

## ON INCIDENCE COLORING OF COMPLETE MULTIPARTITE AND SEMICUBIC BIPARTITE GRAPHS<sup>1</sup>

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### Abstract

In the paper, we show that the incidence chromatic number  $\chi_i$  of a complete  $k$ -partite graph is at most  $\Delta+2$  (i.e., proving the *incidence coloring conjecture* for these graphs) and it is equal to  $\Delta+1$  if and only if the smallest part has only one vertex (i.e.,  $\Delta = n-1$ ). Formally, for a complete  $k$ -partite graph  $G = K_{r_1, r_2, \dots, r_k}$  with the size of the smallest part equal to  $r_1 \geq 1$  we have

$$\chi_i(G) = \begin{cases} \Delta(G) + 1 & \text{if } r_1 = 1, \\ \Delta(G) + 2 & \text{if } r_1 > 1. \end{cases}$$

In the paper we prove that the incidence 4-coloring problem for semicubic bipartite graphs is  $\mathcal{NP}$ -complete, thus we prove also the  $\mathcal{NP}$ -completeness of  $L(1, 1)$ -labeling problem for semicubic bipartite graphs. Moreover, we observe that the incidence 4-coloring problem is  $\mathcal{NP}$ -complete for cubic graphs, which was proved in the paper [12] (in terms of generalized dominating sets).

**Keywords:** incidence coloring, complete multipartite graphs, semicubic graphs, subcubic graphs,  $\mathcal{NP}$ -completeness,  $L(1, 1)$ -labelling.

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## 1. INTRODUCTION

In the following we consider connected simple graphs only, and use standard notations in graph theory. Let  $n$  be a positive integer and  $G = (V, E)$  be any  $n$ -vertex graph of the maximum degree<sup>2</sup>  $\Delta(G) > 0$ . A pair  $(u, \{u, v\})$  is an *incidence* of  $G$  if and only if  $u, v \in V$  and  $\{u, v\} \in E$ . The set of all incidences of  $G$  will be denoted by  $I(G)$ . To shorten the notation, we will write  $uv$  instead of  $(u, \{u, v\})$ . We will say that incidence  $uv$  *leads from*  $u$  *to*  $v$ . We will say that incidences  $uv \neq wx$  are *adjacent* in  $G$  if and only if one of the following holds: (1)  $u = w$ ; (2)  $u = x$  and  $v = w$ ; (3) ( $u \neq x$  and  $v = w$ ) or ( $u = x$  and  $v \neq w$ ), which is equivalent to  $u = x$  or  $u = w$  or  $v = w$ . Obviously, if  $uv$  is adjacent to  $wx$ , then  $v \neq x$ .

A function  $c: I(G) \rightarrow \mathbb{N}$  is an *incidence coloring* of  $G$  if and only if  $c(uv) \neq c(wx)$  for all adjacent incidences  $uv$  and  $wx$ . The *incidence coloring number* of  $G$ , denoted by  $\chi_i(G)$ , is the smallest integer  $k$  such that there is an incidence coloring  $c$  of  $G$  using exactly  $k$  colors. By the *incidence  $k$ -coloring*, we mean an incidence coloring  $c$  of  $G$  with  $k$  colors (i.e.,  $k = |c(I(G))|$ ), and by the *incidence  $k$ -coloring problem* we mean a decision problem of the existence of the incidence  $k$ -coloring in a graph  $G$ .

The notion of the incidence coloring was introduced in [3]. In [10] the author observed that the problem of incidence graph coloring is a special case of the star arboricity problem, i.e., the problem of partitioning of a set of arcs of a symmetric digraph into the smallest number of forests of directed stars. That problem was studied in [1, 2, 10].

The following bounds are well-known (see [3, 17]).

**Proposition 1.** *For every graph  $G$  of order  $n \geq 2$  and  $\Delta(G) > 0$  there is*

$$\Delta(G) + 1 \leq \chi_i(G) \leq n.$$

The upper bound  $\chi_i(G) \leq 2\Delta(G)$  for every graph  $G$  has been proved in [3]. This bound has been improved in [10], where the author proved that  $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$  for every graph  $G$ .

### 1.1. Motivation and our results

In [3] the authors conjectured that  $\chi_i(G) \leq \Delta(G) + 2$  holds for every graph  $G$  (*incidence coloring conjecture*, shortly ICC). This was disproved by Guiduli in [10] who showed that Paley graphs have incidence coloring number at least  $\Delta + \Omega(\log \Delta)$ . For the following classes of graphs the incidence coloring number is at most  $\Delta + 2$ : trees and cycles [3], complete graphs [3] and complete bipartite

<sup>2</sup>We sometimes write  $\Delta$  instead of  $\Delta(G)$  whenever  $G$  is clear from the context.

graphs [5] and [3] (proof corrected in [17]). In fact, for all of them the exact value is equal to  $\Delta + 1$  or  $\Delta + 2$  and the optimal coloring is constructed in polynomial time. In [7] the authors proved that any partial 2-tree (i.e.,  $K_4$ -minor free graph) admits  $\Delta + 2$  incidence coloring, hence for all outerplanar graphs ICC holds. In [8] the authors proved that every planar graph with girth at least 11 or with girth at least 6 and maximum degree at least 5 has incidence coloring number at most  $\Delta + 2$ . Recently, the conjecture was proved for pseudo-Halin graphs [11], some powers of cycles [15] and hypercubes [16].

In the paper [19] the authors claim that the incidence coloring conjecture holds for complete multipartite graphs, but the coloring presented in the proof of Theorem 3.1 in [19] is incorrect and presented without a full proof. In Section 2 we will show that the incidence coloring number of complete  $k$ -partite graphs is at most  $\Delta + 2$ , and is equal to  $\Delta + 1$  if and only if the size of the smallest part equals 1. We present an  $O(n^2)$ -time algorithm giving an optimal coloring (i.e., with the minimum number of colors).

In [6] the authors proved that ICC holds for some subclasses of cubic graphs (e.g. Hamiltonian cubic graphs). In [18] the author proved that ICC holds for cubic graphs having a Hamiltonian path and for bridgeless cubic graphs of high girth. At last, in [14] the author proved that ICC holds for subcubic graphs. In [13] the authors proved  $\mathcal{NP}$ -completeness of the incidence 4-colorability of *semicubic* graphs (i.e., graphs with  $\Delta = 3$  and vertices of degree equal to 1 or 3). By the paper [12] we conclude that the incidence 4-coloring problem is  $\mathcal{NP}$ -complete for cubic graphs. The complexity of this problem was unknown for (semicubic) bipartite graphs. In Section 3 we will show that incidence 4-colorability of semicubic bipartite graphs is  $\mathcal{NP}$ -complete.

## 2. INCIDENCE COLORING OF COMPLETE MULTIPARTITE GRAPHS

In this section we present an algorithm (formula) giving a coloring of a multipartite graph using at most  $\Delta + 2$  colors.

In [19] the authors presented Theorem 3.1 claiming that the incidence coloring conjecture holds for complete multipartite graphs. The coloring  $\sigma$  presented in the proof of Theorem 3.1 is incorrect and in fact there is no proof that this coloring is a proper incidence coloring and uses at most  $\Delta + 2$  colors. In the coloring definition (3.2) [19] the authors use the formula  $\sum_{m=0}^{t-1} (n_m + s)$ , but  $n_0$  is undefined, so we believe it should be corrected to  $m = 1$ . But in this case, following the notation from [19], for  $k \geq 3$  let  $t = k - 1$  and  $s = n_t$ . Take any  $j < t$ , hence for  $n_j \geq n_t$  we can put  $i = s$ , thus we get  $\sigma(v_s^j, v_s^j v_s^{k-1}) = \sum_{m=1}^{k-2} (n_m + s) = \sum_{m=1}^{k-1} n_m + (k-3)n_{k-1} = \Delta + (k-3)n_{k-1} > \Delta + 2$ , for  $k \geq 6$  or  $k = 5$  and  $n_{k-1} > 1$ .



In the following, we present a different coloring than the coloring  $\sigma$  from [19].

Let  $G = K_{r_1, r_2, \dots, r_k}$  be a complete  $k$ -partite graph with  $V(G) = V_1 \cup \dots \cup V_k$ , where integer  $k \geq 1$ ,  $|V_i| = r_i$ , for each  $i \in \{1, \dots, k\}$ , and all  $V_i$  are independent sets and pairwise disjoint.

**Theorem 2.** *For any complete multipartite graph  $G = K_{r_1, r_2, \dots, r_k}$  with  $k \geq 2$ , there is*

$$\chi_i(G) = \Delta(G) + 1 \text{ if and only if } r_1 = 1,$$

where  $r_1 = \min\{r_1, \dots, r_k\}$ .

**Proof.** ( $\Rightarrow$ ) Let  $c$  be an incidence coloring of  $G$  that uses  $\Delta + 1$  colors. Suppose that  $r_1 > 1$ . Let  $u \neq v$  be two vertices that belong to  $V_1$ . Then

- $u$  is of degree  $\Delta$  and  $c$  uses exactly  $\Delta + 1$  colors, so all incidences that lead to  $u$  must have the same color, say  $a$ ;
- $v$  is of degree  $\Delta$  and  $c$  uses exactly  $\Delta + 1$  colors, so all incidences that lead to  $v$  must have the same color, say  $b$ ;
- $a \neq b$  since  $u$  and  $v$  have identical neighborhoods;
- incidences that lead from  $u$  also lead to vertices that are adjacent to  $v$ , so colors of incidences leading from  $u$  must differ from  $b$ .

Hence  $c$  uses at least  $\Delta + 2$  colors:  $a$ ,  $b$  and  $\Delta$  other colors on incidences that lead from  $u$ , a contradiction.

( $\Leftarrow$ ) It follows immediately from Proposition 1, since  $\Delta(G) = |V(G)| - r_1$ . ■

**Theorem 3.** *For any complete multipartite graph  $G = K_{r_1, r_2, \dots, r_k}$  with  $k \geq 2$ , there is*

$$\chi_i(G) \leq \Delta(G) + 2.$$

**Proof.** Let us assume that  $1 \leq r_1 \leq r_2 \leq \dots \leq r_k$  and  $V(G) = V_1 \cup \dots \cup V_k$ ,  $|V_i| = r_i$ , and sets from  $\{V_i\}_{i \in \{1, \dots, k\}}$  are independent sets and pairwise disjoint. Let  $n = |V(G)|$ . It suffices to show that there is an incidence coloring of  $G$  that uses at most  $\Delta + 2$  colors. Before we do this, we have to introduce some notations.

Let  $s_i: V_i \rightarrow \{1, 2, \dots, r_i\}$  be any numbering of vertices of  $V_i$  for  $i \in \{1, \dots, k\}$ . Let  $p: V(G) \rightarrow \{1, 2, \dots, k\}$  be a function such that  $p(u) = i$  if and only if  $u \in V_i$ . Let

$$V(G) \ni u \mapsto s(u) = r_k + 1 - s_{p(u)}(u) \in \{1, 2, \dots, r_k\}$$

and

$$\{1, 2, \dots, r_k\} \ni j \mapsto l(j) = \sum_{i=1}^{j-1} (|s^{-1}(\{i\})| - \lfloor |s^{-1}(\{i\})|/k \rfloor) \in \mathbb{N} \cup \{0\}.$$

It is easy to see the following properties.

- (1)  $u = v$  if and only if  $p(u) = p(v)$  and  $s(u) = s(v)$ ;
- (2)  $1 \leq |s^{-1}(\{1\})| \leq |s^{-1}(\{2\})| \leq \dots \leq |s^{-1}(\{r_k\})| = k$ ;
- (3)  $|s^{-1}(\{i\})| = k$  if and only if  $r_k \geq i \geq r_k - r_1 + 1$ ;
- (4)  $k + 1 - |s^{-1}(\{s(v)\})| \leq p(v) \leq k$ .

Now we are ready to construct the required incidence coloring  $c$ . We define it in two steps. First, we define it on incidences  $uv$  such that  $s(u) = s(v)$ :

$$c(uv) = \begin{cases} \Delta + 2 & \text{if } p(v) = k, \\ k + 1 - p(v) + l(s(v)) & \text{if } p(u) > p(v) > k + 1 - |s^{-1}(\{s(v)\})|, \\ k - p(v) + l(s(v)) & \text{if } p(u) < p(v) < k, \\ \Delta + 1 & \text{if } p(v) = k + 1 - |s^{-1}(\{s(v)\})| \neq k. \end{cases}$$

Next, we extend it to other incidences by the formula:

$$c(uv) = \begin{cases} k + 1 - p(v) + l(s(v)) & \text{if } p(u) < p(v), \\ k - p(v) + l(s(v)) & \text{if } p(u) > p(v). \end{cases}$$

Since  $p(u) \neq p(v)$  for all incidences  $uv$ , the above formula determines the value of  $c(uv)$  for all incidences of  $G$ . To complete the proof, it suffices to show that  $c(I(G)) \subseteq \{1, 2, \dots, \Delta + 2\}$  and  $c$  is an incidence coloring of  $G$ .

It is easy to see that  $c \geq 1$ . On the other hand,  $\Delta = n - r_1 = n - |\{i: |s^{-1}(\{i\})| = k\}| = \sum_{i=1}^{r_k} (|s^{-1}(\{i\})| - \lfloor |s^{-1}(\{i\})|/k \rfloor) = |s^{-1}(\{r_k\})| - 1 + l(r_k) \geq |s^{-1}(\{s(v)\})| - 1 + l(s(v)) \geq k - p(v) + l(s(v))$ , so  $c \leq \Delta + 2$ . Moreover,  $k - p(v) + l(s(v)) = \Delta$  if and only if  $s(v) = r_k$  and  $p(v) = 1$ . As an easy consequence we get that  $c^{-1}(\{\Delta + 1\})$  and  $c^{-1}(\{\Delta + 2\})$  are independent sets.

Suppose that  $c$  is not an incidence coloring of  $G$ . Then  $c(uv) = c(wx) \leq \Delta$  for some adjacent incidences  $uv, wx$ . Without loss of generality we assume  $s(x) \geq s(v)$ . There are several cases to consider.

- $s(x) \geq s(v) + 2$  and  $|s^{-1}(\{s(v) + 1\})| = k$ .

Then  $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) \geq l(s(v) + 2) \geq |s^{-1}(\{s(v)\})| + |s^{-1}(\{s(v)+1\})| - 2 + l(s(v)) \geq |s^{-1}(\{s(v)\})| + l(s(v))$ . By (4) we have  $|s^{-1}(\{s(v)\})| + l(s(v)) \geq k + 1 - p(v) + l(s(v)) \geq c(uv)$ . Since  $c(wx) = c(uv)$ , we get  $p(x) = k$  and  $c(wx) = k - p(x) + l(s(x))$ , a contradiction.

- $s(x) \geq s(v) + 2$  and  $|s^{-1}(\{s(v) + 1\})| < k$ .

Then  $|s^{-1}(\{s(v)\})| < k$  and  $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) \geq l(s(v) + 2) = |s^{-1}(\{s(v)\})| + |s^{-1}(\{s(v) + 1\})| + l(s(v)) > |s^{-1}(\{s(v)\})| + l(s(v)) \geq k + 1 - p(v) + l(s(v)) \geq c(uv)$ , a contradiction.

- $s(x) = s(v) + 1$  and  $c(uv) = k - p(v) + l(s(v))$ .

Then  $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) = l(s(v) + 1) \geq |s^{-1}(\{s(v)\})| - 1 + l(s(v)) \geq k - p(v) + l(s(v)) = c(uv)$ . These inequalities must be equalities since  $c(wx) = c(uv)$ . This gives  $p(x) = k$  and  $c(wx) = k - p(x) + l(s(x))$ , a contradiction.

•  $s(x) = s(v) + 1$  and  $c(uv) = k + 1 - p(v) + l(s(v))$  and  $|s^{-1}(\{s(v)\})| = k$ .

Then  $p(v) \geq 2$  and  $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) = l(s(v) + 1) = |s^{-1}(\{s(v)\})| - 1 + l(s(v)) = k - 1 + l(s(v)) \geq k + 1 - p(v) + l(s(v)) = c(uv)$ . These inequalities must be equalities since  $c(wx) = c(uv)$ . This gives  $p(x) = k$  and  $c(wx) = k - p(x) + l(s(x))$ , a contradiction.

•  $s(x) = s(v) + 1$  and  $c(uv) = k + 1 - p(v) + l(s(v))$  and  $|s^{-1}(\{s(v)\})| < k$ .

Then  $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) = l(s(v) + 1) = |s^{-1}(\{s(v)\})| + l(s(v)) \geq k + 1 - p(v) + l(s(v)) = c(uv)$ . These inequalities must be equalities since  $c(wx) = c(uv)$ . This gives  $p(x) = k$  and  $c(wx) = k - p(x) + l(s(x))$ , a contradiction.

•  $s(x) = s(v)$ .

$s(x) = s(v)$  implies  $p(x) \neq p(v)$ . Without loss of generality we assume that  $p(x) > p(v)$ . Then  $c(wx) \leq k + 1 - p(x) + l(s(x)) \leq k - p(v) + l(s(v)) = c(uv)$ , which gives  $c(wx) = k + 1 - p(x) + l(s(x))$ ,  $c(uv) = k - p(v) + l(s(v))$  and  $p(x) = p(v) + 1$ . There are 4 subcases to consider.

(a)  $s(u) = s(v)$ ,  $p(u) < p(v)$  and  $s(w) = s(x)$ ,  $p(w) > p(x)$ . Then  $p(w) > p(x) > p(v) > p(u)$ , which shows that  $u \neq x$ ,  $u \neq w$  and  $v \neq w$ , a contradiction.

(b)  $s(u) = s(v)$ ,  $p(u) < p(v)$  and  $s(w) \neq s(x)$ ,  $p(w) < p(x)$ . Then  $p(u) < p(x)$ ,  $s(u) \neq s(w)$  and  $s(v) \neq s(w)$ , which shows that  $u \neq x$ ,  $u \neq w$  and  $v \neq w$ , a contradiction.

(c)  $s(u) \neq s(v)$ ,  $p(u) > p(v)$  and  $s(w) = s(x)$ ,  $p(w) > p(x)$ . Then  $s(u) \neq s(x)$ ,  $s(u) \neq s(w)$  and  $p(w) > p(v)$ , which shows that  $u \neq x$ ,  $u \neq w$  and  $v \neq w$ , a contradiction.

(d)  $s(u) \neq s(v)$ ,  $p(u) > p(v)$  and  $s(w) \neq s(x)$ ,  $p(w) < p(x)$ . Then  $s(u) \neq s(x)$  which shows that  $u \neq x$ . If  $u = w$ , then  $p(v) < p(u) = p(w) < p(x) = p(v) + 1$ , which is impossible. Then  $v = w$  and  $s(x) \neq s(w) = s(v) = s(x)$ , a contradiction. ■

**Corollary 4.** Let  $G = K_{r_1, r_2, \dots, r_k}$  be a complete  $k$ -partite graph with  $k \geq 2$  and let  $r_1 = \min\{r_1, \dots, r_k\}$ . Then

$$\chi_i(G) = \begin{cases} \Delta(G) + 1 & \text{if } r_1 = 1, \\ \Delta(G) + 2 & \text{if } r_1 > 1. \end{cases}$$

### 3. $\mathcal{NP}$ -COMPLETENESS OF THE INCIDENCE 4-COLORING OF SEMICUBIC BIPARTITE GRAPHS

In this section we discuss the complexity results of the incidence 4-coloring problems and prove that the incidence 4-coloring problem for semicubic bipartite graphs is  $\mathcal{NP}$ -complete. By semicubic graphs we mean graphs with  $\Delta = 3$  and vertices of degree equal to 1 or 3.

**Theorem 5** [13]. *The incidence 4-coloring problem for semicubic graphs is  $\mathcal{NP}$ -complete.*

In fact, the authors observed in [13] that for semicubic graphs the problem of 2-distance coloring (i.e., a proper vertex coloring such that all vertices having a common neighbour are of distinct colors) is equivalent to the incidence 4-coloring problem. Indeed, for any incidence 4-coloring of a semicubic graph, the colors of incidences leading to a common vertex (say  $v$ ) are equal (say  $a$ ), hence we can assign the color  $a$  to the vertex  $v$ . Thus from the definition of adjacent incidences we get a proper 2-distance vertex coloring.

**Proposition 6** [13]. *For semicubic graphs the incidence 4-coloring problem is equivalent to the 2-distance 4-coloring problem.*

By an  $L(p, q)$ -labelling [4] we mean an assignment of nonnegative integers to the vertices of a graph such that adjacent vertices are labelled using colors at least  $p$  apart, and vertices having a common neighbour are labelled using colors at least  $q$  apart. By [4] any 2-distance vertex coloring of a graph is the same as its  $L(1, 1)$ -labelling, thus we have the following result.

**Proposition 7.** *For semicubic graphs the incidence 4-coloring problem is equivalent to the  $L(1, 1)$ -labelling problem with 4 colors.*

In [12] the authors introduced the concept of generalized dominating sets as follows. For a given graph  $G = (V, E)$  and two subsets  $\sigma$  and  $\rho$  of nonnegative integers, by a  $(\sigma, \rho)$ -set we mean any subset  $S \subset V$  such that for any  $v \in S$  we have  $|N(v) \cap S| \in \sigma$  and for any  $v \notin S$  there is  $|N(v) \cap S| \in \rho$ . By a  $(k, \sigma, \rho)$ -partition of  $V$  we mean a partition  $V_1 \cup \dots \cup V_k = V$  such that each  $V_i$  is the  $(\sigma, \rho)$ -set, for  $i = 1, 2, \dots, k$ . In [4] the author observed that any  $(k, \{0\}, \{0, 1\})$ -partition is equivalent to an  $L(1, 1)$ -labelling with  $k$  colors, thus we get the following.

**Proposition 8.** *For semicubic graphs the incidence 4-coloring problem is equivalent to the  $(4, \{0\}, \{0, 1\})$ -partition problem.*

In [12] the authors proved that the  $(4, \{0\}, \{0, 1\})$ -partition problem is  $\mathcal{NP}$ -complete for cubic graphs, thus by Proposition 8 we have the following theorem.

**Theorem 9.** *The incidence 4-coloring problem of cubic graphs is  $\mathcal{NP}$ -complete.*

In the following, we use the  $\overline{\text{X3C}}$  problem, which is  $\mathcal{NP}$ -complete [9].

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$\overline{\text{X3C}}$

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**Instance:** A subcubic bipartite graph  $G = (V \cup M, E)$  without pendant vertices, such that  $|V| = 3q$  and for every vertex  $m \in M$  we have  $\deg(m) = 3$  and  $m$  is adjacent to three vertices from  $V$ .

**Question:** Is there a subset  $M' \subset M$  of cardinality  $|M'| = q$  dominating all vertices in  $V$ ?

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**Theorem 10.** *The incidence 4-coloring problem for semicubic bipartite graphs is  $\mathcal{NP}$ -complete.*

**Proof.** The proof proceeds by the reduction from the problem  $\overline{\text{X3C}}$ . Let  $G = (V \cup M, E)$  be a subcubic bipartite graph such that  $|V| = 3q$  and for every vertex  $m \in M$  we have  $\deg(m) = 3$  and  $m$  is adjacent to exactly three vertices from  $V$ . We construct a semicubic bipartite graph  $G^*$  such that there is a subset  $M' \subset M$  of cardinality  $|M'| = q$  dominating all vertices in  $V$  if and only if there is a 2-distance 4-coloring of graph  $G^*$ , which by Proposition 6 is equivalent to the existence of an incidence 4-coloring of graph  $G^*$ .

Let  $n_2$  and  $n_3$  be the number of vertices in  $V$  of degree 2 and 3, respectively. Let us consider graphs  $H$  and  $H_i$  (for  $i = 2, 3, \dots$ ), shown in Figures 1, 2 and 3. Let  $H$  be a graph shown in Figure 1 (on the left-hand side) consisting of white vertices only (i.e., without vertices  $x$  and  $y$ ) and edges between them.

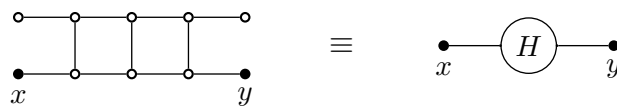


Figure 1. An auxiliary graph  $H$  ( $x, y \notin V(H)$ ).

Let  $H_2$  be a graph shown in Figure 2 (on the left-hand side) consisting of two isomorphic and disjoint copies of graph  $H$  with attached two white vertices, i.e., vertex  $y$  and its pendant neighbour. We assume that two vertices  $x_1$  and  $x_2$  do not belong to  $H_2$ . Let  $H_2^*$  be a graph shown in Figure 4, i.e., the graph  $H_2$  with attached two vertices  $x_1$  and  $x_2$ .

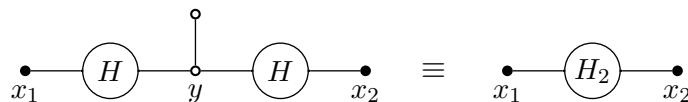


Figure 2. An auxiliary graph  $H_2$  ( $x_1, x_2 \notin V(H_2)$ ).



For each integer  $i \geq 2$ , let  $H_{i+1}$  be a graph shown in Figure 3 and constructed as follows: Take an isomorphic copy of graph  $H_i^*$ , i.e., the graph  $H_i$  with attached pendant vertices  $x'_1, \dots, x'_i$  (shown on the left-hand side in Figure 3) and add two disjoint isomorphic copies of graph  $H_2$  with attached two pendant vertices to each of them (in the manner as shown in Figure 2). Further, as shown in Figure 3, identify the vertex  $x'_i$  with two joined pendant vertices, and label by  $x_i$  and  $x_{i+1}$  the two others. Then, relabel  $x'_k$  with  $x_k$  for each  $k \in \{1, \dots, i-1\}$ . We assume that  $x_1, \dots, x_{i+1} \notin V(H_{i+1})$ .

Let  $H_{i+1}^*$  be a graph obtained from the graph  $H_{i+1}$  by attaching pendant vertices  $x_1, \dots, x_{i+1}$ , as shown in Figure 3 (on the right-hand side). For each integer  $i \geq 2$ , the graph  $H_i^*$  is bipartite and the vertices  $x_1, \dots, x_i$  are in the same partition. Moreover, the graph  $H_i^*$  is semicubic and 2-distance 4-colorable.

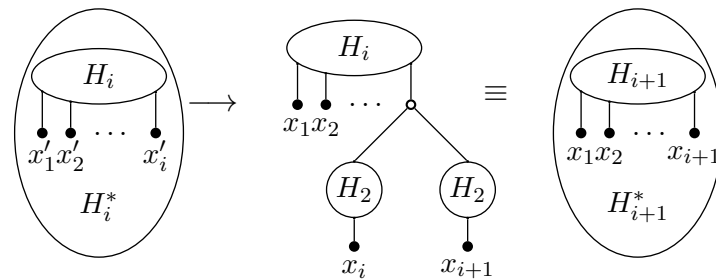


Figure 3. The iterative construction of auxiliary graphs  $H_{i+1}$  and  $H_{i+1}^*$  (for  $i = 2, 3, \dots$ ).

**Observation 11.** For every graph  $H_i^*$  ( $i \geq 2$ ), in every 2-distance 4-coloring of graph  $H_i^*$  the colors assigned to vertices  $x_1, \dots, x_i$  are equal.

**Proof.** Let  $i = 2$  and let  $c$  be any 2-distance 4-coloring of graph  $H_2^*$ . The graph  $H_2^*$  contains as a subgraph two copies of graph  $H$ . By a simple analysis, we leave it to the reader, we can prove that  $c(x_1) = c(y)$  and analogously  $c(y) = c(x_2)$ . By induction, the thesis follows for every  $i \geq 2$ .  $\square$

**Observation 12.** For every graph  $H_i^*$  ( $i \geq 2$ ), if we precolor vertices  $x_1, \dots, x_i$  with one color, say 1, and the neighbors of  $x_1, \dots, x_i$  with arbitrary colors from the set  $\{2, 3, 4\}$ , then we can extend this precoloring to a 2-distance 4-coloring of the whole graph  $H_i^*$ .

**Proof.** Let  $i = 2$  and let  $v_1$  and  $v_2$  be vertices neighboring in the graph  $H_2^*$  with vertices  $x_1$  and  $x_2$ , respectively. Let  $w_1$  be a neighbor of the interior vertex  $y$  (see Figure 2) that is at distance 2 from  $v_1$ , and, analogously, let  $w_2$  be a neighbor of  $y$  at distance 2 from  $w_2$ , which is shown in Figure 4.

Now, without loss of generality, let us assume that we precolor vertices  $x_1$  and  $x_2$  with color 1, and  $v_1$  with color 2, and  $v_2$  with color  $p$ , that may equal



either 2 or 3. In both cases, we color the vertex  $w_1$  with 3 and the vertex  $w_2$  with color 4, what is extendible to the whole graph  $H_2^*$ , which we leave to the reader. By induction, we have the thesis for every integer  $i \geq 2$ .  $\square$

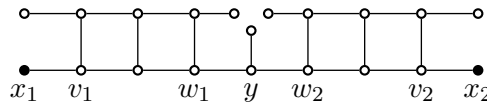


Figure 4. Graph  $H_2^*$ .

Let us consider graphs  $A_2$  and  $A_3$ , shown in Figures 5 and 6. By a detailed (but simple) analysis of graphs  $A_2$  and  $A_3$  we have the following results.

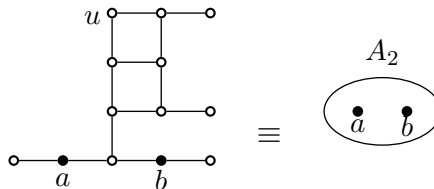


Figure 5. Graph  $A_2$ .

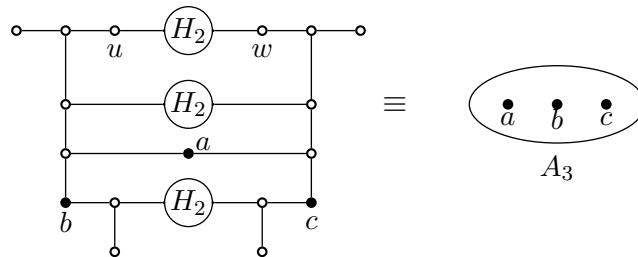


Figure 6. Graph  $A_3$ .

**Observation 13.** (i) *In every 2-distance 4-coloring of the graph  $A_2$  the colors assigned to vertices  $a$  and  $b$  are different and one of them is equal to the color of vertex  $u$ .*

(ii) *Any precoloring of vertices  $\{a, b, u\}$  of the graph  $A_2$ , where the colors assigned to vertices  $a$  and  $b$  are different, and either  $a$  or  $b$  has the same color as  $u$ , we can extend to a 2-distance 4-coloring of the graph  $A_2$ .*

**Observation 14.** (i) *In every 2-distance 4-coloring of the graph  $A_3$  the colors assigned to vertices  $a$ ,  $b$  and  $c$  are different and one of them is equal to the color of vertex  $u$ .*

(ii) Any precoloring of vertices  $\{a, b, c, u\}$  of the graph  $A_3$ , where the colors assigned to vertices  $a, b, c$  are different, and either  $a, b$  or  $c$  has the same color as  $u$ , we can extend to a 2-distance 4-coloring of graph  $A_3$ .

We will transform (in polynomial time) the graph  $G$  into  $G^*$  in four steps:

1. each vertex  $v \in V$  of degree 2 and neighbors  $m_1, m_2 \in M$  replace with a graph  $A_2(v)$  (isomorphic to  $A_2$ ) and add two edges  $\{m_1, a\}$  and  $\{m_2, b\}$ ,
2. each vertex  $v \in V$  of degree 3 and neighbors  $m_1, m_2, m_3 \in M$  replace with a graph  $A_3(v)$  (isomorphic to  $A_3$ ) and add three edges  $\{m_1, a\}$ ,  $\{m_2, b\}$  and  $\{m_3, c\}$ ; graphs of both types ( $A_2(v)$  or  $A_3(v)$ ) we call further  $A$ -graphs,
3. each vertex  $m \in M$  replace with a graph  $H_3^*(m)$  (isomorphic to  $H_3^*$ ) and identify three neighbors of  $m$  (in an  $A$ -graph) with vertices  $x_1, x_2, x_3 \in V(H_3^*(m))$ ,
4. attach a graph  $H_p^*$ , where  $p = 2n_3 + n_2$  and uniquely identify the pendant vertices  $x_1, \dots, x_p \in V(H_p^*)$  with vertices  $u$  and  $w$  in all  $A$ -graphs.

It is easy to observe that the graph  $G^*$  is a semicubic bipartite graph. By Observation 11 and Observations 13(i) and 14(i) we have the following.

**Observation 15.** *In every 2-distance 4-coloring of the graph  $G^*$  the same color (say 1) is assigned to vertices  $u$  and  $w$  in all  $A$ -graphs, and in every  $A$ -graph there is exactly one vertex of  $a, b, c$  colored with 1.*

( $\Rightarrow$ ) Suppose,  $M' \subset M$  dominates all vertices in  $V$  and  $|M'| = q$ . We construct a 2-distance 4-coloring of graph the  $G^*$  as follows: (1) for every  $m \in M'$  color with 1 vertices  $x_1, x_2, x_3$  from the graph  $H_3^*(m)$ , (2) color with 1 vertices  $u$  and  $w$  in all  $A$ -graphs. Let us notice that after removing set of vertices  $M'$  from the graph  $G$ , each vertex from  $V$  in the result graph is of degree 1 or 2, thus (3) for every  $m \in M \setminus M'$  we can color vertices  $x_1, x_2, x_3$  from the graph  $H_3^*(m)$  with either 2, 3 or 4 (by Brooks theorem). By Observation 12 and Observations 13(ii) and 14(ii) we can extend this precoloring to the 2-distance 4-coloring of the whole graph  $G^*$ .

( $\Leftarrow$ ) Let  $c$  be any 2-distance 4-coloring of the graph  $G^*$ . By Observation 15 the colors assigned to vertices  $u$  and  $w$  in all  $A$ -graphs are equal (say 1). Moreover, by Observation 15 there is exactly one vertex from  $\{a, b\}$  in every graph  $A_2(v)$  and exactly one vertex from  $\{a, b, c\}$  in every graph  $A_3(v)$  colored with 1, thus the set of all vertices  $m \in M$  such that the corresponding graph  $H_3^*(m)$  has vertices  $x_1, x_2, x_3$  colored with 1, is the solution to the  $\overline{X3C}$  problem. ■

By Proposition 7 we have the following.

**Corollary 16.** *The  $L(1, 1)$ -labelling problem with 4 colors for bipartite semicubic graphs is  $\mathcal{NP}$ -complete.*



The complexity of the incidence 4-coloring problem (and equivalently, the  $L(1,1)$ -labelling problem with 4 colors) for cubic bipartite graphs remains unknown.

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