



Turán numbers for odd wheels

Tomasz Dzido^a, Andrzej Jastrzębski^{b,*}

^a Institute of Informatics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland

^b Department of Algorithms and System Modeling, Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, 80-233 Gdańsk, Poland

ARTICLE INFO

Article history:

Received 28 May 2015

Received in revised form 3 October 2017

Accepted 5 October 2017

Available online 6 November 2017

Keywords:

Turán numbers

Odd wheels

ABSTRACT

The Turán number $\text{ex}(n, G)$ is the maximum number of edges in any n -vertex graph that does not contain a subgraph isomorphic to G . A *wheel* W_n is a graph on n vertices obtained from a C_{n-1} by adding one vertex w and making w adjacent to all vertices of the C_{n-1} . We obtain two exact values for small wheels:

$$\text{ex}(n, W_5) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} \right\rfloor,$$

$$\text{ex}(n, W_7) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} + 1 \right\rfloor.$$

Given that $\text{ex}(n, W_6)$ is already known, this paper completes the spectrum for all wheels up to 7 vertices. In addition, we present the construction which gives us the lower bound $\text{ex}(n, W_{2k+1}) > \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$ in general case.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

In this paper, all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. Let $d_G(v)$ be the degree of vertex v in G , $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of vertices of G , $\omega(G)$ be the clique number of a graph G and $\chi(G)$ be the chromatic number of graph G . Define $G[S]$ to be a subgraph of G induced by a set of vertices $S \subseteq V(G)$ and $G[S, R]$ to be a bipartite subgraph of G with the bipartition $\{S, R\}$. $G_1 \cup G_2$ denotes the graph which consists of two disconnected subgraphs G_1 and G_2 . We will use $G_1 + G_2$ to denote the join of G_1 and G_2 defined as $G_1 \cup G_2$ together with all edges between G_1 and G_2 . C_m denotes the cycle of length m . A *wheel* W_n is a graph on n vertices obtained from a C_{n-1} by adding one vertex w and making w adjacent to all vertices of the C_{n-1} .

The *Turán number* $\text{ex}(n, G)$ is the maximum number of edges in any n -vertex graph that does not contain a subgraph isomorphic to G . A graph on n vertices is said to be *extremal with respect to* G if it does not contain a subgraph isomorphic to G and has exactly $\text{ex}(n, G)$ edges. $\text{EX}(n, G)$ is the set of all extremal graphs of order n with respect to G .

A main motivation for proving results for Turán numbers is that they are often useful in Ramsey Theory where the original extremal statements would not suffice (see [3] for example). Our goal is to determine the Turán numbers of wheels W_k for odd k . We describe families of extremal graphs for $k = 5, 7$ and present a very simple lower bound for all odd k .

* Corresponding author.

E-mail addresses: tdz@inf.ug.edu.pl (T. Dzido), jendrek@eti.pg.edu.pl (A. Jastrzębski).

2. Known results

First, we recall the result which was proved by Mantel in 1907.

Theorem 1 (Mantel, [5]). *The maximum number of edges in an n -vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$.*

By **Theorem 1** and since $W_3 = C_3$, it is easy to have the property that for all integers $n, n \geq 3$, $\text{ex}(n, W_3) = \lfloor \frac{n^2}{4} \rfloor$. The famous Turán's theorem may be stated as follows.

Theorem 2 (Turán, [8]). *Let G be any subgraph of K_n such that G is K_{r+1} -free. Then the number of edges in G is $e(G) = \lfloor \frac{(r-1)n^2}{2r} \rfloor$. In particular, $\text{ex}(n, K_4) = \lfloor \frac{n^2}{3} \rfloor$.*

As a special case, for $r = 2$, one obtains Mantel's theorem. Since $W_4 = K_4$, we obtain that for all integers $n, n \geq 3$, $\text{ex}(n, W_4) = \lfloor \frac{n^2}{3} \rfloor$. In 1964 Erdős proved the following theorem.

Theorem 3 (Erdős, [4]). *Let G be any graph such that $|E(G)| \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor + 1$. Then G contains a W_5 .*

By **Theorem 3** we immediately obtain the upper bound for $\text{ex}(n, W_5)$, namely $\text{ex}(n, W_5) \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor + 1$. The first author [2] proved that for all $k \geq 3$ and $n \geq 6k - 10$, if G is a graph that contains no subgraph isomorphic to W_{2k} , then $\text{ex}(n, W_{2k}) = \lfloor \frac{n^2}{3} \rfloor$. In addition, he showed that $\text{ex}(n, W_6) = \lfloor \frac{n^2}{3} \rfloor$.

If G is an arbitrary graph whose chromatic number is $r > 2$, then by Erdős–Stone–Simonovits theorem [7] we have that $\text{ex}(n, G) = (\frac{r-2}{r-1} + o(1)) \binom{n}{2}$. This result determines the asymptotic behavior of $\text{ex}(n, W_k)$.

It is interesting that exact values for $\text{ex}(n, C_4)$ and $\text{ex}(n, C_6)$, i.e. for rims of wheels W_5 and W_7 remain unknown in general. Even in the case of the C_4 cycle values are known only for $n \leq 32$ (the last result being $\text{ex}(32, C_4) = 92$, obtained in 2009 by Shao, Xu and Xu), whereas for larger n only the upper or lower bounds are known.

3. Progress on $\text{ex}(n, W_{2k+1})$

3.1. $\text{ex}(n, W_5)$

If G and H have maximum degree 1, then the join $G+H$ does not contain W_5 . So define M_n by taking $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ and adding a maximum matching within each partite set.

Lemma 4. *The graph M_n does not contain a W_5 as a subgraph.*

Proof. Every subgraph induced on 3 vertices of W_5 is connected. If i, j, k have the same parity then, by definition of M_n , graph $M_n[v_i, v_j, v_k]$ has at most one edge, so it is a disconnected graph. If we assume that M_n has a subgraph W_5 , then at least 3 vertices of this subgraph W_5 are indexed by numbers which have the same parity (we denote the vertices of M_n as in the definition). A graph induced in W_5 by these three vertices is connected, but a graph induced in M_n by these vertices is not connected. This means that M_n does not contain a subgraph W_5 . \square

Theorem 5. *The graph M_n is an extremal graph with respect to W_5 .*

Proof. We know that $M_1 = K_1, M_2 = K_2, M_3 = K_3$ and $M_4 = K_4$ are extremal. Assume that each M_n is extremal for $n < N$. We will show that M_N is also extremal. Let G be an extremal graph of order N . Let H be a 4-vertex subgraph of G with maximum possible number of edges. \square

Lemma 6. *A graph G of order 5 contains W_5 as a subgraph if and only if $\delta(G) \geq 3$.*

Proof. If G contains W_5 , it must be a spanning subgraph and so $\delta(G) \geq 3$. If $\delta(G) \geq 3$, then G contains a vertex of degree 4 and G contains a W_5 . \square

Consider the graph $G \setminus V(H)$. From **Lemma 6** we know that each vertex from $G \setminus V(H)$ is adjacent to at most 2 vertices from H . If any $v \in G \setminus V(H)$ was adjacent to three vertices of H , then the graph $G[V(H) \cup \{v\}]$ would contain W_5 as a subgraph or a 4-vertex subgraph with a greater number of edges than H . From the above it follows that

$$\begin{aligned} e(G) &\leq e(H) + 2 \cdot |V(G \setminus V(H))| + e(G \setminus V(H)) \\ &\leq \binom{4}{2} + 2 \cdot (N - 4) + \text{ex}(N - 4, W_5) = e(M_N). \end{aligned}$$

If G is extremal, then M_N does not contain W_5 . In addition, $e(M_N) \geq e(G)$, so M_N is also extremal. \square

Table 1

The values of $ex(n, W_7)$ and $|EX(n, W_7)|$ for all $7 \leq n \leq 26$.

| | | | | | | | | | | |
|----------------|----|----|-----|-----|-----|-----|-----|-----|-----|-----|
| n | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $ex(n, W_7)$ | 17 | 21 | 25 | 31 | 37 | 43 | 50 | 57 | 65 | 73 |
| $ EX(n, W_7) $ | 2 | 1 | 5 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| n | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| $ex(n, W_7)$ | 82 | 91 | 101 | 111 | 122 | 133 | 145 | 157 | 170 | 183 |
| $ EX(n, W_7) $ | 1 | 2 | 1 | 2 | 1 | 3 | 2 | 3 | 1 | 2 |

Corollary 7.

$$ex(n, W_5) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} \right\rfloor.$$

Bataineh, Jaradat and Jaradat [1] presented a very extensive characterization of all extremal W_5 -free graphs.

3.2. $ex(n, W_7)$

It is not hard to verify that if G has maximum degree 1 and H has maximum degree 2 and does not contain P_5 , then the join $G + H$ does not contain W_7 . So let G_m be the graph formed from m isolated vertices by adding a maximum matching. Further, let H_m be any 2-regular m -vertex graph formed by the disjoint union of copies of 3- or 4-cycles. (It can be checked that H_m exists for $m \geq 6$.) Then define the graph N_n as $G_{k-1} + H_{k+1}$ if $n = 2k$, and $G_k + H_{k+1}$ if $n = 2k + 1$. It can be checked that N_n has $k^2 + k + 1$ edges if $n = 2k$, and $k^2 + 2k + 2$ edges if $n = 2k + 1$.

From this construction we see that $ex(2k, W_7) \geq k^2 + k + 1$ and $ex(2k + 1, W_7) \geq k^2 + 2k + 2$.

Theorem 8. For all $k \geq 5$, if $ex(2k, W_7) = k^2 + k + 1$, then $ex(2k + 1, W_7) \leq k^2 + 2k + 2$.

Proof. Let G be a graph of order $2k + 1$ which does not contain W_7 and assume that $e(G) = k^2 + 2k + 3$.

Observe that $\delta(G) \geq e(G) - ex(2k, W_7) = k + 2$. Since $e(G) \geq \frac{(2k+1)(k+2)}{2} > k^2 + 2k + 3 = e(G)$ for all $k \geq 5$, we deduce the result. \square

Theorem 9. For all $k \geq 5$, $ex(2k, W_7) = k^2 + k + 1$.

Proof. The cases $5 \leq k \leq 8$ were checked by computational calculations (see Table 1).

Suppose that $k > 8$ is the smallest number such that $ex(2k, W_7) > k^2 + k + 1$, then for all $5 \leq l < k$ we have $ex(2l, W_7) = l^2 + l + 1$ and by Theorem 8 $ex(2l + 1, W_7) = l^2 + 2l + 2$.

Let G be a graph of order $2k$ with $e(G) = k^2 + k + 2$ edges and G does not contain W_7 as a subgraph. We see that $\delta(G) \geq e(G) - ex(2k - 1, W_7) = k + 1$. If $\delta(G) \geq k + 2$, then $e(G) \geq \frac{2k(k+2)}{2} > e(G)$ for all $k > 2$. So we have $\delta(G) = k + 1$.

The remaining part of the proof is divided into four cases according to the value of $\omega(G)$. Clearly $\omega(G) < 7$.

Case 1. $\omega(G) = 6$

Let K be a clique of order 6 in G and $W = V(G) \setminus V(K)$. To avoid W_7 , every vertex in W is joined to K by at most two edges. We have

$$\binom{6}{2} + 2(2k - 6) + ex(2k - 6, W_7) = k^2 - k + 10 < e(G),$$

a contradiction.

Case 2. $\omega(G) = 5$

Let $K = \{v_1, v_2, v_3, v_4, v_5\}$ be a maximum clique and $W = V(G) \setminus K$. Consider the edges of the bipartite graph $H = G[K, W]$. Let $W^4 = \{v \in W : d_H(v) = 4\}$, $W^3 = \{v \in W : d_H(v) = 3\}$ and $W^r = W - W^4 - W^3$, obviously if $v \in W^r$ then $d_H(v) < 3$.

One can easily verify that if $|W^4| \geq 2$, then we immediately have W_7 . If $|W^4| = 1$, then to avoid W_7 in G we have that $|W^3| = 0$. Since $e(H) \leq 4 + 2(2k - 6) < 5(k - 3) = 5(\delta(G) - 4) \leq e(H)$ for $k > 7$, we obtain that in fact $W^4 = \emptyset$. Note that W^3 in G is an independent set and each edge in $G[K, W^3]$ is adjacent to the same three vertices of K , say $\{v_1, v_2, v_3\}$. From $\delta(G) = k + 1$, it follows that $|W^r| + 3 \geq \delta(G)$, so $|W^3| \leq k - 3$. In fact $|W^3| = k - 3$ because of the inequality $e(G) \leq 10 + 3|W^3| + 2|W^r| + ex(2k - 5, W_7) = k^2 + 5 + |W^3|$.

Note that for every vertex v in W^3 we have that $d_G(v) = k + 1$. The bipartite graph $G[W^r, W^3]$ is complete, therefore $\Delta(G[W^r]) \leq 2$. If not, then we have W_7 in $G[W]$. Hence, $e(G[W]) \leq |W^3||W^r| + \frac{2|W^r|}{2} = k^2 - 4k + 4$ and $e(G) \leq 10 + 3|W^3| + 2|W^r| + e(G[W]) \leq k^2 + k + 1$, a contradiction.

We have $W^4 = W^3 = \emptyset$, $|W^r| = 2k - 5$ but $e(G[K, W]) \leq 2(2k - 5) < 5(k - 3) = 5(\delta(G) - 4) < e(G[K, W])$ for $k > 5$, a contradiction.

Case 3. $\omega(G) = 4$

Let $K = \{v_1, v_2, v_3, v_4\}$ be a maximum clique and $W = V(G) \setminus K$.

Let U_i be the set of vertices from W such that they are adjacent to all vertices from $V(K) \setminus \{v_i\}$. This means that if $v \in U_i$ then $d_{G[K, W]}(v) = 3$. To avoid K_5 all U_i are independent. Let the remaining vertices of W be W^r .

First observe that if U_i, U_j, U_l are not empty for $i \neq j \neq l \in \{1, 2, 3, 4\}$, then we immediately have W_7 . Without loss of generality, let us assume that U_3, U_4 are empty. Observe that if $|U_1 \cup U_2| > 2$, then the set $U_1 \cup U_2$ is independent.

Subcase 3.1 $U_1 = U_2 = \emptyset$

We have $e(G) \leq \text{ex}(2k - 4, W_7) + 6 + 2(2k - 4) = k^2 + k + 1 < e(G)$, a contradiction.

Subcase 3.2 $|U_1 \cup U_2| = 1$

Without loss of generality, let $w \in U_1$. To avoid a contradiction similar to the previous subcase, for all vertices $v \in W^r$ we have $d_{G[K, W]}(v) = 2$. This means that one vertex from K has degree $k + 2$ and the remaining three vertices have degree $k + 1$ in G , so at least one vertex from W has degree greater than or equal to $k + 2$ in G .

Let X be all vertices from W^r adjacent to w and $Y = W^r \setminus X$. Obviously $|X| \geq k - 2$. It is not hard to see that if $G[X]$ contains P_4 or K_3 as a subgraph, then $G[K \cup U_1 \cup X]$ contains W_7 as a subgraph. If $|X| \geq 4$, then there exist at least 3 vertices of degree 1 in $G[X]$. These vertices are adjacent to all vertices in Y , therefore $\Delta(G[Y]) \leq 2$, $|X| = k - 2$, $|Y| = k - 3$, subsequently $\delta(G[X]) = 1$, $\delta(G[Y]) \geq 1$ and $\Delta(G[Y]) \leq 2$, so each vertex from Y is adjacent to all or all except one vertex from X .

If there exists a vertex $p \in Y$ such that $d_G(p) > k + 1$, then $d_{G[Y]}(p) = 2$ and p is adjacent to every vertex in X . Let p_1, p_2 be the vertices adjacent to p in Y . If there exists P_3 in $G[X]$, then one end-vertex of the path is adjacent to p_1 and the other to p_2 , then the graph induced by the path, p_1, p_2, p and an additional vertex from X adjacent to p_1 and p_2 contains W_7 as a subgraph. Contrary, there exist two independent edges in $G[X]$ such that their vertices are adjacent to p_1 or p_2 . These edges with p_1, p_2 and p induce a graph with W_7 as a subgraph.

If there exists a vertex $p \in X$ such that $d_G(p) > k + 1$, then $d_{G[X]}(p) \geq 2$. If $d_{G[X]}(p) = 2$ then p is adjacent to every vertex in Y . Let p_1 and p_2 be the adjacent vertices to p in X . Note that p_1, p_2 have degree 1 in $G[X]$. There exist two independent edges in $G[Y]$. Since p, p_1 and p_2 are adjacent to vertices incident to these independent edges, then they both with w induce a graph with a subgraph W_7 . If $d_{G[X]}(p) > 2$, then vertex w , three vertices adjacent to p in X and two vertices adjacent to p in Y induce a graph with a subgraph W_7 .

From the above arguments, every vertex of W has degree $k + 1$ in G , so $e(G[W]) < \text{ex}(2k - 5, W_7)$, a contradiction.

Subcase 3.3 $|U_1 \cup U_2| = 2$

Let $w_1, w_2 \in U_1$. There exists a vertex $p \in W^r$ adjacent to w_1 and two vertices of K , v_1 and another vertex. A graph induced by $K \cup U_1$ and p contains W_7 as a subgraph.

Let $w_1 \in U_1, w_2 \in U_2$ and Q_1, Q_2 be the set of neighbors of w_1, w_2 in W^r , respectively. Every vertex of W^r is adjacent to at least one vertex of K .

Let $s_1 \in Q_1 \cap Q_2$ such that s_1 is adjacent to a vertex in K and $s_2 \in Q_1$ is adjacent to two vertices in K . The graph induced by $K \cup U_1 \cup U_2 \cup \{s_1, s_2\}$ contains a subgraph W_7 .

If there are no vertices in $Q_1 \cap Q_2$ adjacent to one vertex in K then every vertex in Q_1 or Q_2 is adjacent to two vertices in K . Without loss of generality, let Q_1 be such a set. It is easy to see that the set Q_1 is independent. The maximal degree of w_1 in $G[K \cup U]$ is 4. From the assumption $\delta(G) \geq k + 1$, we conclude $|Q_1| \geq k - 3$. Let $X = W^r \setminus Q_1$. Since each vertex from Q_1 has degree at least $k + 1$ in G and Q_1 is independent, we conclude $|X| \geq k - 3, |Q_1| = |X| = k - 3$ and w_1 is adjacent to w_2 . If $Q_1 \neq Q_2$, then a vertex from $Q_2 \setminus Q_1$, any two vertices from Q_1 , vertices w_1, w_2 and K induce a graph which contains W_7 as a subgraph. Since $k \geq 7$, we have that $\Delta(G[X]) \leq 2$. From all previous considerations we have $e(G) \leq 6 + 7 + 2(2k - 6) + 2(k - 3) + (k - 3) + (k - 3)(k - 3) = k^2 + k + 1$, a contradiction.

Subcase 3.4 $|U_1 \cup U_2| > 2$

Let $W^2 = \{v \in W : d_{G[K, W]}(v) = 2\}$, $W^1 = \{v \in W : d_{G[K, W]}(v) \leq 1\}$ and $U = U_1 \cup U_2$. At least one of the sets U_1, U_2 has order greater than or equal to 2, say U_1 is such a set. Let $u_1, u_2 \in U_1$. If there exist vertices $w_1, w_2 \in W^2$ (not necessarily different) such that u_1 is adjacent to w_1 and u_2 is adjacent to w_2 , then the graph $G[K \cup \{u_1, u_2, w_1, w_2\}]$ contains W_7 . In the opposite case, one of the vertices u_1, u_2 is not adjacent to any vertex from W^2 and since U is an independent set, we have $|W^1| \geq k - 2$. By the inequalities $e(K) + 3|U| + 2|W^2| + |W^1| + \text{ex}(2k - 4, W_7) \geq e(G)$ and $|U| + |W^2| + |W^1| = 2k - 4$, we have $|U| \geq |W^1| + 1$, so $|U| + |W^1| \geq 2k - 3$, a contradiction.

Case 4. $\omega(G) = 3$

Let $K = \{v_1, v_2, v_3\}$ be the clique in G and the remaining vertices are W . Let U_i be a set of all vertices from W such that they are adjacent to vertices $K - v_i$. This means that if $v \in U_i$ then $d_{G[K, W]}(v) = 2$. To avoid K_4 all U_i are independent. Let the remaining vertices of W be W^r and $U_1 \cup U_2 \cup U_3 = U$.

First observe that if there is a $K_2 \cup K_2$ between U_i and U_j where $i \neq j \in \{1, 2, 3\}$, then we immediately have W_7 . Since $3(\delta(G) - 2) \leq e(G[K, W]) \leq (2k - 3 - |U|) + 2|U|$, we have $|U| \geq k$. There exists a vertex in U adjacent to at most two vertices in U . This vertex is adjacent to at least $k - 3$ vertices in W^r . The equalities $|U| = k$ and $|W^r| = k - 3$ are obtained by the above inequalities and the property $|W^r| + |U| = 2k - 3$.

If there is a vertex of degree at most 1 in U , then we have a contradiction with $\delta(G) = k + 1$. Since graphs $G[U_i \cup U_j]$ do not contain $K_2 \cup K_2$, the only graph with the property is $K_{k-2, 1, 1}$.

Note that all vertices of degree 2 in U are joined to every vertex of W^T but none of the vertices of degree $k - 1$ in U are joined to any of vertices W^T . Moreover, to avoid W_7 we have $\Delta(G[W^T]) \leq 2$, so none of the vertices in G has degree greater than $k + 1$, a contradiction. \square

Corollary 10.

$$ex(n, W_7) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} + 1 \right\rfloor.$$

At the end of this subsection we enumerate all of the extremal graphs for $7 \leq n \leq 26$. An important property to generate these graphs is that if they exist, then they can be selected from the sets of all W_7 -free graphs with the number of edges greater than or equal to $\lceil \frac{n^2}{4} + \frac{n}{2} - 1 \rceil$. The sets were generated using the modified McKay's graph generation program `geng` [6].

For the cases when $n \in \{7, 8, 9\}$, the example of the extremal graph is $C_4 + (K_2 \cup (n - 6)K_1)$. More precisely, the sets $EX(n, W_7)$ for these values of n are as follows:

- $EX(7, W_7) = \{C_4 + (K_2 \cup K_1), K_2 + (K_4 \cup K_1)\}$
- $EX(8, W_7) = \{C_4 + (K_2 \cup 2K_1)\}$
- $EX(9, W_7) = \{C_4 + (K_2 \cup 3K_1), (K_3 \cup K_2) + (K_2 \cup 2K_1), (C_4 \cup K_1) + (K_2 \cup 2K_1), C_5 + 4K_1, 2C_3 + (K_2 \cup K_1)\}$.

3.3. $ex(n, W_{2k+1})$, where $n \geq 2k + 1$ and $k \geq 4$

Let us recall that we denote by aG the graph consisting of a disconnected subgraphs G . It is not hard to see that the graph $(K_2 \cup aK_1) + bK_k$ does not contain W_{2k+1} as a subgraph for all $a, b \in \mathbb{N}$. We will try to maximize the number of its edges. We need to determine the number of disconnected copies of K_k . Consider the situation when $b = \lfloor \frac{n+k+1}{2k} \rfloor$. In this case, $a = n - 2 - k \lfloor \frac{n+k+1}{2k} \rfloor$ and $ex(n, W_{2k+1}) \geq e(K_k)b + kb(n - kb) + 1$.

Theorem 11. Assume that $k \geq 4$ and $n \geq 2k + 1$. Then

$$ex(n, W_{2k+1}) \geq \left\lfloor \frac{n+k+1}{2k} \right\rfloor \left(\binom{k}{2} + kn - \left\lfloor \frac{n+k+1}{2k} \right\rfloor \right) + 1 > \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

Acknowledgments

We would like to thank the anonymous referees for their careful reading of this manuscript and many helpful comments. This research was partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

References

- [1] M. Bataineh, M. Jaradat, A. Jaradat, Edge maximal graphs containing no specific wheels, *Jordan J. Math. Statist.* 8 (2) (2015) 107–120.
- [2] T. Dzido, A note on Turán numbers for even wheels, *Graphs Combin.* 29 (5) (2013) 1305–1309.
- [3] T. Dzido, M. Kubale, K. Piwakowski, On some Ramsey and Turán-type numbers for paths and cycles, *Electron. J. Combin.* 13 (2006) #R55.
- [4] P. Erdős, Extremal problems in graph theory, *Theory of graphs and its applications*, in: Proc. Sympos. Smolenice, 1964, pp. 29–36.
- [5] W. Mantel, Problem 28, soln. by H. Gouventak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W.A. Wythoff, *Wiskundige Opgaven* 10 (1907) 60–61.
- [6] B.D. McKay, A. Piperno, Practical graph isomorphism, {II}, *J. Symbolic Comput.* 60 (2014).
- [7] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: *Theory of Graphs*, Academic Press, New York, 1968, pp. 279–319.
- [8] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* 48 (1941) 436–452.