

## TOTAL DOMINATION VERSUS PAIRED-DOMINATION IN REGULAR GRAPHS

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### Abstract

A subset  $S$  of vertices of a graph  $G$  is a dominating set of  $G$  if every vertex not in  $S$  has a neighbor in  $S$ , while  $S$  is a total dominating set of  $G$  if every vertex has a neighbor in  $S$ . If  $S$  is a dominating set with the additional property that the subgraph induced by  $S$  contains a perfect matching, then  $S$  is a paired-dominating set. The domination number, denoted  $\gamma(G)$ , is the minimum cardinality of a dominating set of  $G$ , while the minimum cardinalities of a total dominating set and paired-dominating set are the total domination number,  $\gamma_t(G)$ , and the paired-domination number,  $\gamma_{pr}(G)$ , respectively. For  $k \geq 2$ , let  $G$  be a connected  $k$ -regular graph. It is known [Schaudt, *Total domination versus paired domination*, Discuss. Math. Graph Theory **32** (2012) 435–447] that  $\gamma_{pr}(G)/\gamma_t(G) \leq (2k)/(k+1)$ . In the special case when  $k = 2$ , we observe that  $\gamma_{pr}(G)/\gamma_t(G) \leq 4/3$ , with equality if and only if  $G \cong C_5$ . When  $k = 3$ , we show that  $\gamma_{pr}(G)/\gamma_t(G) \leq 3/2$ , with equality if and only if  $G$  is the Petersen graph. More generally for  $k \geq 2$ , if  $G$  has girth at least 5 and satisfies  $\gamma_{pr}(G)/\gamma_t(G) = (2k)/(k+1)$ , then we show that  $G$  is a diameter-2 Moore graph. As a consequence of this result, we prove that for  $k \geq 2$  and  $k \neq 57$ , if  $G$  has girth at least 5, then

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$\gamma_{\text{pr}}(G)/\gamma_t(G) \leq (2k)/(k+1)$ , with equality if and only if  $k = 2$  and  $G \cong C_5$  or  $k = 3$  and  $G$  is the Petersen graph.

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## 1. INTRODUCTION

In this paper we continue the study of total domination and paired-domination in graphs. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [15, 16].

A vertex  $v$  is said to *dominate* a vertex  $u$  in a graph  $G$  if  $u = v$  or if  $u$  and  $v$  are neighbors in  $G$ . A *dominating set* of  $G$  is a subset  $S$  of vertices of  $G$  such that every vertex outside  $S$  is dominated by at least one vertex in  $S$ . A *total dominating set*, abbreviated TD-set, of  $G$  is a set  $S$  of vertices of  $G$  such that every vertex in  $V(G)$  is adjacent to at least one vertex in  $S$ . The *total domination number* of  $G$ , denoted by  $\gamma_t(G)$ , is the minimum cardinality of a TD-set of  $G$ . We refer to a minimum total dominating set of  $G$  as a  $\gamma_t(G)$ -set. For a recent book on total domination in graphs we refer the reader to [21]. A survey of total domination in graphs can also be found in [20].

A set of edges in a graph  $G$  is *independent* if no two edges in it are adjacent in  $G$ ; that is, an independent edge set is a set of edges without common vertices. A *matching* in a graph  $G$  is a set of independent edges in  $G$ . The *matching number* of a graph  $G$ , denoted  $\alpha'(G)$ , is the maximum cardinality of a matching in  $G$ . A *perfect matching*  $M$  is a matching such that every vertex of  $G$  is incident to an edge of  $M$ .

A *paired-dominating set* of  $G$  is a dominating set  $S$  of  $G$  with the additional property that the subgraph  $G[S]$  induced by  $S$  contains a perfect matching  $M$  (not necessarily induced). The *paired-domination number* of  $G$ , denoted by  $\gamma_{\text{pr}}(G)$ , is the minimum cardinality of a paired-dominating set in  $G$ . Paired-domination was introduced by Haynes and Slater [17, 18] as a model for assigning backups to guards for security purposes, and is well-studied in graph theory. Recent papers on paired-domination can be found, for example, in [1, 2, 4–6, 8–12, 14, 19, 23].

Let  $S \subseteq V(G)$  be a subset of vertices in  $G$  and for the following definitions let  $v$  be a vertex in  $S$ . The *open neighborhood* of  $v$  is the set  $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$  and the *closed neighborhood of  $v$*  is  $N_G[v] = \{v\} \cup N_G(v)$ . The *open neighborhood of  $S$*  is the set  $N_G(S) = \bigcup_{v \in S} N_G(v)$  and its *closed neighborhood* is the set  $N_G[S] = N_G(S) \cup S$ . If the graph  $G$  is clear from the context, we often omit it in the given expressions. For example, we write  $V$  rather than  $V(G)$ , and  $N(v)$  rather than  $N_G(v)$ .



The *S-external private neighborhood* of  $v$ , abbreviated  $\text{epn}(v, S)$ , is the set of all vertices outside  $S$  that are adjacent to  $v$  but to no other vertex of  $S$ ; that is, if  $w \in \text{epn}(v, S)$ , then  $w \in V \setminus S$  and  $N_G(w) \cap S = \{v\}$ . We define an *S-external private neighbor* of  $v$  to be a vertex in  $\text{epn}(v, S)$ .

The *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $(u, v)$ -path in  $G$ . The maximum distance among all pairs of vertices of  $G$  is the *diameter* of  $G$ , denoted by  $\text{diam}(G)$ . We say that  $G$  is a *diameter-2 graph* if  $\text{diam}(G) = 2$ .

A graph  $G$  is *k-regular* if every vertex has degree  $k$  in  $G$ . A *regular graph* is a graph that is  $k$ -regular for some integer  $k \geq 0$ . We remark that 3-regular graphs are also called *cubic graphs* in the literature. The *girth*,  $g(G)$ , of a graph  $G$  is the length of a shortest cycle in  $G$ . We use the standard notation  $[k] = \{1, 2, \dots, k\}$ .

## 2. MAIN RESULT

Schaudt [23] established the following upper bound on the ratio of the paired-domination number versus the total domination number.

**Theorem 1** ([23]). *If  $G$  is a graph with no isolated vertex and maximum degree  $\Delta$ , then*

$$\gamma_{\text{pr}}(G) \leq \left( \frac{2\Delta}{\Delta + 1} \right) \gamma_t(G).$$

As remarked by Schaudt [23], the upper bound of Theorem 1 is tight for all  $\Delta \geq 2$ , as may be seen by letting  $G$  be the graph obtained from a star  $K_{1,\Delta}$  by subdividing every edge exactly once. Such a graph  $G$  satisfies  $\gamma_{\text{pr}}(G) = 2\Delta$  and  $\gamma_t(G) = \Delta + 1$ . We observe that for this extremal family of graphs, the difference between the maximum and minimum degrees is large. In this paper, our focus is therefore on regular graphs.

We wish to determine the connected  $k$ -regular graphs that achieve equality in Schaudt's Theorem 1. We shall prove the following result.

**Theorem 2.** *For  $k \geq 2$  and  $k \neq 57$ , if  $G$  is a connected  $k$ -regular graph of girth at least 5, then  $\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} \leq \frac{2k}{k+1}$ , with equality if and only if*

- (a)  $k = 2$  and  $G \cong C_5$ , or
- (b)  $k = 3$  and  $G$  is the Petersen graph.

## 3. PRELIMINARY LEMMAS

We shall need the following preliminary lemma about the matching number of a graph.

**Lemma 3.** *If  $G$  is a graph of order  $n$  with no isolated vertex and maximum degree  $\Delta$ , then  $\alpha'(G) \geq \frac{n}{\Delta+1}$ , with equality if and only if every component of  $G$  is isomorphic to  $K_{1,\Delta}$ , or  $\Delta = 2$  and every component of  $G$  is isomorphic to  $K_{1,2}$  or  $K_3$ .*

**Proof.** We proceed by induction on  $n \geq \Delta + 1$ . If  $n = \Delta + 1$ , then  $K_{1,\Delta}$  is a spanning subgraph of  $G$  and  $\alpha'(G) \geq 1 = \frac{n}{\Delta+1}$ . Further, if  $\alpha'(G) = 1$ , then  $G = K_{1,\Delta}$  or  $G = K_3$ . This establishes the base case. Suppose that  $n \geq \Delta + 2$ , and that the result is true for all connected graphs of order less than  $n$ . Let  $G$  be a graph of order  $n$  with no isolated vertex. Further, let  $\delta = \delta(G)$  and  $\Delta = \Delta(G)$ . By linearity, we may assume that  $G$  is connected, for otherwise we can apply the result to each component of  $G$ . If  $\Delta = 1$ , then  $G = K_2$ , contradicting the fact that  $n \geq \Delta + 2 = 3$ . Hence,  $\Delta \geq 2$ . Among all edges of  $G$ , let  $v_1v_2$  be chosen so that  $d(v_1) = \delta$ , and, subject to this condition,  $d(v_1) + d(v_2)$  is a minimum.

Let  $L$  denote the set of isolated vertices in  $G - \{v_1, v_2\}$ , and let  $|L| = \ell \geq 0$ . Since  $G$  has no isolated vertex, every vertex in  $L$  is adjacent to at least one of  $v_1$  and  $v_2$ . If a vertex, say  $u$ , in  $L$  is adjacent to  $v_1$  but not to  $v_2$ , then  $d(u) = 1$ , implying that  $\delta = 1$ . However, both  $u$  and  $v_2$  are neighbors of  $v_1$ , and so  $d(v_1) \geq 2$ , contradicting the fact that  $d(v_1) = \delta$ . Hence, every vertex in  $L$  is adjacent to  $v_2$ , implying that  $L \subseteq N(v_2) \setminus \{v_1\}$  and  $\ell \leq \Delta - 1$ .

Let  $G'$  be the isolate-free graph obtained from  $G$  by deleting  $v_1$  and  $v_2$ , and deleting the resulting isolated vertices that belong to the set  $L$ . Let  $G'$  have order  $n'$ . As observed earlier,  $G \setminus \{v_1, v_2\}$  contains at most  $\Delta - 1$  isolated vertices, and so  $n' \geq n - (\Delta + 1)$ . Let  $\Delta' = \Delta(G')$ , and note that  $\Delta' \leq \Delta$ . Applying the inductive hypothesis to each component of  $G'$ , we have that

$$\alpha'(G) \geq 1 + \alpha'(G') \geq 1 + \frac{n'}{\Delta'+1} \geq 1 + \frac{n - (\Delta + 1)}{\Delta + 1} = \frac{n}{\Delta + 1}.$$

Suppose that  $\alpha'(G) = \frac{n}{\Delta+1}$ . Then we must have equality throughout the above inequality chain. In particular,  $\ell = \Delta - 1$ , implying that  $d(v_2) = \Delta$  and  $N(v_2) = L \cup \{v_1\}$ . If in this case, a vertex, say  $w$ , in  $L$  is adjacent to  $v_1$ , then  $d(w) = 2$ , implying by our choice of the edge  $v_1v_2$  that  $d(v_2) = \Delta = 2$  and  $G = K_3$ . Therefore, we may assume that if  $\ell = \Delta - 1$ , then  $v_1$  has no neighbor in  $L$ , implying that  $G = K_{1,\Delta}$ . This completes the proof of the lemma. ■

As a consequence of Lemma 3, we have the following relationship between the paired-domination number and the total domination number of a graph. As remarked earlier, the upper bound in Lemma 4 is precisely Schaudt's Theorem 1 obtained in [23]. However, we present here a slightly stronger result and a different proof of Schaudt's bound in order to characterize the regular graphs that achieve equality in this upper bound.



**Lemma 4.** *If  $G$  is a graph with no isolated vertex and maximum degree  $\Delta$ , then*

$$\gamma_{\text{pr}}(G) \leq \left(\frac{2\Delta}{\Delta+1}\right) \gamma_t(G).$$

*Further, if  $\gamma_{\text{pr}}(G) = \left(\frac{2\Delta}{\Delta+1}\right) \gamma_t(G)$ , then every minimum total dominating set in  $G$  induces a graph whose components are isomorphic to  $K_{1,\Delta}$ .*

**Proof.** Let  $S$  be a minimum TD-set in  $G$ , and consider the subgraph  $H$  of  $G$  induced by  $S$ , that is,  $H = G[S]$ . We note that  $H$  has order  $n(H) = |S|$ . Further,  $H$  has no isolated vertex, and  $\Delta(H) \leq \Delta(G) = \Delta$ . Let  $M$  be a maximum matching in  $H$ , and let  $V(M)$  be the set of vertices incident with an edge of  $M$ . Thus,  $|M| = \alpha'(H)$  and  $|V(M)| = 2|M|$ . By Lemma 3,

$$\alpha'(H) \geq \frac{n(H)}{\Delta(H)+1} \geq \frac{|S|}{\Delta+1}.$$

By the maximality of the matching  $M$ , the set  $S \setminus V(M)$  is an independent set. By the minimality of the TD-set  $S$ , every vertex,  $v$ , in  $S \setminus V(M)$  has a neighbor,  $v'$ , outside  $S$  that is adjacent in  $G$  to  $v$  but to no other vertex of  $S$ . We call such a vertex  $v'$  an external  $S$ -private neighbor of  $v$ . Let

$$S' = \bigcup_{v \in S \setminus V(M)} \{v'\},$$

where the set  $S'$  is chosen in such a way that for each vertex  $v \in S \setminus V(M)$  we choose exactly one external  $S$ -private neighbor  $v'$ . We now consider the set  $D = S \cup S'$ . Since  $S \subseteq D$ , the set  $D$  is a superset of a dominating set of  $G$  and is therefore itself a dominating set of  $G$ . Further, since  $G[D]$  contains a perfect matching, the set  $D$  is a paired-dominating set of  $G$ . Thus,

$$\begin{aligned} \gamma_{\text{pr}}(G) &\leq |S| + |S'| = |S| + (|S| - |V(M)|) = 2|S| - 2\alpha'(H) \\ &\leq 2|S| - \left(\frac{2}{\Delta+1}\right) |S| = \left(\frac{2\Delta}{\Delta+1}\right) |S| = \left(\frac{2\Delta}{\Delta+1}\right) \gamma_t(G). \end{aligned}$$

Suppose that  $\gamma_{\text{pr}}(G) = \left(\frac{2\Delta}{\Delta+1}\right) \gamma_t(G)$ . Then, we must have equality throughout the above inequality chain. In particular,  $\alpha'(H) = |S|/(\Delta+1)$ . Thus, by Lemma 3, every component of  $H$  is isomorphic to  $K_{1,\Delta}$ , or  $\Delta = 2$  and every component of  $H$  is isomorphic to  $K_{1,2}$  or  $K_3$ . If  $\Delta = 2$  and some component, say  $C$ , of  $H$  is isomorphic to  $K_3$ , then the component  $C$  is also a component of  $G$ . Every minimum TD-set of  $G$  contains exactly two vertices from every  $K_3$ -component of  $G$ . However, the set  $S$  contains all three vertices from the  $K_3$ -component  $C$ , a contradiction. Hence, every component of  $G$  is isomorphic to  $K_{1,\Delta}$ . ■

As a special case of Lemma 4, we have the following result.

**Lemma 5.** *For  $k \geq 1$ , if  $G$  is a  $k$ -regular graph, then*

$$\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} \leq \frac{2k}{k+1}.$$

*Further, if  $\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} = \frac{2k}{k+1}$ , then every minimum total dominating set in  $G$  induces a graph whose components are isomorphic to  $K_{1,k}$ .*

#### 4. SMALL VALUES OF $k$

In this section, we wish to determine the connected  $k$ -regular graphs achieving equality in the upper bound of Lemma 5 for small values of  $k$ . For  $k \geq 1$ , let  $G$  be a connected  $k$ -regular graph. By Lemma 5,  $\gamma_{\text{pr}}(G)/\gamma_t(G) \leq (2k)/(k+1)$ . We wish to characterize such graphs  $G$  satisfying  $\gamma_{\text{pr}}(G)/\gamma_t(G) = (2k)/(k+1)$ .

If  $k = 1$ , then  $G = K_2$  and  $\gamma_{\text{pr}}(G) = \gamma_t(G) = 2$ , and so  $\gamma_{\text{pr}}(G)/\gamma_t(G) = 1 = (2k)/(k+1)$ . Hence, it is only of interest to consider the case when  $k \geq 2$ .

Suppose that  $k = 2$  and that the connected  $k$ -regular graph  $G$  has order  $n$ . In this case,  $G \cong C_n$  and by Lemma 5,  $\gamma_{\text{pr}}(G)/\gamma_t(G) \leq (2k)/(k+1) = 4/3$ . Suppose that  $\gamma_{\text{pr}}(G)/\gamma_t(G) = 4/3$ . By Lemma 5, every minimum TD-set in the cycle  $G$  induces a graph whose components are isomorphic to  $K_{1,2}$ . However, this is only the case when  $G \cong C_5$ , since if  $G \cong C_n$  and  $n \neq 5$ , then we can always find a minimum TD-set in  $G$  that induces a graph with at least one component isomorphic to  $K_2$ . We state this formally as follows.

**Theorem 6.** *If  $G$  is a 2-regular connected graph, then*

$$\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} \leq \frac{4}{3}, \text{ with equality if and only if } G \cong C_5.$$

We next consider the case when  $k \geq 3$ . For this purpose, we shall need the following two results on the paired-domination number of a cubic graph.

**Theorem 7** ([4]). *If  $G$  is a cubic graph of order  $n$ , then  $\gamma_{\text{pr}}(G) \leq \frac{3}{5}n$ .*

**Theorem 8** ([14]). *If  $G$  is a connected cubic graph of order  $n$  satisfying  $\gamma_{\text{pr}}(G) = \frac{3}{5}n$ , then  $G$  is the Petersen graph (illustrated in Figure 1).*

We shall prove the following result.

**Theorem 9.** *If  $G$  is a connected cubic graph, then*

$$\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} \leq \frac{3}{2}, \text{ with equality if and only if } G \text{ is the Petersen graph.}$$



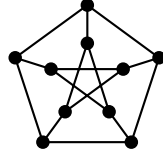


Figure 1. The Petersen graph.

**Proof.** If  $G$  is a cubic graph, then by Lemma 5,  $\gamma_{\text{pr}}(G)/\gamma_t(G) \leq 3/2$ . If  $G$  is the Petersen graph, then  $\gamma_{\text{pr}}(G) = 6$  and  $\gamma_t(G) = 4$ , and so the Petersen graph achieves the  $3/2$ -ratio for the paired-domination number versus the total domination number. It suffices for us to prove that the Petersen graph is the unique such graph.

Suppose that  $G$  is a connected cubic graph of order  $n$  satisfying  $\gamma_{\text{pr}}(G)/\gamma_t(G) = 3/2$ . By Lemma 5, every minimum TD-set in  $G$  induces a graph whose components are isomorphic to  $K_{1,3}$ . Let  $S$  be a minimum TD-set in  $G$ . Thus,  $G[S]$  is the disjoint union of copies of  $K_{1,3}$ ; that is,  $G[S] = \ell K_{1,3}$  for some integer  $\ell \geq 1$ . We proceed further with the following claim.

**Claim A.**  $G[S] = K_{1,3}$ .

**Proof.** We wish to show that  $\ell = 1$ . Suppose, to the contrary, that  $\ell \geq 2$ . Let  $G_1, G_2, \dots, G_\ell$  be the components of  $G[S]$ , and so  $G_i \cong K_{1,3}$  for all  $i \in [\ell]$ . Further, let  $V(G_i) = \{v_i, v_{i1}, v_{i2}, v_{i3}\}$ , where  $v_i$  is the central vertex of the star  $G_i$ . By the minimality of the TD-set  $S$ , every vertex  $v_{ij}$ , where  $i \in [\ell]$  and  $j \in [3]$ , has an  $S$ -external private neighbor; that is,  $\text{epn}(v_{ij}, S) \neq \emptyset$ . For each such vertex  $v_{ij}$ , let  $v'_{ij} \in \text{epn}(v_{ij}, S)$ . Thus,  $v'_{ij} \in V(G) \setminus S$  and  $N(v'_{ij}) \cap S = \{v_{ij}\}$ .

Let  $P$  be a shortest path in  $G$  that joins a vertex from one component of  $G[S]$  to a vertex from another component of  $G[S]$ . Renaming components and vertices of  $G[S]$  if necessary, we may assume that  $P$  is a  $(v_{11}, v_{21})$ -path. Thus, if  $P'$  is an arbitrary path in  $G$  that starts at a vertex in  $V(G_i)$  and ends at a vertex in  $V(G_j)$ , where  $1 \leq i, j \leq \ell$  and  $i \neq j$ , then  $P'$  has length at least that of  $P$ . By the minimality of the path  $P$ , every internal vertex of  $P$  belongs to  $V(G) \setminus S$ .

The path  $P$  has length at least 2, since vertices in different components of  $G[S]$  are not adjacent. We show that the path  $P$  has length 2 or 3. Suppose, to the contrary, that  $P$  has length at least 4. Let  $v_{11}xy$  be the subpath of  $P$  consisting of the first three vertices on  $P$ . By supposition,  $y \neq v_{21}$ . Since  $S$  is a TD-set of  $G$ , there is a vertex  $z \in S$  that is adjacent to  $y$  in  $G$ . If  $z \in V(G_1)$ , then the path  $zy$  followed by the subpath of  $P$  from  $y$  to  $v_{21}$  is a shorter path than  $P$  joining vertices from different components of  $G[S]$ , a contradiction. Hence,  $z \notin V(G_1)$ , and so  $v_{11}xyz$  is a path of length 3 joining vertices from different components of  $G[S]$ , contradicting our choice of the path  $P$ . Therefore,  $P$  has length 2 or 3.



Let  $X = N_G(v_{11}) \setminus \{v_1\} = \{x_1, x_2\}$ , where  $x_1$  is the neighbor of  $v_{11}$  on the path  $P$ . We note that  $|X| = 2$  and  $X \subset V(G) \setminus S$ . Further, no vertex in  $X$  is adjacent to a vertex  $v_j$  for any  $j \in [\ell]$ . If  $P$  has length 2, let  $x_1^* = x_1$ , while if  $P$  has length 3, let  $x_1^*$  be the common neighbor of  $x_1$  and  $v_{21}$  on the path  $P$ . In both cases, we note that  $v_{21}x_1^*$  is an edge of  $G$ . We now build a paired-dominating set  $S^*$  of  $G$  as follows.

Initially, we let  $S^*$  be obtained from  $S$  by removing the  $\ell-1$  vertices  $v_2, \dots, v_\ell$ , removing the vertex  $v_{11}$ , and adding the vertex  $x_1^*$ ; that is,  $S^* = (S \setminus \{v_{11}, v_2, \dots, v_\ell\}) \cup \{x_1^*\}$ .

**Subclaim A.1.** *The vertex  $x_2$  is not dominated by  $S^*$ .*

**Proof.** Suppose that  $x_2$  is dominated by  $S^*$ . In this case, we add to  $S^*$  the vertices  $v'_{ij}$  for all  $i$  and  $j$ , where  $i \in [\ell]$ ,  $j \in [3]$ , and  $(i, j) \notin \{(1, 1), (1, 2), (2, 1)\}$ . The resulting set  $S^*$  is a paired-dominating set of  $G$ , with  $v_1$  and  $v_{12}$  paired,  $v_{21}$  and  $x_1^*$  paired, and with  $v_{ij}$  and  $v'_{ij}$  paired for all  $i$  and  $j$ , where  $i \in [\ell]$ ,  $j \in [3]$ , and  $(i, j) \notin \{(1, 1), (1, 2), (2, 1)\}$ . Further,  $|S^*| = 6\ell - 2$ , implying that  $\gamma_{\text{pr}}(G) \leq |S^*| < 6\ell$ . Recall that  $\gamma_t(G) = 4\ell$ . Thus,  $\gamma_{\text{pr}}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$ , a contradiction.  $\square$

By Subclaim A.1, the vertex  $x_2$  is not dominated by  $S^*$ . Let  $x_2^*$  be a neighbor of  $x_2$  different from  $v_{11}$  and  $x_1$ . Since  $x_2$  is not dominated by  $S^*$ , we note that  $x_2^* \notin S^*$ . Since  $S$  is a TD-set of  $G$  and  $x_2^* \notin X$ , there is a vertex in  $S \setminus \{v_{11}\}$  that is adjacent to  $x_2^*$ .

**Subclaim A.2.** *The vertex  $v_{21}$  is not adjacent to  $x_2^*$ .*

**Proof.** Suppose that  $v_{21}$  is adjacent to  $x_2^*$ , and so  $N(v_{21}) = \{v_2, x_1^*, x_2^*\}$ . If  $x_1x_2$  is an edge, then we note that  $x_1^* \neq x_1$ , since  $x_2$  is not dominated by  $S^*$ . Thus,  $P$  is the path  $v_{11}x_1x_1^*v_{21}$ . The set  $S' = (S \setminus \{v_{11}, v_{21}\}) \cup \{x_1, x_2\}$  is a minimum TD-set in  $G$  that induces a graph with at least one component (namely the component containing the edge  $x_1x_2$ ) that is not isomorphic to  $K_{1,3}$ , a contradiction. Therefore,  $x_1x_2$  is not an edge. Thus the third neighbor of  $x_2$ , say  $x_2^{**}$ , that is different from  $v_{11}$  and  $x_2^*$ , is not the vertex  $x_1$ . Since  $x_2$  is not dominated by  $S^*$ , we note that  $x_2^{**} \notin S^*$ . Thus there is a vertex in  $S \setminus \{v_{11}\}$  that is adjacent to  $x_2^{**}$ . Let  $v_{st}$  be such a vertex in  $S \setminus \{v_{11}\}$ . Since  $x_2^{**}$  is not adjacent to  $v_{21}$ , we note that  $(s, t) \notin \{(1, 1), (2, 1)\}$ . We now add the vertex  $x_2^{**}$  to  $S^*$ .

If  $(s, t) \in \{(1, 2), (1, 3)\}$ , say  $(s, t) = (1, 2)$ , then we add to  $S^*$  the vertex  $v'_{ij}$  for all  $i$  and  $j$ , where  $i \in [\ell] \setminus \{1\}$ ,  $j \in [3]$ , and  $(i, j) \neq (2, 1)$ . The resulting set  $S^*$  is a paired-dominating set of  $G$ , with  $v_1$  and  $v_{13}$  paired,  $v_{12}$  and  $x_2^{**}$  paired,  $v_{21}$  and  $x_1^*$  paired, and with  $v_{ij}$  and  $v'_{ij}$  paired for all  $i$  and  $j$ , where  $i \in [\ell] \setminus \{1\}$ ,  $j \in [3]$ , and  $(i, j) \neq (2, 1)$ . Further,  $|S^*| = 6\ell - 2$ , implying that  $\gamma_{\text{pr}}(G) < 6\ell$ . Thus,  $\gamma_{\text{pr}}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$ , a contradiction.



If  $(s, t) \notin \{(1, 2), (1, 3)\}$ , then we add to  $S^*$  the vertices  $v'_{ij}$  for all  $i$  and  $j$ , where  $i \in [\ell]$ ,  $s \in [3]$ , and  $(i, j) \notin \{(1, 1), (1, 2), (2, 1), (s, t)\}$ . The resulting set  $S^*$  is a paired-dominating set of  $G$ , with  $v_1$  and  $v_{12}$  paired,  $v_{21}$  and  $x_1^*$  paired,  $v_{st}$  and  $x_2^{**}$  paired, and with  $v_{ij}$  and  $v'_{ij}$  paired for all  $i$  and  $j$ , where  $i \in [\ell]$ ,  $s \in [3]$ , and  $(i, j) \notin \{(1, 1), (1, 2), (2, 1), (s, t)\}$ . Further,  $|S^*| = 6\ell - 2$ , implying that  $\gamma_{pr}(G) < 6\ell$ . Thus,  $\gamma_{pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$ , a contradiction.  $\square$

By Subclaim A.2, the vertex  $v_{21}$  is not adjacent to  $x_2^*$ . Let  $v_{i'j'}$  be the vertex in  $S \setminus \{v_{11}, v_{21}\}$  that is adjacent to  $x_2^*$ . Thus,  $(i', j') \notin \{(1, 1), (2, 1)\}$ .

If  $(i', j') \in \{(1, 2), (1, 3)\}$ , say  $(i', j') = (1, 2)$ , then we add to  $S^*$  the vertex  $v'_{ij}$  for all  $i$  and  $j$ , where  $i \in [\ell] \setminus \{1\}$ ,  $j \in [3]$ , and  $(i, j) \neq (2, 1)$ . The resulting set  $S^*$  is a paired-dominating set of  $G$ , with  $v_1$  and  $v_{13}$  paired,  $v_{12}$  and  $x_2^*$  paired,  $v_{21}$  and  $x_1^*$  paired, and with  $v_{ij}$  and  $v'_{ij}$  paired for all  $i$  and  $j$ , where  $i \in [\ell] \setminus \{1\}$ ,  $j \in [3]$ , and  $(i, j) \neq (2, 1)$ . Further,  $|S^*| = 6\ell - 2$ , implying that  $\gamma_{pr}(G) < 6\ell$ . Thus,  $\gamma_{pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$ , a contradiction.

If  $(i', j') \notin \{(1, 2), (1, 3)\}$ , then we add to  $S^*$  the vertices  $v'_{ij}$  for all  $i$  and  $j$ , where  $i \in [\ell]$ ,  $s \in [3]$ , and  $(i, j) \notin \{(1, 1), (1, 2), (2, 1), (i', j')\}$ . The resulting set  $S^*$  is a paired-dominating set of  $G$ , with  $v_1$  and  $v_{12}$  paired,  $v_{21}$  and  $x_1^*$  paired,  $v_{i'j'}$  and  $x_2^*$  paired, and with  $v_{ij}$  and  $v'_{ij}$  paired for all  $i$  and  $j$ , where  $i \in [\ell]$ ,  $s \in [3]$ , and  $(i, j) \notin \{(1, 1), (1, 2), (2, 1), (i', j')\}$ . Further,  $|S^*| = 6\ell - 2$ , implying that  $\gamma_{pr}(G) < 6\ell$ . Thus,  $\gamma_{pr}(G)/\gamma_t(G) < 6\ell/4\ell = 3/2$ , a contradiction. This completes the proof of Claim A.  $\square$

By Claim A,  $G[S] = K_{1,3}$ . In particular, we note that  $\gamma_t(G) = 4$ . Since  $\gamma_{pr}(G)/\gamma_t(G) = 3/2$ , this implies that  $\gamma_{pr}(G) = 6$ . Recall that  $S = \{v_1, v_{11}, v_{12}, v_{13}\}$ , where  $v_1$  is the central vertex of the star  $G_1 = G[S]$ . Since  $G$  is a connected, cubic graph of order  $n$ , and every vertex in  $G$  is within distance 2 from  $v_1$ , we note that  $n \leq 10$ . By Theorem 7 and the fact that  $n \leq 10$ , we get

$$6 = \gamma_{pr}(G) \leq 3n/5 \leq 6.$$

We must have equality throughout this inequality chain. In particular,  $\gamma_{pr}(G) = 3n/5$ . Thus, by Theorem 8,  $G$  is the Petersen graph. This completes the proof of Theorem 9.  $\blacksquare$

### 5. GENERAL VALUES OF $k$

In this section, we wish to determine the connected  $k$ -regular graphs that achieve equality in the upper bound of Lemma 5 for general values of  $k$ , given the requirement that the girth of the graph is at least 5.

In order to state our next result, we recall that the diameter-2 graphs of girth 5 are precisely the diameter-2 Moore graphs. It is shown (see [22, 25]) that Moore graphs are  $k$ -regular and that diameter-2 Moore graphs have order

$n = k^2 + 1$  and exist for  $k = 2, 3, 7$  and possibly 57, but for no other degrees. (It is currently unknown whether there exists such a Moore graph for  $k = 57$ ). The diameter-2 Moore graphs for the first three values of  $k$  are unique, namely

- the 5-cycle (2-regular graph on  $n = 5$  vertices),
- the Petersen graph (3-regular graph on  $n = 10$  vertices),
- the Hoffman-Singleton graph (7-regular graph on  $n = 50$  vertices).

We show next that if we impose a girth condition, then every connected, regular graph achieving equality in the upper bound of Lemma 5 is a diameter-2 Moore graph.

**Theorem 10.** *For  $k \geq 2$ , if  $G$  is a connected  $k$ -regular graph of girth at least 5 satisfying  $\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} = \frac{2k}{k+1}$ , then  $G$  is a diameter-2 Moore graph.*

**Proof.** When  $k = 2$  and  $k = 3$ , the result follows from Theorem 6 and Theorem 9, respectively (even without the girth condition). Hence, we may assume in what follows that  $k \geq 4$ . By Lemma 4, every minimum TD-set in  $G$  induces a graph whose components are isomorphic to  $K_{1,k}$ . Let  $S$  be a minimum TD-set in  $G$ . Thus,  $G[S]$  is the disjoint union of copies of  $K_{1,k}$ ; that is,  $G[S] = \ell K_{1,k}$  for some integer  $\ell \geq 1$ . We show that  $\ell = 1$ ; that is,  $G[S] = K_{1,k}$ .

Suppose, to the contrary, that  $\ell \geq 2$ . Let  $G_1, G_2, \dots, G_\ell$  be the components of  $G[S]$ , and so  $G_i \cong K_{1,k}$  for all  $i \in [\ell]$ . Further, let  $V(G_i) = \{v_i, v_{i1}, v_{i2}, \dots, v_{ik}\}$ , where  $v_i$  is the central vertex of the star  $G_i$ . By the minimality of the TD-set  $S$ , every vertex  $v_{ij}$ , where  $i \in [\ell]$  and  $j \in [k]$ , has an  $S$ -external private neighbor; that is,  $\text{epn}(v_{ij}, S) \neq \emptyset$ . For each such vertex  $v_{ij}$ , let  $v'_{ij} \in \text{epn}(v_{ij}, S)$ . Thus,  $v'_{ij} \in V(G) \setminus S$  and  $N(v'_{ij}) \cap S = \{v_{ij}\}$ .

Let  $P$  be a shortest path in  $G$  that joins a vertex from one component of  $G[S]$  to a vertex from another component of  $G[S]$ . Renaming components and vertices of  $G[S]$  if necessary, we may assume that  $P$  is a  $(v_{11}, v_{21})$ -path. Analogously as in the proof of Theorem 9, the path  $P$  has length 2 or 3.

Let  $X = N_G(v_{11}) \setminus \{v_1\} = \{x_1, x_2, \dots, x_{k-1}\}$ , where  $x_1$  is the neighbor of  $v_{11}$  on the path  $P$ . We note that  $|X| = k - 1$  and that  $X \subset V(G) \setminus S$ . Further, the girth condition and the choice of the path  $P$  implies that no vertex in  $X$  is adjacent to a vertex in  $V(G_1)$ , except for the vertex  $v_{11}$ .

If  $P$  has length 2, let  $x_1^* = x_1$ , while if  $P$  has length 3, let  $x_1^*$  be the common neighbor of  $x_1$  and  $v_{21}$  on the path  $P$ . In both cases, we note that  $v_{21}x_1^*$  is an edge of  $G$ . Let  $y_1^* = v_{21}$ . We now build a paired-dominating set  $S^*$  of  $G$  as follows. Initially, we let  $S^*$  be obtained from  $S$  by removing the  $\ell - 1$  vertices  $v_2, \dots, v_\ell$ , removing the vertex  $v_{11}$ , and adding the vertex  $x_1^*$ ; that is,  $S^* = (S \setminus \{v_{11}, v_2, \dots, v_\ell\}) \cup \{x_1^*\}$ .

We now consider the vertices  $x_2, x_3, \dots, x_{k-1}$  in turn. For  $i \in [k-1]$ , let  $N_i$  be the set of  $k-1$  neighbors of  $x_i$  different from  $v_{11}$ , and so  $N_i = N_G(x_i) \setminus \{v_{11}\}$ . Since  $G$  has girth at least 5, we note that  $N_i$  is an independent set and  $N_i \cap V(G_1) = \emptyset$ .

Further,  $N_i \cap N_j = \emptyset$  for  $i, j \in [k]$  and  $i \neq j$ .

If  $x_2$  is dominated by  $S^*$ , then we add no new vertex to  $S^*$  associated with  $x_2$ , and we consider the next vertex  $x_3$  in the list. If  $x_2$  is not dominated by  $S^*$ , then we consider the set  $N_2$ . Since  $G$  has girth at least 5, at most one vertex in  $N_2$  is a neighbor of  $y_1^*$ . Let  $x_2^*$  be a vertex in  $N_2$  that is not adjacent to  $y_1^*$ . Since  $x_2$  is not dominated by  $S^*$ , we note that  $x_2^* \notin S^*$ . Let  $y_2^*$  be a vertex in  $S$  that is adjacent to  $x_2^*$ . We note that  $y_1^* \neq y_2^*$ . We now add the vertex  $x_2^*$  to the set  $S^*$ .

Next, we consider the vertex  $x_3$ . If  $x_3$  is dominated by  $S^*$ , then we add no new vertex to  $S^*$  associated with  $x_3$ , and we consider the next vertex  $x_4$  in the list. If  $x_3$  is not dominated by  $S^*$ , then we consider the set  $N_3$ . Since  $G$  has girth at least 5, at most one vertex in  $N_3$  is a neighbor of  $y_1^*$  and at most one vertex in  $N_3$  is a neighbor of  $y_2^*$ , if  $y_2^*$  exists. Hence, since  $|N_3| = k - 1 > 2$ , there is a vertex  $x_3^*$  in  $N_3$  that is not adjacent to  $y_1^*$  and is not adjacent to  $y_2^*$ , if it exists. Since  $x_3$  is not dominated by  $S^*$ , we note that  $x_3^* \notin S^*$ . Let  $y_3^*$  be a vertex in  $S$  that is adjacent to  $x_3^*$ . We note that  $y_1^*, y_2^*$  and  $y_3^*$  are distinct vertices, if they exist. We now add the vertex  $x_3^*$  to the set  $S^*$ .

We continue in the fashion until finally we consider the last vertex on the list, namely the vertex  $x_{k-1}$ . If  $x_{k-1}$  is dominated by  $S^*$ , then we add no new vertex to  $S^*$  associated with  $x_{k-1}$ . If  $x_{k-1}$  is not dominated by  $S^*$ , then we consider the set  $N_{k-1}$ . Since  $G$  has girth at least 5 and  $|N_{k-1}| = k - 1 > k - 2$ , and since at most  $k - 2$  vertices  $y_1^*, y_2^*, \dots, y_{k-2}^*$  have been identified with the previous vertices  $x_1, x_2, \dots, x_{k-2}$  on the list, there is a vertex  $x_{k-1}^*$  in  $N_{k-1}$  that is not adjacent to any previously defined vertex  $y_j^*$ , where  $j \in [k - 2]$ . Since  $x_{k-1}$  is not dominated by  $S^*$ , we note that  $x_{k-1}^* \notin S^*$ . Let  $y_{k-1}^*$  be a vertex in  $S$  that is adjacent to  $x_{k-1}^*$ . We now add the vertex  $x_{k-1}^*$  to the set  $S^*$ .

Let  $Y$  be the set of all vertices  $y_j^*$  defined previously for  $j \in [k - 1]$ . We note that  $y_1^* \in Y$ , and so  $|Y| \geq 1$ . Further, at most  $k - 1$  such vertices  $y_j^*$  exist, and so  $|Y| \leq k - 1$ . We note that if  $y_j^*$  exists for some  $j \in [k - 1]$ , then  $x_j^* y_j^*$  is an edge of  $G$ . Since the vertex  $y_1^* = v_{21}$  is associated with the vertex  $x_1$ , and since at most  $k - 2$  vertices in the set  $\{v_{12}, v_{13}, \dots, v_{1k}\}$  (of cardinality  $k - 1$ ) are identified with the remaining  $k - 2$  vertices in  $X \setminus \{x_1\}$ , we may assume, renaming vertices if necessary, that  $v_{12}$  is not identified with any vertex in  $X$ .

For each vertex  $v_{ij}$  where  $i \in [\ell]$  and  $j \in [k]$ , and where  $v_{ij} \notin Y \cup \{v_{11}, v_{12}\}$ , we add to  $S^*$  the vertex  $v'_{ij}$ . The resulting set  $S^*$  is a paired-dominating set of  $G$ , with  $v_1$  and  $v_{12}$  paired, with each vertex  $y_j^* \in Y$  paired with the vertex  $x_j^*$ , and with  $v_{ij}$  and  $v'_{ij}$  paired for all  $i$  and  $j$ , where  $i \in [\ell]$  and  $j \in [k]$ , and where  $v_{ij} \notin Y \cup \{v_{11}, v_{12}\}$ . Further,  $|S^*| = (\ell \cdot 2k) - 2$ , implying that  $\gamma_{\text{pr}}(G) \leq |S^*| < \ell \cdot 2k$ . Recall that  $\gamma_t(G) = \ell(k + 1)$ . Thus,  $\gamma_{\text{pr}}(G)/\gamma_t(G) < (2k)/(k + 1)$ , a contradiction. Therefore,  $\ell = 1$ .

Since  $G[S] = K_{1,k}$ , we note that  $\gamma_t(G) = k + 1$ . Further,  $S = \{v_1, v_{11}, v_{12}, \dots, v_{1k}\}$ , where  $v_1$  is the central vertex of the star  $G_1 = G[S]$ . Since  $G$  is a  $k$ -regular

graph of girth at least 5, and since every vertex in  $G$  is within distance 2 from  $v_1$ , we note that  $n = k^2 + 1$ . This in turn, together with the girth and the regularity conditions, imply that every vertex in  $G$  has  $k$  neighbors and exactly  $k(k - 1)$  vertices at distance exactly 2 from it, and that  $G$  has girth 5. Therefore,  $G$  is a diameter-2 Moore graph. ■

As shown by Robertson [24] (see also Bondy and Murty [3], p. 239), the Hoffman-Singleton graph can be constructed from the five 5-cycles  $P_1, P_2, \dots, P_5$  and the 5-cycles  $Q_1, Q_2, \dots, Q_5$  illustrated in Figure 2 with vertex  $i$  of the 5-cycle  $P_j$  joined to vertex  $(i + jk) \pmod{5}$  of the 5-cycle  $Q_k$ . We call each cycle  $P_j$ ,  $j \in [5]$ , a  $P$ -cycle and each cycle  $Q_k$ ,  $k \in [5]$ , a  $Q$ -cycle.

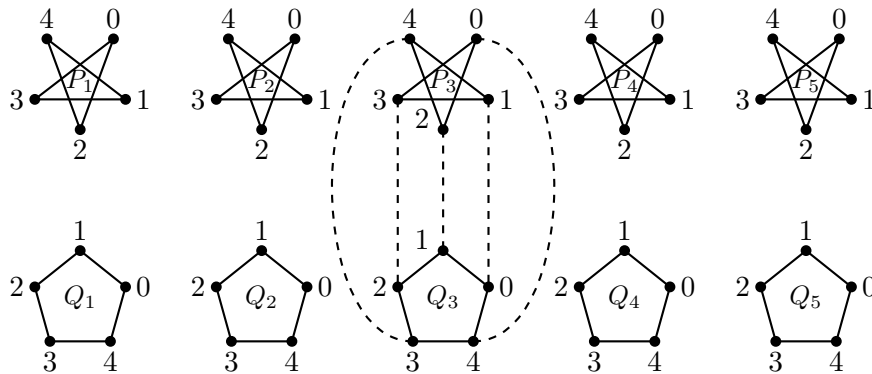


Figure 2. The Hoffman-Singleton graph, where vertex  $i$  in  $P_j$  is joined to vertex  $i + jk \pmod{5}$  in  $Q_k$ .

As observed by Goddard [13], there is a perfect matching between each  $P$ -cycle and each  $Q$ -cycle. Further, the vertices of any  $P$ -cycle and  $Q$ -cycle combined dominate the graph, and induce a graph that contains a perfect matching. Thus,  $V(P_j) \cup V(Q_k)$  is a paired-dominating set of  $G$  for any  $j \in [5]$  and  $k \in [5]$ . For example,  $V(P_3) \cup V(Q_3)$  is a paired-dominating set of  $G$ , with the vertices 0, 1, 2, 3, 4 in  $P_3$  paired with the vertices 4, 0, 1, 2, 3, respectively, in  $Q_3$ . Thus, the Hoffman-Singleton graph  $G$  satisfies  $\gamma_{pr}(G) \leq 10$ . It is known [7] that the Hoffman-Singleton graph  $G$  satisfies  $\gamma_t(G) = 8$ . We state this formally as follows.

**Remark 11.** If  $G$  is the Hoffman-Singleton graph, then  $\frac{\gamma_{pr}(G)}{\gamma_t(G)} \leq \frac{5}{4} < \frac{7}{4} = \frac{2k}{k+1}$ , where here  $k = 7$ .

Theorem 2 is an immediate consequence of Theorem 10 and Remark 11.

### 6. CLOSING CONJECTURE

We believe the girth condition can be dropped in Theorem 2 and pose the following conjecture that we have yet to settle.

**Conjecture 12.** For  $k \geq 2$  and  $k \neq 57$ , if  $G$  is a connected  $k$ -regular graph, then  $\frac{\gamma_{\text{pr}}(G)}{\gamma_t(G)} \leq \frac{2k}{k+1}$ , with equality if and only if

- (a)  $k = 2$  and  $G \cong C_5$ , or
- (b)  $k = 3$  and  $G$  is the Petersen graph.

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