Discussiones Mathematicae
Graph Theory xx (xxxx) 1–15
doi:10.7151/dmgt.2026

TOTAL DOMINATION VERSUS PAIRED-DOMINATION IN REGULAR GRAPHS

Joanna Cyman[1], Magda Dettlaff[1]

Michael A. Henning[2*], Magdalena Lemańska[1]

and

Joanna Raczek[1]

[1] *Department of Mathematics*
Gdańsk University of Technology

[2] *Department of Pure and Applied Mathematics*
University of Johannesburg
South Africa

e-mail: {joana,magda,mdettlaff,gardenial}@mif.pg.gda.pl
mahenning@uj.ac.za

Abstract

A subset S of vertices of a graph G is a dominating set of G if every vertex not in S has a neighbor in S, while S is a total dominating set of G if every vertex has a neighbor in S. If S is a dominating set with the additional property that the subgraph induced by S contains a perfect matching, then S is a paired-dominating set. The domination number, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G, while the minimum cardinalities of a total dominating set and paired-dominating set are the total domination number, $\gamma_t(G)$, and the paired-domination number, $\gamma_{\rm pr}(G)$, respectively. For $k \ge 2$, let G be a connected k-regular graph. It is known [Schaudt, *Total domination versus paired domination*, Discuss. Math. Graph Theory **32** (2012) 435–447] that $\gamma_{\rm pr}(G)/\gamma_t(G) \le (2k)/(k+1)$. In the special case when $k = 2$, we observe that $\gamma_{\rm pr}(G)/\gamma_t(G) \le 4/3$, with equality if and only if $G \cong C_5$. When $k = 3$, we show that $\gamma_{\rm pr}(G)/\gamma_t(G) \le 3/2$, with equality if and only if G is the Petersen graph. More generally for $k \ge 2$, if G has girth at least 5 and satisfies $\gamma_{\rm pr}(G)/\gamma_t(G) = (2k)/(k+1)$, then we show that G is a diameter-2 Moore graph. As a consequence of this result, we prove that for $k \ge 2$ and $k \ne 57$, if G has girth at least 5, then

*Research supported in part by the South African National Research Foundation and the University of Johannesburg.

$\gamma_{\rm pr}(G)/\gamma_t(G) \le (2k)/(k+1)$, with equality if and only if $k=2$ and $G \cong C_5$ or $k=3$ and G is the Petersen graph.

Keywords: domination, total domination, paired-domination.

2010 Mathematics Subject Classification: 05C69, 05C99.

1. Introduction

In this paper we continue the study of total domination and paired-domination in graphs. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [15, 16].

A vertex v is said to *dominate* a vertex u in a graph G if $u = v$ or if u and v are neighbors in G. A *dominating set* of G is a subset S of vertices of G such that every vertex outside S is dominated by at least one vertex in S. A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. We refer to a minimum total dominating set of G as a $\gamma_t(G)$*-set*. For a recent book on total domination in graphs we refer the reader to [21]. A survey of total domination in graphs can also be found in [20].

A set of edges in a graph G is *independent* if no two edges in it are adjacent in G; that is, an independent edge set is a set of edges without common vertices. A *matching* in a graph G is a set of independent edges in G. The *matching number* of a graph G, denoted $\alpha'(G)$, is the maximum cardinality of a matching in G. A *perfect matching* M is a matching such that every vertex of G is incident to an edge of M.

A *paired-dominating set* of G is a dominating set S of G with the additional property that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). The *paired-domination number* of G, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a paired-dominating set in G. Paired-domination was introduced by Haynes and Slater [17, 18] as a model for assigning backups to guards for security purposes, and is well-studied in graph theory. Recent papers on paired-domination can be found, for example, in [1, 2, 4–6, 8–12, 14, 19, 23].

Let $S \subseteq V(G)$ be a subset of vertices in G and for the following definitions let v be a vertex in S. The *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. The *open neighborhood* of S is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we often omit it in the given expressions. For example, we write V rather than $V(G)$, and $N(v)$ rather than $N_G(v)$.

The S-*external private neighborhood* of v, abbreviated $\mathrm{epn}(v, S)$, is the set of all vertices outside S that are adjacent to v but to no other vertex of S; that is, if $w \in \mathrm{epn}(v, S)$, then $w \in V \setminus S$ and $N_G(w) \cap S = \{v\}$. We define an S-*external private neighbor* of v to be a vertex in $\mathrm{epn}(v, S)$.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. We say that G is a *diameter*-2 *graph* if $\mathrm{diam}(G) = 2$.

A graph G is k-*regular* if every vertex has degree k in G. A *regular graph* is a graph that is k-regular for some integer $k \geq 0$. We remark that 3-regular graphs are also called *cubic graphs* in the literature. The *girth*, $g(G)$, of a graph G is the length of a shortest cycle in G. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

2. Main Result

Schaudt [23] established the following upper bound on the ratio of the paired-domination number versus the total domination number.

Theorem 1 ([23])**.** *If G is a graph with no isolated vertex and maximum degree Δ, then*

$$\gamma_{\mathrm{pr}}(G) \leq \left(\frac{2\Delta}{\Delta + 1}\right) \gamma_t(G).$$

As remarked by Schaudt [23], the upper bound of Theorem 1 is tight for all $\Delta \geq 2$, as may be seen by letting G be the graph obtained from a star $K_{1,\Delta}$ by subdividing every edge exactly once. Such a graph G satisfies $\gamma_{\mathrm{pr}}(G) = 2\Delta$ and $\gamma_t(G) = \Delta + 1$. We observe that for this extremal family of graphs, the difference between the maximum and minimum degrees is large. In this paper, our focus is therefore on regular graphs.

We wish to determine the connected k-regular graphs that achieve equality in Schaudt's Theorem 1. We shall prove the following result.

Theorem 2. *For $k \geq 2$ and $k \neq 57$, if G is a connected k-regular graph of girth at least 5, then $\frac{\gamma_{\mathrm{pr}}(G)}{\gamma_t(G)} \leq \frac{2k}{k+1}$, with equality if and only if*

(a) *$k = 2$ and $G \cong C_5$, or*

(b) *$k = 3$ and G is the Petersen graph.*

3. Preliminary Lemmas

We shall need the following preliminary lemma about the matching number of a graph.

Lemma 3. *If G is a graph of order n with no isolated vertex and maximum degree Δ, then $\alpha'(G) \geq \frac{n}{\Delta+1}$, with equality if and only if every component of G is isomorphic to $K_{1,\Delta}$, or $\Delta = 2$ and every component of G is isomorphic to $K_{1,2}$ or K_3.*

Proof. We proceed by induction on $n \geq \Delta + 1$. If $n = \Delta + 1$, then $K_{1,\Delta}$ is a spanning subgraph of G and $\alpha'(G) \geq 1 = \frac{n}{\Delta+1}$. Further, if $\alpha'(G) = 1$, then $G = K_{1,\Delta}$ or $G = K_3$. This establishes the base case. Suppose that $n \geq \Delta + 2$, and that the result is true for all connected graphs of order less than n. Let G be a graph of order n with no isolated vertex. Further, let $\delta = \delta(G)$ and $\Delta = \Delta(G)$. By linearity, we may assume that G is connected, for otherwise we can apply the result to each component of G. If $\Delta = 1$, then $G = K_2$, contradicting the fact that $n \geq \Delta + 2 = 3$. Hence, $\Delta \geq 2$. Among all edges of G, let v_1v_2 be chosen so that $d(v_1) = \delta$, and, subject to this condition, $d(v_1) + d(v_2)$ is a minimum.

Let L denote the set of isolated vertices in $G - \{v_1, v_2\}$, and let $|L| = \ell \geq 0$. Since G has no isolated vertex, every vertex in L is adjacent to at least one of v_1 and v_2. If a vertex, say u, in L is adjacent to v_1 but not to v_2, then $d(u) = 1$, implying that $\delta = 1$. However, both u and v_2 are neighbors of v_1, and so $d(v_1) \geq 2$, contradicting the fact that $d(v_1) = \delta$. Hence, every vertex in L is adjacent to v_2, implying that $L \subseteq N(v_2) \setminus \{v_1\}$ and $\ell \leq \Delta - 1$.

Let G' be the isolate-free graph obtained from G by deleting v_1 and v_2, and deleting the resulting isolated vertices that belong to the set L. Let G' have order n'. As observed earlier, $G \setminus \{v_1, v_2\}$ contains at most $\Delta - 1$ isolated vertices, and so $n' \geq n - (\Delta + 1)$. Let $\Delta' = \Delta(G')$, and note that $\Delta' \leq \Delta$. Applying the inductive hypothesis to each component of G', we have that

$$\alpha'(G) \geq 1 + \alpha'(G') \geq 1 + \frac{n'}{\Delta' + 1} \geq 1 + \frac{n - (\Delta + 1)}{\Delta + 1} = \frac{n}{\Delta + 1}.$$

Suppose that $\alpha'(G) = \frac{n}{\Delta+1}$. Then we must have equality throughout the above inequality chain. In particular, $\ell = \Delta - 1$, implying that $d(v_2) = \Delta$ and $N(v_2) = L \cup \{v_1\}$. If in this case, a vertex, say w, in L is adjacent to v_1, then $d(w) = 2$, implying by our choice of the edge v_1v_2 that $d(v_2) = \Delta = 2$ and $G = K_3$. Therefore, we may assume that if $\ell = \Delta - 1$, then v_1 has no neighbor in L, implying that $G = K_{1,\Delta}$. This completes the proof of the lemma. ∎

As a consequence of Lemma 3, we have the following relationship between the paired-domination number and the total domination number of a graph. As remarked earlier, the upper bound in Lemma 4 is precisely Schaudt's Theorem 1 obtained in [23]. However, we present here a slightly stronger result and a different proof of Schaudt's bound in order to characterize the regular graphs that achieve equality in this upper bound.

Lemma 4. *If G is a graph with no isolated vertex and maximum degree Δ, then*

$$\gamma_{\rm pr}(G) \le \left(\frac{2\Delta}{\Delta+1}\right)\gamma_t(G).$$

Further, if $\gamma_{\rm pr}(G) = \left(\frac{2\Delta}{\Delta+1}\right)\gamma_t(G)$, then every minimum total dominating set in G induces a graph whose components are isomorphic to $K_{1,\Delta}$.

Proof. Let S be a minimum TD-set in G, and consider the subgraph H of G induced by S, that is, $H = G[S]$. We note that H has order $n(H) = |S|$. Further, H has no isolated vertex, and $\Delta(H) \le \Delta(G) = \Delta$. Let M be a maximum matching in H, and let $V(M)$ be the set of vertices incident with an edge of M. Thus, $|M| = \alpha'(H)$ and $|V(M)| = 2|M|$. By Lemma 3,

$$\alpha'(H) \ge \frac{n(H)}{\Delta(H)+1} \ge \frac{|S|}{\Delta+1}.$$

By the maximality of the matching M, the set $S \setminus V(M)$ is an independent set. By the minimality of the TD-set S, every vertex, v, in $S \setminus V(M)$ has a neighbor, v', outside S that is adjacent in G to v but to no other vertex of S. We call such a vertex v' an external S-private neighbor of v. Let

$$S' = \bigcup_{v \in S\setminus V(M)} \{v'\},$$

where the set S' is chosen in such a way that for each vertex $v \in S \setminus V(M)$ we choose exactly one external S-private neighbor v'. We now consider the set $D = S \cup S'$. Since $S \subseteq D$, the set D is a superset of a dominating set of G and is therefore itself a dominating set of G. Further, since $G[D]$ contains a perfect matching, the set D is a paired-dominating set of G. Thus,

$$\begin{aligned}\gamma_{\rm pr}(G) &\le |S| + |S'| = |S| + (|S| - |V(M)|) = 2|S| - 2\alpha'(H)\\ &\le 2|S| - \left(\tfrac{2}{\Delta+1}\right)|S| = \left(\tfrac{2\Delta}{\Delta+1}\right)|S| = \left(\tfrac{2\Delta}{\Delta+1}\right)\gamma_t(G).\end{aligned}$$

Suppose that $\gamma_{\rm pr}(G) = \left(\frac{2\Delta}{\Delta+1}\right)\gamma_t(G)$. Then, we must have equality throughout the above inequality chain. In particular, $\alpha'(H) = |S|/(\Delta+1)$. Thus, by Lemma 3, every component of H is isomorphic to $K_{1,\Delta}$, or $\Delta = 2$ and every component of H is isomorphic to $K_{1,2}$ or K_3. If $\Delta = 2$ and some component, say C, of H is isomorphic to K_3, then the component C is also a component of G. Every minimum TD-set of G contains exactly two vertices from every K_3-component of G. However, the set S contains all three vertices from the K_3-component C, a contradiction. Hence, every component of G is isomorphic to $K_{1,\Delta}$. ■

As a special case of Lemma 4, we have the following result.

Lemma 5. *For $k \geq 1$, if G is a k-regular graph, then*

$$\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} \leq \frac{2k}{k+1}.$$

Further, if $\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} = \frac{2k}{k+1}$, then every minimum total dominating set in G induces a graph whose components are isomorphic to $K_{1,k}$.

4. Small Values of k

In this section, we wish to determine the connected k-regular graphs achieving equality in the upper bound of Lemma 5 for small values of k. For $k \geq 1$, let G be a connected k-regular graph. By Lemma 5, $\gamma_{\rm pr}(G)/\gamma_t(G) \leq (2k)/(k+1)$. We wish to characterize such graphs G satisfying $\gamma_{\rm pr}(G)/\gamma_t(G) = (2k)/(k+1)$.

If $k = 1$, then $G = K_2$ and $\gamma_{\rm pr}(G) = \gamma_t(G) = 2$, and so $\gamma_{\rm pr}(G)/\gamma_t(G) = 1 = (2k)/(k+1)$. Hence, it is only of interest to consider the case when $k \geq 2$.

Suppose that $k = 2$ and that the connected k-regular graph G has order n. In this case, $G \cong C_n$ and by Lemma 5, $\gamma_{\rm pr}(G)/\gamma_t(G) \leq (2k)/(k+1) = 4/3$. Suppose that $\gamma_{\rm pr}(G)/\gamma_t(G) = 4/3$. By Lemma 5, every minimum TD-set in the cycle G induces a graph whose components are isomorphic to $K_{1,2}$. However, this is only the case when $G \cong C_5$, since if $G \cong C_n$ and $n \neq 5$, then we can always find a minimum TD-set in G that induces a graph with at least one component isomorphic to K_2. We state this formally as follows.

Theorem 6. *If G is a 2-regular connected graph, then*

$$\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} \leq \frac{4}{3}, \text{ with equality if and only if } G \cong C_5.$$

We next consider the case when $k \geq 3$. For this purpose, we shall need the following two results on the paired-domination number of a cubic graph.

Theorem 7 ([4])**.** *If G is a cubic graph of order n, then $\gamma_{\rm pr}(G) \leq \frac{3}{5}n$.*

Theorem 8 ([14])**.** *If G is a connected cubic graph of order n satisfying $\gamma_{\rm pr}(G) = \frac{3}{5}n$, then G is the Petersen graph* (*illustrated in Figure* 1).

We shall prove the following result.

Theorem 9. *If G is a connected cubic graph, then*

$$\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} \leq \frac{3}{2}, \text{ with equality if and only if } G \text{ is the Petersen graph.}$$

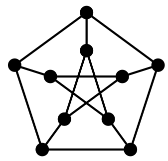

Figure 1. The Petersen graph.

Proof. If G is a cubic graph, then by Lemma 5, $\gamma_{\rm pr}(G)/\gamma_t(G) \le 3/2$. If G is the Petersen graph, then $\gamma_{\rm pr}(G) = 6$ and $\gamma_t(G) = 4$, and so the Petersen graph achieves the 3/2-ratio for the paired-domination number versus the total domination number. It suffices for us to prove that the Petersen graph is the unique such graph.

Suppose that G is a connected cubic graph of order n satisfying $\gamma_{\rm pr}(G)/\gamma_t(G) = 3/2$. By Lemma 5, every minimum TD-set in G induces a graph whose components are isomorphic to $K_{1,3}$. Let S be a minimum TD-set in G. Thus, $G[S]$ is the disjoint union of copies of $K_{1,3}$; that is, $G[S] = \ell K_{1,3}$ for some integer $\ell \ge 1$. We proceed further with the following claim.

Claim A. $G[S] = K_{1,3}$.

Proof. We wish to show that $\ell = 1$. Suppose, to the contrary, that $\ell \ge 2$. Let $G_1, G_2, \ldots, G_\ell$ be the components of $G[S]$, and so $G_i \cong K_{1,3}$ for all $i \in [\ell]$. Further, let $V(G_i) = \{v_i, v_{i1}, v_{i2}, v_{i3}\}$, where v_i is the central vertex of the star G_i. By the minimality of the TD-set S, every vertex v_{ij}, where $i \in [\ell]$ and $j \in [3]$, has an S-external private neighbor; that is, $\mathrm{epn}(v_{ij}, S) \ne \emptyset$. For each such vertex v_{ij}, let $v'_{ij} \in \mathrm{epn}(v_{ij}, S)$. Thus, $v'_{ij} \in V(G) \setminus S$ and $N(v'_{ij}) \cap S = \{v_{ij}\}$.

Let P be a shortest path in G that joins a vertex from one component of $G[S]$ to a vertex from another component of $G[S]$. Renaming components and vertices of $G[S]$ if necessary, we may assume that P is a (v_{11}, v_{21})-path. Thus, if P' is an arbitrary path in G that starts at a vertex in $V(G_i)$ and ends at a vertex in $V(G_j)$, where $1 \le i, j \le \ell$ and $i \ne j$, then P' has length at least that of P. By the minimality of the path P, every internal vertex of P belongs to $V(G) \setminus S$.

The path P has length at least 2, since vertices in different components of $G[S]$ are not adjacent. We show that the path P has length 2 or 3. Suppose, to the contrary, that P has length at least 4. Let $v_{11}xy$ be the subpath of P consisting of the first three vertices on P. By supposition, $y \ne v_{21}$. Since S is a TD-set of G, there is a vertex $z \in S$ that is adjacent to y in G. If $z \in V(G_1)$, then the path zy followed by the subpath of P from y to v_{21} is a shorter path than P joining vertices from different components of $G[S]$, a contradiction. Hence, $z \notin V(G_1)$, and so $v_{11}xyz$ is a path of length 3 joining vertices from different components of $G[S]$, contradicting our choice of the path P. Therefore, P has length 2 or 3.

Let $X = N_G(v_{11}) \setminus \{v_1\} = \{x_1, x_2\}$, where x_1 is the neighbor of v_{11} on the path P. We note that $|X| = 2$ and $X \subset V(G) \setminus S$. Further, no vertex in X is adjacent to a vertex v_j for any $j \in [\ell]$. If P has length 2, let $x_1^* = x_1$, while if P has length 3, let x_1^* be the common neighbor of x_1 and v_{21} on the path P. In both cases, we note that $v_{21}x_1^*$ is an edge of G. We now build a paired-dominating set S^* of G as follows.

Initially, we let S^* be obtained from S by removing the $\ell-1$ vertices $v_2, \ldots, v_\ell$, removing the vertex v_{11}, and adding the vertex x_1^*; that is, $S^* = (S \setminus \{v_{11}, v_2, \ldots, v_\ell\}) \cup \{x_1^*\}$.

Subclaim A.1. *The vertex x_2 is not dominated by S^*.*

Proof. Suppose that x_2 is dominated by S^*. In this case, we add to S^* the vertices v'_{ij} for all i and j, where $i \in [\ell]$, $j \in [3]$, and $(i,j) \notin \{(1,1),(1,2),(2,1)\}$. The resulting set S^* is a paired-dominating set of G, with v_1 and v_{12} paired, v_{21} and x_1^* paired, and with v_{ij} and v'_{ij} paired for all i and j, where $i \in [\ell]$, $j \in [3]$, and $(i,j) \notin \{(1,1),(1,2),(2,1)\}$. Further, $|S^*| = 6\ell - 2$, implying that $\gamma_{\rm pr}(G) \le |S^*| < 6\ell$. Recall that $\gamma_t(G) = 4\ell$. Thus, $\gamma_{\rm pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$, a contradiction. $\square$

By Subclaim A.1, the vertex x_2 is not dominated by S^*. Let x_2^* be a neighbor of x_2 different from v_{11} and x_1. Since x_2 is not dominated by S^*, we note that $x_2^* \notin S^*$. Since S is a TD-set of G and $x_2^* \notin X$, there is a vertex in $S \setminus \{v_{11}\}$ that is adjacent to x_2^*.

Subclaim A.2. *The vertex v_{21} is not adjacent to x_2^*.*

Proof. Suppose that v_{21} is adjacent to x_2^*, and so $N(v_{21}) = \{v_2, x_1^*, x_2^*\}$. If x_1x_2 is an edge, then we note that $x_1^* \neq x_1$, since x_2 is not dominated by S^*. Thus, P is the path $v_{11}x_1x_1^*v_{21}$. The set $S' = (S \setminus \{v_{11}, v_{21}\}) \cup \{x_1, x_2\}$ is a minimum TD-set in G that induces a graph with at least one component (namely the component containing the edge x_1x_2) that is not isomorphic to $K_{1,3}$, a contradiction. Therefore, x_1x_2 is not an edge. Thus the third neighbor of x_2, say x_2^{**}, that is different from v_{11} and x_2^*, is not the vertex x_1. Since x_2 is not dominated by S^*, we note that $x_2^{**} \notin S^*$. Thus there is a vertex in $S \setminus \{v_{11}\}$ that is adjacent to x_2^{**}. Let v_{st} be such a vertex in $S \setminus \{v_{11}\}$. Since x_2^{**} is not adjacent to v_{21}, we note that $(s,t) \notin \{(1,1),(2,1)\}$. We now add the vertex x_2^{**} to S^*.

If $(s,t) \in \{(1,2),(1,3)\}$, say $(s,t) = (1,2)$, then we add to S^* the vertex v'_{ij} for all i and j, where $i \in [\ell] \setminus \{1\}$, $j \in [3]$, and $(i,j) \neq (2,1)$. The resulting set S^* is a paired-dominating set of G, with v_1 and v_{13} paired, v_{12} and x_2^{**} paired, v_{21} and x_1^* paired, and with v_{ij} and v'_{ij} paired for all i and j, where $i \in [\ell] \setminus \{1\}$, $j \in [3]$, and $(i,j) \neq (2,1)$. Further, $|S^*| = 6\ell - 2$, implying that $\gamma_{\rm pr}(G) < 6\ell$. Thus, $\gamma_{\rm pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$, a contradiction.

If $(s,t) \notin \{(1,2),(1,3)\}$, then we add to S^* the vertices v'_{ij} for all i and j, where $i \in [\ell]$, $s \in [3]$, and $(i,j) \notin \{(1,1),(1,2),(2,1),(s,t)\}$. The resulting set S^* is a paired-dominating set of G, with v_1 and v_{12} paired, v_{21} and x_1^* paired, v_{st} and x_2^{**} paired, and with v_{ij} and v'_{ij} paired for all i and j, where $i \in [\ell]$, $s \in [3]$, and $(i,j) \notin \{(1,1),(1,2),(2,1),(s,t)\}$. Further, $|S^*| = 6\ell - 2$, implying that $\gamma_{\rm pr}(G) < 6\ell$. Thus, $\gamma_{\rm pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$, a contradiction. □

By Subclaim A.2, the vertex v_{21} is not adjacent to x_2^*. Let $v_{i'j'}$ be the vertex in $S \setminus \{v_{11}, v_{21}\}$ that is adjacent to x_2^*. Thus, $(i',j') \notin \{(1,1),(2,1)\}$.

If $(i',j') \in \{(1,2),(1,3)\}$, say $(i',j') = (1,2)$, then we add to S^* the vertex v'_{ij} for all i and j, where $i \in [\ell] \setminus \{1\}$, $j \in [3]$, and $(i,j) \neq (2,1)$. The resulting set S^* is a paired-dominating set of G, with v_1 and v_{13} paired, v_{12} and x_2^* paired, v_{21} and x_1^* paired, and with v_{ij} and v'_{ij} paired for all i and j, where $i \in [\ell] \setminus \{1\}$, $j \in [3]$, and $(i,j) \neq (2,1)$. Further, $|S^*| = 6\ell - 2$, implying that $\gamma_{\rm pr}(G) < 6\ell$. Thus, $\gamma_{\rm pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$, a contradiction.

If $(i',j') \notin \{(1,2),(1,3)\}$, then we add to S^* the vertices v'_{ij} for all i and j, where $i \in [\ell]$, $s \in [3]$, and $(i,j) \notin \{(1,1),(1,2),(2,1),(i',j')\}$. The resulting set S^* is a paired-dominating set of G, with v_1 and v_{12} paired, v_{21} and x_1^* paired, $v_{i'j'}$ and x_2^* paired, and with v_{ij} and v'_{ij} paired for all i and j, where $i \in [\ell]$, $s \in [3]$, and $(i,j) \notin \{(1,1),(1,2),(2,1),(i',j')\}$. Further, $|S^*| = 6\ell - 2$, implying that $\gamma_{\rm pr}(G) < 6\ell$. Thus, $\gamma_{\rm pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$, a contradiction. This completes the proof of Claim A. □

By Claim A, $G[S] = K_{1,3}$. In particular, we note that $\gamma_t(G) = 4$. Since $\gamma_{\rm pr}(G)/\gamma_t(G) = 3/2$, this implies that $\gamma_{\rm pr}(G) = 6$. Recall that $S = \{v_1, v_{11}, v_{12}, v_{13}\}$, where v_1 is the central vertex of the star $G_1 = G[S]$. Since G is a connected, cubic graph of order n, and every vertex in G is within distance 2 from v_1, we note that $n \leq 10$. By Theorem 7 and the fact that $n \leq 10$, we get

$$6 = \gamma_{\rm pr}(G) \leq 3n/5 \leq 6.$$

We must have equality throughout this inequality chain. In particular, $\gamma_{\rm pr}(G) = 3n/5$. Thus, by Theorem 8, G is the Petersen graph. This completes the proof of Theorem 9. ■

5. GENERAL VALUES OF k

In this section, we wish to determine the connected k-regular graphs that achieve equality in the upper bound of Lemma 5 for general values of k, given the requirement that the girth of the graph is at least 5.

In order to state our next result, we recall that the diameter-2 graphs of girth 5 are precisely the diameter-2 Moore graphs. It is shown (see [22, 25]) that Moore graphs are k-regular and that diameter-2 Moore graphs have order

$n = k^2 + 1$ and exist for $k = 2, 3, 7$ and possibly 57, but for no other degrees. (It is currently unknown whether there exists such a Moore graph for $k = 57$). The diameter-2 Moore graphs for the first three values of k are unique, namely

- the 5-cycle (2-regular graph on $n = 5$ vertices),
- the Petersen graph (3-regular graph on $n = 10$ vertices),
- the Hoffman-Singleton graph (7-regular graph on $n = 50$ vertices).

We show next that if we impose a girth condition, then every connected, regular graph achieving equality in the upper bound of Lemma 5 is a diameter-2 Moore graph.

Theorem 10. *For $k \ge 2$, if G is a connected k-regular graph of girth at least 5 satisfying $\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} = \frac{2k}{k+1}$, then G is a diameter-2 Moore graph.*

Proof. When $k = 2$ and $k = 3$, the result follows from Theorem 6 and Theorem 9, respectively (even without the girth condition). Hence, we may assume in what follows that $k \ge 4$. By Lemma 4, every minimum TD-set in G induces a graph whose components are isomorphic to $K_{1,k}$. Let S be a minimum TD-set in G. Thus, $G[S]$ is the disjoint union of copies of $K_{1,k}$; that is, $G[S] = \ell K_{1,k}$ for some integer $\ell \ge 1$. We show that $\ell = 1$; that is, $G[S] = K_{1,k}$.

Suppose, to the contrary, that $\ell \ge 2$. Let $G_1, G_2, \ldots, G_\ell$ be the components of $G[S]$, and so $G_i \cong K_{1,k}$ for all $i \in [\ell]$. Further, let $V(G_i) = \{v_i, v_{i1}, v_{i2}, \ldots, v_{ik}\}$, where v_i is the central vertex of the star G_i. By the minimality of the TD-set S, every vertex v_{ij}, where $i \in [\ell]$ and $j \in [k]$, has an S-external private neighbor; that is, $\mathrm{epn}(v_{ij}, S) \ne \emptyset$. For each such vertex v_{ij}, let $v'_{ij} \in \mathrm{epn}(v_{ij}, S)$. Thus, $v'_{ij} \in V(G) \setminus S$ and $N(v'_{ij}) \cap S = \{v_{ij}\}$.

Let P be a shortest path in G that joins a vertex from one component of $G[S]$ to a vertex from another component of $G[S]$. Renaming components and vertices of $G[S]$ if necessary, we may assume that P is a (v_{11}, v_{21})-path. Analogously as in the proof of Theorem 9, the path P has length 2 or 3.

Let $X = N_G(v_{11}) \setminus \{v_1\} = \{x_1, x_2, \ldots, x_{k-1}\}$, where x_1 is the neighbor of v_{11} on the path P. We note that $|X| = k - 1$ and that $X \subset V(G) \setminus S$. Further, the girth condition and the choice of the path P implies that no vertex in X is adjacent to a vertex in $V(G_1)$, except for the vertex v_{11}.

If P has length 2, let $x_1^* = x_1$, while if P has length 3, let x_1^* be the common neighbor of x_1 and v_{21} on the path P. In both cases, we note that $v_{21}x_1^*$ is an edge of G. Let $y_1^* = v_{21}$. We now build a paired-dominating set S^* of G as follows. Initially, we let S^* be obtained from S by removing the $\ell - 1$ vertices $v_2, \ldots, v_\ell$, removing the vertex v_{11}, and adding the vertex x_1^*; that is, $S^* = (S \setminus \{v_{11}, v_2, \ldots, v_\ell\}) \cup \{x_1^*\}$.

We now consider the vertices $x_2, x_3, \ldots, x_{k-1}$ in turn. For $i \in [k-1]$, let N_i be the set of $k-1$ neighbors of x_i different from v_{11}, and so $N_i = N_G(x_i) \setminus \{v_{11}\}$. Since G has girth at least 5, we note that N_i is an independent set and $N_i \cap V(G_1) = \emptyset$.

Further, $N_i \cap N_j = \emptyset$ for $i, j \in [k]$ and $i \neq j$.

If x_2 is dominated by S^*, then we add no new vertex to S^* associated with x_2, and we consider the next vertex x_3 in the list. If x_2 is not dominated by S^*, then we consider the set N_2. Since G has girth at least 5, at most one vertex in N_2 is a neighbor of y_1^*. Let x_2^* be a vertex in N_2 that is not adjacent to y_1^*. Since x_2 is not dominated by S^*, we note that $x_2^* \notin S^*$. Let y_2^* be a vertex in S that is adjacent to x_2^*. We note that $y_1^* \neq y_2^*$. We now add the vertex x_2^* to the set S^*.

Next, we consider the vertex x_3. If x_3 is dominated by S^*, then we add no new vertex to S^* associated with x_3, and we consider the next vertex x_4 in the list. If x_3 is not dominated by S^*, then we consider the set N_3. Since G has girth at least 5, at most one vertex in N_3 is a neighbor of y_1^* and at most one vertex in N_3 is a neighbor of y_2^*, if y_2^* exists. Hence, since $|N_3| = k - 1 > 2$, there is a vertex x_3^* in N_2 that is not adjacent to y_1^* and is not adjacent to y_2^*, if it exists. Since x_3 is not dominated by S^*, we note that $x_3^* \notin S^*$. Let y_3^* be a vertex in S that is adjacent to x_3^*. We note that y_1^*, y_2^* and y_3^* are distinct vertices, if they exists. We now add the vertex x_3^* to the set S^*.

We continue in the fashion until finally we consider the last vertex on the list, namely the vertex x_{k-1}. If x_{k-1} is dominated by S^*, then we add no new vertex to S^* associated with x_{k-1}. If x_{k-1} is not dominated by S^*, then we consider the set N_{k-1}. Since G has girth at least 5 and $|N_{k-1}| = k - 1 > k - 2$, and since at most $k-2$ vertices $y_1^*, y_2^*, \ldots, y_{k-2}^*$ have been identified with the previous vertices $x_1, x_2, \ldots, x_{k-2}$ on the list, there is a vertex x_{k-1}^* in N_{k-1} that is not adjacent to any previously defined vertex y_j^*, where $j \in [k-2]$. Since x_{k-1} is not dominated by S^*, we note that $x_{k-1}^* \notin S^*$. Let y_{k-1}^* be a vertex in S that is adjacent to x_{k-1}^*. We now add the vertex x_{k-1}^* to the set S^*.

Let Y be the set of all vertices y_j^* defined previously for $j \in [k-1]$. We note that $y_1^* \in Y$, and so $|Y| \geq 1$. Further, at most $k-1$ such vertices y_j^* exists, and so $|Y| \leq k-1$. We note that if y_j^* exists for some $j \in [k-1]$, then $x_j^* y_j^*$ is an edge of G. Since the vertex $y_1^* = v_{21}$ is associated with the vertex x_1, and since at most $k-2$ vertices in the set $\{v_{12}, v_{13}, \ldots, v_{1k}\}$ (of cardinality $k-1$) are identified with the remaining $k-2$ vertices in $X \setminus \{x_1\}$, we may assume, renaming vertices if necessary, that v_{12} is not identified with any vertex in X.

For each vertex v_{ij} where $i \in [\ell]$ and $j \in [k]$, and where $v_{ij} \notin Y \cup \{v_{11}, v_{12}\}$, we add to S^* the vertex v'_{ij}. The resulting set S^* is a paired-dominating set of G, with v_1 and v_{12} paired, with each vertex $y_j^* \in Y$ paired with the vertex x_j^*, and with v_{ij} and v'_{ij} paired for all i and j, where $i \in [\ell]$ and $j \in [k]$, and where $v_{ij} \notin Y \cup \{v_{11}, v_{12}\}$. Further, $|S^*| = (\ell \cdot 2k) - 2$, implying that $\gamma_{\rm pr}(G) \leq |S^*| < \ell \cdot 2k$. Recall that $\gamma_t(G) = \ell(k+1)$. Thus, $\gamma_{\rm pr}(G)/\gamma_t(G) < (2k)/(k+1)$, a contradiction. Therefore, $\ell = 1$.

Since $G[S] = K_{1,k}$, we note that $\gamma_t(G) = k+1$. Further, $S = \{v_1, v_{11}, v_{12}, \ldots, v_{1k}\}$, where v_1 is the central vertex of the star $G_1 = G[S]$. Since G is a k-regular graph of girth at least 5, and since every vertex in G is within distance 2 from v_1, we note that $n = k^2 + 1$. This in turn, together with the girth and the regularity conditions, imply that every vertex in G has k neighbors and exactly $k(k-1)$ vertices at distance exactly 2 from it, and that G has girth 5. Therefore, G is a diameter-2 Moore graph. ■

As shown by Robertson [24] (see also Bondy and Murty [3], p. 239), the Hoffman-Singleton graph can be constructed from the five 5-cycles $P_1, P_2, \ldots, P_5$ and the 5-cycles $Q_1, Q_2, \ldots, Q_5$ illustrated in Figure 2 with vertex i of the 5-cycle P_j joined to vertex $(i + jk) \pmod 5$ of the 5-cycle Q_k. We call each cycle P_j, $j \in [5]$, a *P-cycle* and each cycle Q_k, $k \in [5]$, a *Q-cycle*.

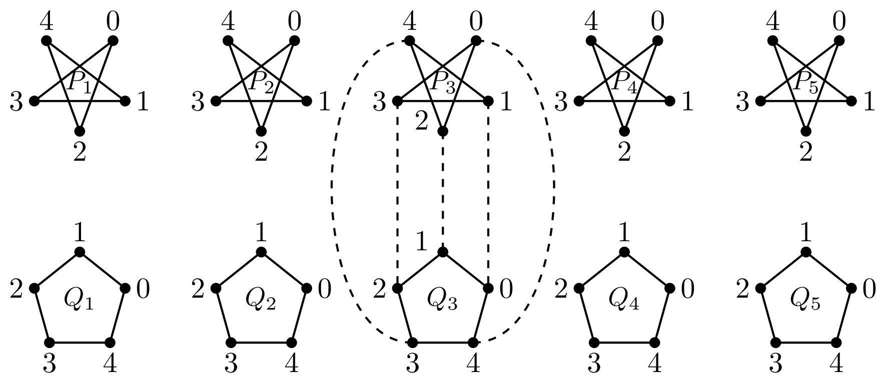

Figure 2. The Hoffman-Singleton graph, where vertex i in P_j is joined to vertex $i + jk \pmod 5$ in Q_k.

As observed by Goddard [13], there is a perfect matching between each P-cycle and each Q-cycle. Further, the vertices of any P-cycle and Q-cycle combined dominate the graph, and induce a graph that contains a perfect matching. Thus, $V(P_j) \cup V(Q_k)$ is a paired-dominating set of G for any $j \in [5]$ and $k \in [5]$. For example, $V(P_3) \cup V(Q_3)$ is a paired-dominating set of G, with the vertices $0, 1, 2, 3, 4$ in P_3 paired with the vertices $4, 0, 1, 2, 3$, respectively, in Q_3. Thus, the Hoffman-Singleton graph G satisfies $\gamma_{\rm pr}(G) \le 10$. It is known [7] that the Hoffman-Singleton graph G satisfies $\gamma_t(G) = 8$. We state this formally as follows.

Remark 11. If G is the Hoffman-Singleton graph, then $\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} \le \frac{5}{4} < \frac{7}{4} = \frac{2k}{k+1}$, where here $k = 7$.

Theorem 2 is an immediate consequence of Theorem 10 and Remark 11.

6. Closing Conjecture

We believe the girth condition can be dropped in Theorem 2 and pose the following conjecture that we have yet to settle.

Conjecture 12. *For $k \geq 2$ and $k \neq 57$, if G is a connected k-regular graph, then $\frac{\gamma_{\rm pr}(G)}{\gamma_t(G)} \leq \frac{2k}{k+1}$, with equality if and only if*

(a) *$k = 2$ and $G \cong C_5$, or*

(b) *$k = 3$ and G is the Petersen graph.*

References

[1] M. Blidia, M. Chellali and T.W. Haynes, *Characterizations of trees with equal paired and double domination numbers*, Discrete Math. **306** (2006) 1840–1845. doi:10.1016/j.disc.2006.03.061

[2] B. Brešar, M.A. Henning and D.F. Rall, *Paired-domination of Cartesian products of graphs*, Util. Math. **73** (2007) 255–265.

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North Holland, New York, 1976).

[4] X.-G. Chen, L. Sun and H.-M. Xing, *Paired-domination numbers of cubic graphs*, Acta Math. Sci. Ser. A Chin. Ed. **27** (2007) 166–170, in Chinese.

[5] T.C.E. Cheng, L.Y. Kang and C.T. Ng, *Paired domination on interval and circular-arc graphs*, Discrete Appl. Math. **155** (2007) 2077–2086. doi:10.1016/j.dam.2007.05.011

[6] T.C.E. Cheng, L.Y. Kang and E. Shan, *A polynomial-time algorithm for the paired-domination problem on permutation graphs*, Discrete Appl. Math. **157** (2009) 262–271. doi:10.1016/j.dam.2008.02.015

[7] W.J. Desormeaux, T.W. Haynes, M.A. Henning and A. Yeo, *Total domination in graphs with diameter* 2, J. Graph Theory **75** (2014) 91–103. doi:10.1002/jgt.21725

[8] W.J. Desormeaux and M.A. Henning, *Paired domination in graphs: a survey and recent results*, Util. Math. **94** (2014) 101–166.

[9] P. Dorbec and S. Gravier, *Paired-domination in P_5-free graphs*, Graphs Combin. **24** (2008) 303–308. doi:10.1007/s00373-008-0792-x

[10] P. Dorbec, S. Gravier and M.A. Henning, *Paired-domination in generalized claw-free graphs*, J. Comb. Optim. **14** (2007) 1–7. doi:10.1007/s10878-006-9022-8

[11] P. Dorbec, B. Hartnell and M.A. Henning, *Paired versus double domination in $K_{1,r}$-free graphs*, J. Comb. Optim. **27** (2014) 688–694. doi:10.1007/s10878-012-9547-y

[12] O. Favaron and M.A. Henning, *Paired-domination in claw-free cubic graphs*, Graphs Combin. **20** (2004) 447–456. doi:10.1007/s00373-004-0577-9

[13] W. Goddard, personal communication at 16th CID Workshop on Graph Theory, September 20–25 (2015) Szklarska Poręba, Poland.

[14] W. Goddard and M.A. Henning,, *A characterization of cubic graphs with paired-domination number three-fifths their order*, Graphs Combin. **25** (2009) 675–692. doi:10.1007/s00373-010-0884-2

[15] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[16] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker, New York, 1998).

[17] T.W. Haynes and P.J. Slater, *Paired-domination and the paired-domatic number*, Congr. Numer. **109** (1995) 65–72.

[18] T.W. Haynes and P.J. Slater, *Paired-domination in graphs*, Networks **32** (1998) 199–206.
doi:10.1002/(SICI)1097-0037(199810)32:3⟨199::AID-NET4⟩3.0.CO;2-F

[19] M.A. Henning, *Graphs with large paired-domination number*, J. Comb. Optim. **13** (2007) 61–78.
doi:10.1007/s10878-006-9014-8

[20] M.A. Henning, *A survey of selected recent results on total domination in graphs*, Discrete Math. **309** (2009) 32–63.
doi:10.1016/j.disc.2007.12.044

[21] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer Monographs in Mathematics, Springer-Verlag, New York, 2013).

[22] A.J. Hoffman and R.R. Singleton, *On Moore graphs with diameter* 2 *and* 3, IBM J. Res. Dev. **4** (1960) 497–504.
doi:10.1147/rd.45.0497

[23] O. Schaudt, *Total domination versus paired domination*, Discuss. Math. Graph Theory **32** (2012) 435–447.
doi:10.7151/dmgt.1614

[24] N. Robertson, Graphs Minimal Under Girth, Valency, and Connectivity Constraints (Dissertation, Waterloo, Ontario, University of Waterloo, 1969).

[25] R.R. Singleton, *There is no irregular Moore graph*, Amer. Math. Monthly **75** (1968) 42–43.
doi:10.2307/2315106

Received 31 August 2016
Revised 11 January 2017
Accepted 11 January 2017