilde{N}_1, \tilde{L}_1)$ is an epimorphism because $g^* \circ f_2^*$ is an isomorphism and g^* is monomorphisms because $f_1 \circ g^*$ is an isomorphism. Therefore

$$H^*_E(\tilde{N}_1, \tilde{L}_1) = H^*_E(N_2, L_2)$$

which is the desired conclusion.

□

3.4 Cup-length

Module structure defined for E-cohomology groups allows us to define cup-length for E-cohomological Conley index. To see this, note that the following Proposition holds.

Proposition 3.4.1. *Let U be an isolating neighbourhood for an $\mathcal{LS}$-flow ϕ and let (N, L), $(\tilde{N}, \tilde{L})$ be two index pairs for* $\mathrm{Inv}(U, \phi)$. *Then*

$$\mathrm{CL}(U; N, L) = \mathrm{CL}(U; \tilde{N}, \tilde{L})$$

Proof. Follow the arguments in the proof of Proposition 3.2.13 and notice that the group morphisms introduced there are in fact morphisms of modules. □

Definition 3.4.2. Let U be an isolated invariant set for an $\mathcal{LS}$-flow ϕ. An E-cohomological cup-length is defined as

$$\mathrm{CL}(U, \phi) = \mathrm{CL}(U; N, L)$$

where (N, L) is an index pair for $S = \mathrm{Inv}(U, \phi)$.

Again, if no confusion can arise, we may also write $\mathrm{CL}(U)$ or $\mathrm{CL}(S)$ for $\mathrm{CL}(U, \phi)$.

Proposition 3.4.3. *Let ϕ_0 and ϕ_1 be related by continuation in U. Then*

$$\mathrm{CL}(U, \phi_0) = \mathrm{CL}(U, \phi_1).$$

Proof. Follow the arguments in the proof of Theorem 3.3.3 and notice that the group morphisms introduced there are in fact morphisms of modules. □

Usefulness of the cup-length is provided by the following proposition.

Proposition 3.4.4. *Let ϕ be an $\mathcal{LS}$-flow generated by a gradient of a function f. Then*

$$\#\mathrm{Crit}(f, U) \geq \mathrm{CL}(U, \phi)$$

where $\mathrm{Crit}(f, U)$ *is the set of critical values of f in U.*

The following has an almost identical proof to that of analogous proposition in the finite-dimensional case (see [DGU11, Thm 4.1]). However we give a sketch for the sake of completeness.

Proof. Suppose $\mathrm{Crit}(f, U) = \{c_1 < c_2 < \ldots < c_k\}$. We start with a Morse filtration. Let M_i be a set of critical points with value c_i and put

$$M_{i,j} := \{x \in S : \omega(x) \cup \alpha(x) \subset M_i \cup M_{i+1} \cup \ldots \cup M_j\}; \quad M_i = M_{i,i}$$

where $S = \operatorname{Inv} U$, and $\alpha(x)$, $\omega(x)$ are α and ω-limits respectively. Let (N_k, N_0) be and index pair for S.
Take regular values $b_l \in (c_l, c_{l+1})$ and put $N_i = N_k \cap f^{-1}(-\infty, b_i)$. Then (N_i, N_{j-1}) is an index pair for $M_{i,j}$. By Lemma 2.3.5 we have

$$\mathrm{CL}(U; N_i, N_0) \leq \mathrm{CL}(U; N_{i-1}, N_0) + \mathrm{CL}(U; N_i, N_{i-1}).$$

On the other hand, by Lemma 2.3.6

$$\mathrm{CL}(U; N_i, N_{i-1}) \leq 1$$

and therefore

$$\mathrm{CL}(U, \phi) \leq k.$$

□

Chapter 4

Finite-dimensional reduction

Definition 4.0.5. Let V be a subspace of $\mathbb{H}$ such that $L(V) \subset V$ and let P_V be the orthogonal projection onto V. We say that an $\mathcal{LS}$-vector field $F = L + K$ is an $\mathcal{LS}_V$-vector field if

$$K = P_V K P_V.$$

An $\mathcal{LS}$-flow is an $\mathcal{LS}_V$-flow if it is generated by an $\mathcal{LS}_V$-vector field.

An $\mathcal{LS}_V$-flow ϕ is a product flow on $V \times V^{\perp}$. Denote by ϕ_V and $\phi_{\perp}$ flows on V and $V^{\perp}$ respectively. $\phi_{\perp}$ is a linear flow generated by L and therefore any isolated invariant set of ϕ is contained in V.

Definition 4.0.6. A flow is an $\mathcal{LS}_{\text{fin}}$-flow if it is an $\mathcal{LS}_V$-flow for some finite-dimensional V.

4.1 Continuation to an $\mathcal{LS}_{\text{fin}}$-flow

In this section we consider an $\mathcal{LS}$-flow ϕ_0 and use a series of continuations to end up with an $\mathcal{LS}_V$-flow for some finite-dimensional V. This will allow us to compare infinite-dimensional on E-cohomological Conley index and Morse cohomology on $\mathbb{H}$ to each other and to their classical versions on V.

Proposition 4.1.1. *Let $F_0 = L + K_0$ be a Lipschitz continuous $\mathcal{LS}$-vector field and let U be an isolated neighbourhood for an induced $\mathcal{LS}$-flow ϕ_0. There exists $\epsilon > 0$ such that if* *an $\mathcal{LS}$-vector field $F_1 = L + K_1$ satisfies*

$$\forall_{x \in U} \, ||K_0(x) - K_1(x)|| < \epsilon$$

then the flows (ϕ_0, U) and (ϕ_1, U) are related by continuation.

Before giving a proof we comment on the assumption.

Remark. Suppose that an $\mathcal{LS}$-vector field F is locally Lipschitz rather than Lipschitz continuous on an isolating neighbourhood U. Then there exists another isolating neighbourhood U' for $S = \operatorname{Inv} U$ such that F is Lipschitz continuous on U'. To see this, note that S is compact and there exists a cover of S by the open balls $\{B_i\}$ in $\mathbb{H}$ such that F is Lipschitz continuous on every B_i. Sum of B_i gives a desired neighbourhood.

proof of Proposition 4.1.1. Fix $p \in U$ and let x_i, $i = 0, 1$, be the solutions to $\dot{x}_i(s) = F_i(x_i(s))$ with $x_i(0) = p$.

$$\begin{aligned} ||x_0(t) - x_1(t)|| = \left| \int_0^t [F_0(x_0(s)) - F_1(x_1(s))] \, ds \right| \leq \\ \leq \int_0^t ||F_0(x_0(s)) - F_0(x_1(s))|| + ||F_0(x_1(s)) - F_1(x_1(s))|| \, ds \leq \\ \leq \int_0^t (c \, ||x_0(s) - x_1(s)|| + \epsilon) \, ds. \end{aligned}$$

By Gronwall's inequality we get

$$||x_0(t) - x_1(t)|| \leq \epsilon T e^{cT}.$$

We can choose ρ, T and ϵ such that (see Lemma 3.3.2 and Lemma 3.2.4)

1. $\operatorname{Inv}(\phi_0, U_\rho) = \operatorname{Inv}(\phi_0, U)$;
2. $G^T_{\phi_0}(U_\rho) \subset \operatorname{Int} U$;
3. $\epsilon T e^{cT} < \frac{\rho}{2}$.

Then U is also an isolating neighbourhood for an $\mathcal{LS}$-flow ϕ_s induced by

$$F_s := L + (1 - s) K_0(\cdot) + s K_1(\cdot).$$

This is due to the following inclusions

$$\operatorname{Inv}(\phi_s, U) \subset G^T_{\phi_s}(U) \subset G^T_{\phi_1}(U_{\frac{\rho}{2}}) \subset G^T_{\phi_0}(U_\rho) \subset \operatorname{Int} U.$$

$\square$

Directly by Proposition 4.1.1 we have the following.

Corollary 4.1.2. *Let $F = L + K$ be a Lipschitz continuous $\mathcal{LS}$-vector field, ϕ_0 an $\mathcal{LS}$-flow generated by F and U an isolating neighbourhood for ϕ_0. Moreover, let $\{E_n\}$ be an approximating scheme for L (see Definition 3.1.5) and let P_n be an orthogonal projection onto E_n. Then, for n sufficiently large, ϕ_0 is related by continuation on U to the flow generated by the vector field $F_n = L + P_n K$.*

Note that from Proposition 4.1.1 we cannot conclude that an $\mathcal{LS}$-flow generated by $L + K$ can be continued to a flow generated by $L + P_n K P_n$. To see this, pick a nonzero vector $v \in \mathbb{H}$ and put $K(x) = ||x||\, v$. Then on a unit ball $||K - P_n K P_n||_{sup} = 1$.
To show that such flows are actually related by continuation we adopt different technique which go back to [GIP99, Lemma 4.1].

Lemma 4.1.3. *(see [Mau91, pp.241-243]) Let Ω be open subset of a Banach space X and let V be an open subset of Banach space Y. Moreover, suppose that $f : \Omega \times V \to X$ is continuous, has derivative in X direction and that the map*

$$U \times V \to X \ni (x, y) \mapsto f'_X(x, y) \in L(X, X)$$

is continuous.
Then f induces a continuous family of local flows.

Proposition 4.1.4. *Let $F = L + K$ be a continuously differentiable $\mathcal{LS}$-vector field and let U be an isolating neighbourhood. Then, for n sufficiently large, flows generated by F and $F_n = L + P_n K P_n$ are related by continuation on U.*

Proof. By Corollary 4.1.2 we can assume that $P_n K = K$. Define $H : [0, 1] \times U \to \mathbb{H}$ by $H(\lambda, \cdot) = L + K((1 + n)(1 - n\lambda)P_{n+1} + n[(n + 1)\lambda - 1]P_n)$ for $\lambda \in (\frac{1}{n+1}, \frac{1}{n}]$ and $H(0, \cdot) = F$. If H induces a continuous family of local flows then there exists $s > 0$ such that U is an isolating neighbourhood for $H(\lambda, \cdot)$ provided $\lambda \in [0, s)$. If this is the case, it is enough to take n such that $\frac{1}{n} < s$.

To show that H indeed induces a continuous family of local flows we will check that H satisfies the assumptions of the Proposition 4.1.3. Therefore we examine continuity of the map $(\lambda, x) \mapsto D_X H(\lambda, x)$ is continuous. The only nontrivial case is to check $D_X H(\lambda_n, x_n) \to D_X H(0, x_0)$ when $(x_n, \lambda_n) \to (x_0, 0)$. It is enough to compute the limit with $\lambda_n = \frac{1}{n}$. We have

$$
\begin{aligned}
||D_X H(\lambda_n, x_n) - D_X H(0, x_0)|| &= ||DK(P_n(x_n))P_n - DK(x_0)|| = \\
&= ||DK(P_n(x_n))P_n - DK(x_0)P_n + DK(x_0)P_n - DK(x_0)|| \leq \\
&\leq ||DK(P_n(x_n))P_n - DK(x_0)P_n|| + ||DK(x_0)P_n - DK(x_0)|| \leq \\
&\leq ||DK(P_n(x_n)) - DK(x_0)|| + ||DK(x_0)P_n - DK(x_0)||
\end{aligned}
$$

$P_n(x_n)$ converges to x_0 so $||DK(P_n(x_n)) - DK(x_0)||$ converges to 0 by the assumption that K is C^1. On the other hand,

$$
\begin{aligned}
||DK(x_0)P_n - DK(x_0)|| &= ||[DK(x_0)P_n - DK(x_0)]^*|| = \\
&= ||P_n[DK(x_0)]^* - [DK(x_0)]^*|| \to 0
\end{aligned}
$$

since the adjoint $[DK(x_0)]^*$ is compact.

□

Corollary 4.1.5. *Let ϕ be a flow generated by a continuously differentiable $\mathcal{LS}$-vector field and let U be an isolating neighbourhood for ϕ. Then ϕ is related by continuation to an $\mathcal{LS}_{fin}$-flow.*

4.2 Comparison with $\mathcal{LS}$-index

Suppose we have an $\mathcal{LS}$-flow on $\mathbb{H}$ and an isolated invariant set U. Then we can take an index pair in U and consider intersections of this pair with finite-dimensional subspaces to get the E-cohomological Conley index. On the other hand, as shown in [GIP99], we can take projections of the underlying $\mathcal{LS}$-vector field and compute classical Conley indices. This paragraph is devoted to show that those two approaches are equivalent. Ideologically, one could say that *it does not matter if we look at finite-dimensional reductions of an index pair or an index pair of a finite-dimensional reduction of the flow.*

Let ϕ be an $\mathcal{LS}$-flow, U be an isolating neighbourhood and $\{E_n\}$ an approximating scheme. For sufficiently large n set $U_n := U \cap E^n$ is an isolating neighbourhood for the flow generated by the field $L + P_nKP_n$. Therefore we get a sequence of Conley indices:

$$h_n = h(U_n, \phi_n) := [X_n, Y_n],$$

where (X_n, Y_n) is an index pair for a finite-dimensional flow $L + P_nKP_n$ and isolated neighbourhood U_n.

Theorem 4.2.1. *[GIP99, p.224] Sequence of homotopy types h_n stabilizes in the following sense. For sufficiently large n*

$$h_{n+k} = S^{\nu(n+k)-\nu(k)}h_n$$

where $\nu(m) = \dim(E_m \cap E^-)$.

As a Corollary, following cohomology groups are well defined.

Definition 4.2.2 (compare [Izy01])**.** The $\mathcal{LS}$-cohmological index is defined by

$$\mathrm{ch}^*_{\mathcal{LS}}(U) = \tilde{H}^{*-\nu(n)}(X_n, Y_n)$$

where n is sufficiently large.

Remark. Above definition slightly differs from the one introduced in [Izy01]. Izydorek defines $\mathrm{ch}_{\mathcal{LS}}$ as a limit with suspensions being the morphisms. However, those morphisms are isomorphisms for n sufficiently large.

Proposition 4.2.3. *Let ϕ be an $\mathcal{LS}$-flow and let U be an isolating neighbourhood. Then the $\mathcal{LS}$-cohomological Conley index and the E-cohomological Conley index are isomorphic, i.e.*

$$\mathrm{ch}^*_{\mathcal{LS}}(U, \phi) = \mathrm{ch_E}^*(U, \phi)$$

Proof. By Corollary 4.1.5 ϕ can be continued to an $\mathcal{LS}_{\mathrm{fin}}$ flow $(\phi_V, \phi_\perp)$. Let (X, Y) be an index pair for the flow ϕ_V in the isolating neighbourhood $U \cap V$. Denote by D^+ (D^-) a unit ball in $V^\perp \cap E^+$ ($V^\perp \cap E_+$). Clearly,

$$(N, L) = (X \times D^+ \times D^-, Y \times D^+ \times D^- \cup X \times D^+ \times \partial D^-)$$

is an index pair for the flow $(\phi_V, \phi_\perp)$. On the other hand,

$$H^*_E(N, L) = \tilde{H}^*(X, Y)$$

and the conclusion follows. $\square$

4.3 Continuation to a gradient

In [Rei90] J.F.Reineck proved that in the case of finite dimensional manifolds any flow generated by a vector field can be continued to the flow generated by a gradient vector field. Moreover, by a perturbation argument, one can continue such a flow to a gradient flow of Morse-Smale function. We now state the analogous theorem for $\mathcal{LS}$-flows. The proof is a direct consequence of the finite-dimensional reduction discussed in the previous section and the classical Reineck's theorem.

Theorem 4.3.1 (Reineck's theorem for $\mathcal{LS}$-flows)**.** *Let $F = L + K$ be a continuously differentiable $\mathcal{LS}$-vector field and let U be an isolating neighbourhood for the induced flow ϕ. Then ϕ is related by continuation on U to a gradient $\mathcal{LS}_{fin}$-flow of a Morse-Smale function.*

Proof. By Corollary 4.1.5 flow ϕ can be continued to an $\mathcal{LS}_{\text{fin}}$-flow $(\phi_V, \phi_\perp)$. On the other hand, by Reineck's Theorem ([Rei90, Thm. 2.1. and Cor.2.2]), ϕ_V can be continued to a gradient flow $\tilde{\phi}_V$ of a Morse-Smale function $f_V : U \cap V \to \mathbb{R}$. This defines the continuation of ϕ to $(\tilde{\phi}_V, \phi_\perp)$ on U. Clearly, $(\tilde{\phi}_V, \phi_\perp)$ is an $\mathcal{LS}$-flow and it is a gradient flow of a Morse-Smale function $f : (U \cap V, U \cap V^\perp) \to \mathbb{R}$ given by

$$f(x, y) = f_V(x) + \frac{1}{2}\langle Ly, y\rangle .$$

□

4.4 Local Morse cohomology

4.4.1 Closed manifolds

Let M be a closed finite-dimensional Riemannian manifold and let f be a Morse-Smale function. Define the chain complex by

$$C^k = \bigoplus_{x \in \mathrm{Crit(f)}; \mathrm{ind}\, x = k} \mathbb{Z}_2\langle x\rangle$$

The following definition of the boundary operator $\partial_k^c : C^k \to C^{k+1}$ is due to E.Witten. Since f is a Morse-Smale function the intersection $W^u(y) \cap W^s(x)$ is a manifold of dimension $\mathrm{ind}\, y - \mathrm{ind}\, x$. If $\mathrm{ind}\, y - \mathrm{ind}\, x = 1$ then $M(y, x) := W^u(y) \cap W^s(x)$ consists of finitely many (unparametrized) flow orbits connecting y and x. Denote by

$n(y,x)$ the number of such trajectories i.e. $n(x,y) = \#(W^u(y) \cap W^s(x))_{/\mathbb{R}}$. The Floer boundary operator is defined by

$$\partial_k^c(x) = \sum_y n(y,x)\langle y \rangle$$

where the sum runs over all critical points of index $k+1$.

Theorem 4.4.1 (R. Thom, S. Smale, J. Milnor, C. Conley, E. Witten)**.**

$$\partial_k^c \circ \partial_{k+1}^c = 0$$

$$H_k(M, \mathbb{Z}_2) = \frac{\ker \partial_k^c}{\operatorname{im} \partial_{k+1}^c}$$

4.4.2 Local Morse cohomology

Above construction can be also used to define local Morse homology. To be more precise let f be a Morse-Smale function on a closed and bounded set $U \subset M$. Suppose that U is an isolating neighbourhood of the flow generated by $-\nabla f$. Then the relation

$$\partial_k^c \circ \partial_{k+1}^c = 0$$

still holds and therefore we can define homology groups

$$H_*^{Morse}(f, U)$$

Define Morse cohomology $H^*_{Morse}(f,U) := H_*^{Morse}(-f,U)$. Again by the work of many authors we have the following.

Theorem 4.4.2 (McCord, Floer, Salamon)**.** *Suppose that f and U are as above. Then*

$$H^*_{Morse}(f,U) = H^*(N,L)$$

where (N,L) is an index pair for $S = \operatorname{Inv} U$.

In particular we have:

Corollary 4.4.3. *Take $H : U \times [0,1] \to \mathbb{R}$ and put $f_t = H(\cdot,t)$. Suppose that f_0 and f_1 are Morse-Smale functions and that U is an isolating neighbourhood for the flow generated by $-\nabla f_t$ for every $t \in [0,1]$. Then*

$$H^*_{Morse}(f_1,U) = H^*_{Morse}(f_2,U)$$

4.4.3 Case of a Hilbert space

Suppose that $f \in V^2(\mathbb{H}, \mathbb{R})$ is Morse-Smale while restricted to U. Moreover, suppose that ∇f generates an $\mathcal{LS}$-flow ϕ, U is an isolating neighbourhood for ϕ.

Theorem 4.4.4. *[Sta15, Main Theorem] Suppose that the above assumptions on (f, U) hold. Then*

$$H^*_{Morse}(f, U) = \mathrm{ch}_{\mathrm{E}}(U, \phi).$$

Before sketching a proof let us discuss local Palais-Smale condition. It is crucial ingredient in showing that the Morse complex is well defined.

Proposition 4.4.5 (local (PS)-condition)**.** *Let f be a $C^{1,1}$ function on $\mathbb{H}$ and suppose that its gradient flow ϕ satisfies (C) compactness property. Suppose that U is a bounded subset of $\mathbb{H}$ and that $\{x_n\} \subset U$ satisfies $|\nabla f(x_n)| \to 0$. Then $\{x_n\}$ contains a convergent subsequence.*

Proof. Let c be a Lipschitz constant of ∇f and let $U \subset B(R)$ where $B(R)$ is a ball in $\mathbb{H}$ of radius R. Without loss of generality, suppose that

$$|\nabla f(x_n)| \leq \frac{e^{-cn}}{n}$$

Then

$$\gamma(t) := |\phi(x_n, t) - \phi(x_n, 0)| \leq \int_0^t |\nabla f(\phi(x_n, s))|\, ds \leq$$
$$\leq \int_0^t |\nabla f(\phi(x_n, s)) - \nabla f(\phi(x_n, 0))| + |\nabla f(\phi(x_n, 0))|\, ds \leq c \int_0^t \gamma(s)\, ds + e^{-cn}$$

By Gronwall inequality we have

$$\gamma(t) \leq e^{-cn} e^{ct} \leq 1$$

for $t \in [0, n]$ and therefore $\phi(x_n, [0, n])$ lies inside the ball of radius $R + 1$ for every n. Analogously, we show that $\phi(x_n, [-n, 0])$ is bounded. By (C) property, x_n converges up to subsequence. □

Sketch of the proof of Thm 4.4.4. We use series of continuations. By Theorem 4.3.1, ϕ can be continued to an $\mathcal{LS}_{\mathrm{fin}}$ gradient flow $(\phi_V, \phi_\perp)$ of a Morse-Smale function $\tilde{f}$. Moreover, $\tilde{f}$ is of the form

$$\tilde{f}(x, y) = f_V(x) + \frac{1}{2} \langle Ly, y \rangle$$

where $(x, y) \in (U \cap V, U \cap V^{\perp})$. Clearly

$$H^*_{Morse}(\tilde{f}, U) = H^*_{Morse}(f_V, U \cap V).$$

On the other hand,

$$H^*_{Morse}(f_V, U \cap V) = \mathrm{ch}(U \cap V, \phi_V)$$

since $U \cap V$ is finite-dimensional. Finally,

$$\mathrm{ch}(U \cap V, \phi_V) = \mathrm{ch}_{\mathrm{E}}(U, \phi)$$

by direct computation. $\square$

Chapter 5

Arnold conjecture on T^{2n}

5.1 Preliminaries

Let ω be the standard symplectic structure and let J be the standard almost complex structure on $\mathbb{R}^{2n}$, i.e.

$$\omega = dx_1dy_1 + dx_2dy_2 + \ldots + dx_ndy_n, \quad J = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}$$

Consider a 1-periodic Hamiltonian, i.e. a function $H \in C^2(S^1 \times \mathbb{R}^{2n}, \mathbb{R})$. Throughout this section, we assume that H satisfies a suitable growth condition, namely that $|H(x)| \leq C\,|x|^2$ at the infinity and that H'' is globally bounded. To H we associate a Hamiltonian vector field X_H which, by the non-degeneracy of ω, is uniquely determined by the equation

$$dH(\cdot) = \omega(X_H, \cdot).$$

In fact, one can easily see that $X_H = J\nabla H$. We seek 1-periodic solutions to the Hamilton equation, i.e.

$$\dot{x} = X_H(x). \tag{5.1}$$

Following Hofer and Zehnder ([HZ12, Ch.3]) we introduce the analytical setting for this problem. Let Ω be the space of smooth loops on $\mathbb{R}^{2n}$, i.e. $\Omega = C^\infty(S^1, \mathbb{R}^{2n})$ and put $e_k = e^{tk2\pi J}$. Any $x \in \Omega$ is represented by its Fourier-series, i.e.

$$x = \sum_{k \in \mathbb{Z}} e_k x_k$$

The Sobolev space $H^{\frac{1}{2}}(S^1,\mathbb{R})$ is defined as the completion of Ω with respect to the scalar product

$$\langle x,y\rangle_s = \langle x_0,y_0\rangle + 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|\,\langle x_k,y_k\rangle\,.$$

Hilbert space $H^{\frac{1}{2}}(S^1,\mathbb{R})$ splits into $2n$-dimensional subspace E_0 and closed infinite-dimensional subspaces E^+ and E^- which in the Fourier-series expansion correspond to $k=0$, $k>0$ and $k<0$ respectively. Denote by P_0, P^+ and P^- the corresponding orthogonal projections. Define a functional $\Phi:\Omega\to\mathbb{R}$ by the formula

$$\Phi_H(x) = a(x)-b(x) := \frac{1}{2}\int_0^1 \langle -J\dot{x}(t),x(t)\rangle\,dt - \int_0^1 H(t,x(t))\,dt.$$

If no confusion can arise, we drop a subscript in Φ_H. Proof of the following proposition can be found in [HZ12, p.76].

Proposition 5.1.1. *The critical points of Φ are periodic solutions to the Hamilton equation.*

This shows the importance of Φ. Note that $-J\dot{x} = -J\frac{d}{dt}(\sum_{k\in\mathbb{Z}} e_k x_k) = 2\pi\sum_{k\in\mathbb{Z}} k e_k x_k$ and therefore we have

$$\begin{aligned}2a(x) = \langle -J\dot{x},x\rangle_{L^2} = 2\pi\left\langle \sum_{k\in\mathbb{Z}} k e_k x_k, \sum_{k\in\mathbb{Z}} e_k x_k\right\rangle_{L^2} &= \\ 2\pi\sum_{k>0}|k|\,|x_k|^2 - 2\pi\sum_{k<0}|k|\,|x_k|^2 = \left|\left|P^+x\right|\right|^2_{\frac{1}{2}} - \left|\left|P^-x\right|\right|^2_{\frac{1}{2}}.& \quad (5.2)\end{aligned}$$

Above computation shows that $a(\cdot)$ extends to $H^{1/2}(S^1,\mathbb{R}^{2n})$. On the other hand, it is easily seen that $b(\cdot)$ extends to $L^2(S^1,\mathbb{R}^{2n})$.

Corollary 5.1.2. *Φ extends to $H^{1/2}(S^1,\mathbb{R}^{2n})$.*

From the formula (5.2) one reads that the $H^{1/2}$-gradient of $a(\cdot)$ is equal to $Lx := P^+x - P^-x$ and therefore it is a linear Fredholm self-adjoint operator. The gradient of $b(\cdot)$ is compact. To see this, take the inclusion $j: H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n}) \to L^2(S^1,\mathbb{R}^{2n})$. Direct computation shows that the adjoint j^* is given by the formula

$$j^*(\sum_{k\in\mathbb{Z}} e_k x_k) = x_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{\lambda_k} e_k x_k.$$

From the above formula we conclude that the range of j^* is contained in $H^1(S^1,\mathbb{R}^{2n})$ and since the inclusion $H^1(S^1,\mathbb{R}^{2n}) \hookrightarrow H^{1/2}(S^1,\mathbb{R}^{2n})$ is compact, so is j^*. Function

b is continuously differentiable on $L^2(S^1, \mathbb{R}^{2n})$. Since $\nabla^{H^{1/2}} b = j^* \nabla^{L^2} b$, we have that $\nabla^{H^{1/2}} b$ is compact. Therefore $\nabla\Phi$ is of the form $L + K$ where L is a linear Fredholm operator and K is a compact petrurbation. In addition, the assumption on the growth of H guarantees that $\nabla\Phi$ is globally Lipschitz.

Corollary 5.1.3. $\nabla\Phi$ *induces an* $\mathcal{LS}$ *flow on* $H^{1/2}(S^1, \mathbb{R}^{2n})$.

5.2 Case of the torus

On a general manifold the space of $H^{\frac{1}{2}}(S^1, M)$ is not well defined. The reason is that $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$ contains non-continuous functions. However, for a torus one can overcome this problem.
Let $\Pi : \mathbb{R}^{2n} \to T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be the universal cover. Smooth Hamiltonians on T^{2n} are in one-to-one correspondence with $\mathbb{Z}^{2n}$-invariant smooth Hamiltonians on $\mathbb{R}^{2n}$, where $\mathbb{Z}^{2n}$ acts on $\mathbb{R}^{2n}$ by translations. By abuse of notation, we denote by H both the Hamiltonian on torus and the Hamiltonian lifted to $\mathbb{R}^{2n}$. Note that a $\mathbb{Z}^{2n}$-invariant Hamiltonian is bounded and in particular it satisfies the growth condition from the previous subsection. The symplectic structure ω, the almost complex structure J and the vector field X_H descend from $\mathbb{R}^{2n}$ to the torus and therefore the Hamilton equation reads exactly as in the Euclidean space (see 5.1).

Theorem 5.2.1 (Arnold conjecture on T^{2n}). *For a given Hamiltonian on* T^{2n}*, there exist at least* $2n + 1$ *contractible periodic solutions to the Hamilton equation* (5.1). *Moreover, if all the periodic solutions are non-degenerate, then there are at least* 2^{2n} *of them.*

Before giving a proof we give more details on the analytical setting. The space of smooth contractible loops $\hat{\Omega}$ can be viewed as $\Omega/\mathbb{Z}^{2n}$ where $\mathbb{Z}^{2n}$ is seen as the subset of constant, integer-valued loops.

Definition 5.2.2. We define the space of *contractible* $H^{1/2}$*-loops on* T^{2n} to be

$$\mathcal{M} := H^{1/2}(S^1, \mathbb{R}^{2n})/\mathbb{Z}^{2n}.$$

As before, constant, positive and negative frequencies give decomposition of $H^{1/2}(S^1, \mathbb{R}^{2n})$ into $E_0 \times E^+ \times E^-$. Therefore

$$\mathcal{M} = E_0/\mathbb{Z}^{2n} \times E^+ \times E^- = T^{2n} \times E^+ \times E^-.$$

Let H be a Hamiltonian on T^{2n} lifted to $\mathbb{R}^{2n}$ and let $\Phi : H^{1/2}(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ be as in previous subsection. Note that Φ is $\mathbb{Z}^{2n}$-invariant and therefore it gives a well-defined function on $\mathcal{M}$. Since we want to apply E-cohomological Conley index we prefer to work with a Hilbert space rather than a Hilbert manifold. For that reason, let us embed $\mathcal{M}$ into $\hat{E} = \mathbb{R}^{4n} \times E^+ \times E^-$ in such a way that every S^1 in $T^{2n} = S^1 \times \ldots \times S^1$ is mapped to a unit circle in $\mathbb{R}^2$. Let $\mathcal{N}$ be a tubular neighbourhood of $\mathcal{M}$ in $\hat{E}$, namely

$$\mathcal{N} = A^{2n} \times E^+ \times E^-$$

where $A = \{(x,y) \in \mathbb{R}^2 | \frac{1}{2} < \sqrt{x^2+y^2} < \frac{3}{2}\}$ is an annulus. Let $\pi : A^{2n} \to T^{2n}$ be the standard projection. We extend Φ to $\mathcal{N}$ by the formula

$$\Psi(x) = \Phi(\pi(x)) + \sum_{i=1}^{2n} \frac{1}{2}(1 - r_i(x))^2.$$

where r_i denotes the i-th polar coordinate A^{2n}. Note that the extension is done in such a way that the critical points of Φ and Ψ are the same. Denote by $\tilde{K}$ the compact operator which is the sum of K and $\nabla(\sum_{i=1}^{2n} \frac{1}{2}(1 - r_i(x))^2)$.

Recall that Φ and therefore also Ψ depend on a chosen Hamiltonian H. Suppose we have a continuous family of Hamiltonians $\{H_\lambda | \lambda \in [0,1]\}$. This gives us a corresponding family of functionals $\{\Psi_\lambda = L + \tilde{K}_\lambda\}$.

Lemma 5.2.3. *For every bounded set B the set $\bigcup_{\lambda \in [0,1]} \nabla \Psi_\lambda^{-1}(B)$ is bounded.*

Proof. Suppose the contrary, i.e. that there exist a sequence $\{(x_n, \lambda_n\}$ and a constant $c > 0$ such that $||x_n|| \to \infty$ and $||\nabla \Psi_{\lambda_n}(x_n)|| < c$. We have

$$c > ||\nabla_{\lambda_n} \Psi(x_n)|| \geq \left|\left|P^+ x_n\right|\right| + \left|\left|P^- x_n\right|\right| - \left|\left|\tilde{K}_{\lambda_n}(x_n)\right|\right|.$$

Since the family $\{H_{\lambda_n}\}$ is uniformly bounded and $\frac{1}{2} < r_i < \frac{3}{2}$ we have that the norm $\left|\left|\tilde{K}_{\lambda_n}(x_n)\right|\right|$ is bounded. On the other hand if $||x_n|| \to \infty$ then $||P^+ x_n|| \to \infty$ or $||P^- x_n|| \to \infty$. A contradiction. □

Proposition 5.2.4. *The set $\bigcup_{\lambda \in [0,1]} Crit(\Psi_\lambda)$ is compact.*

Proof. Let $\{x_n\}$ be a sequence of critical points of Ψ_{λ_n}. By Lemma 5.2.3, $\{x_n\}$ is bounded. Since L is a Fredholm operator, there exists a Fredholm operator T such that $TL = \mathbb{I} + K_1$ where K_1 is compact. We have

$$0 = T\nabla\Psi(x_n) = x_n + (K_1 + \tilde{K}_{\lambda_n})(x_n)$$

that is

$$x_n = -(K_1 + \tilde{K}_{\lambda_n})(x_n) \tag{5.3}$$

By Lemma 5.2.3 the sequence $\{x_n\}$ is bounded. Thus the right hand side of (5.3) converges up to subsequence and so does $\{x_n\}$. □

Let S_λ be the corresponding union of critical points and orbits connecting them, namely

$$S_\lambda = \{x \in \mathcal{N} | \alpha(x), \omega(x) \in \mathrm{Crit}(\Psi_\lambda)\}$$

Proposition 5.2.5. $\bigcup_{\lambda\in[0,1]} S_\lambda$ *is bounded.*

Proof. The proof relies on the standard arguments in the compactness proofs for Floer theory. Reader may compare with finite-dimensional case from [Sch93, pp. 56-57].

Denote by $B(r)$ a ball of radius r in $\hat{E}$. By Lemma 5.2.3 there exists a constant $r_0 > 0$ such that $\bigcup_{\lambda\in[0,1]} \nabla\Psi_\lambda^{-1}(B(1)) \subset B(r_0)$. By Proposition 5.2.4 there exists a constant $r_1 > 0$ such that $|\Psi_\lambda(p)| < \frac{r_1}{2}$ for all p and λ such that $p \in \mathrm{Crit}\Psi_\lambda$. We will show that $S_\lambda \subset B(r_0 + r_1)$ for every λ.

Let u be a trajectory of the $-\nabla\Psi_\lambda$-flow such that $u(0) \in S \setminus B(\frac{r_0}{3})$ and let p and q be points in $\alpha(u(0))$ and $\omega(u(0))$, respectively. Choose $t_0 \in (-\infty, 0)$ such that

- $|\nabla\Psi(u(t_0))| = 1$;
- $|\nabla\Psi(u(t_0))| \geq 1$ for any $t \in [t_0, 0]$.

Denote by d the distance in $\hat{E}$. The following inequalities hold

$$d(0, u(0)) \leq d(0, u(t_0)) + d(u(t_0), u(0)) \leq r_0 + \int_{t_0}^{0} |\dot{u}(s)|\, ds$$

It is enough to show that $\int_{t_0}^{0} |\dot{u}(s)|\, ds \leq r_1$.
Put $l(s) = \int_{t_0}^{s} |\dot{u}(\tau)|\, d\tau$. We have

$$\frac{dl}{ds}(s) = |\dot{u}(s)| = |\nabla\Psi(u(s))|$$

$$\frac{d(\Psi(u(s)))}{ds}(s) = -|\nabla\Psi(u(s))|^2$$

so

$$\frac{dl}{ds}(s) \leq -\frac{d(\Psi(u(s)))}{ds}(s)$$

for every $s \in [t_0, 0]$. Therefore

$$\int_{t_0}^{0} |\dot{u}(s)|\, ds = \int_{t_0}^{0} \frac{dl}{ds}(s)\, ds \leq \int_{t_0}^{0} \frac{d(\Psi(u(s)))}{ds}(s)\, ds =$$
$$= \Psi(t_0) - \Psi(0) \leq \Psi(p) - \Psi(q) \leq r_1.$$

□

proof of the Theorem 5.2.1 (Arnold conjecture).
The idea is to use continuation principle (Corollary 3.3.4) and continue the flow induced by an arbitrary Hamiltonian H to the flow generated by the most generate function, namely by a constant Hamiltonian. For the latter, the computation of the E-cohomological Conley index and its cup-length is straightforward.
For a given Hamiltonian H consider a linear homotopy $H_\lambda(\cdot) = (1-\lambda)H(\cdot)$. We have a family of associated functionals Ψ_λ on $\mathcal{N}$ and thus a family of flows generated by $-\nabla\Psi_\lambda$. By Proposition 5.2.5, there exists R such that $U = A^{2n} \times B(E^+ \times E^-, R)$ is an isolating neighbourhood for S_λ for every $\lambda \in [0,1]$. This means that the flow generated by $-\nabla\Psi_0 = -\nabla\Psi$ continues within U to the flow generated by $\nabla\Psi_1$. The latter one is linear and the direct computation shows $\mathrm{ch}_{\mathrm{E}E}(S, \Psi_1)$ is isomorphic, as a module to reduced cohomology of the $2n$-dimensional torus. The conclusion follows from Proposition 3.4.4 (degenerate case) and Theorem 4.4.4 (non-degenerate case).

□

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