

Generic appearance of objective results in quantum measurementsJ. K. Korbicz,^{1,2,*} E. A. Aguilar,³ P. Ćwikliński,³ and P. Horodecki^{1,2}¹*Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, 80-233 Gdańsk, Poland*²*National Quantum Information Centre in Gdańsk, 81-824 Sopot, Poland*³*Faculty of Mathematics, Physics and Informatics University of Gdańsk, 80-952 Gdańsk, National Quantum Information Centre in Gdańsk, 81-824 Sopot, Poland*

(Received 29 August 2016; published 28 September 2017)

Measurement is of central interest in quantum mechanics as it provides the link between the quantum world and the world of everyday experience. One of the features of everyday experience is its robust, objective character, contrasting the delicate nature of quantum systems. Here we analyze in a completely model-independent way the celebrated von Neumann measurement process, using recent techniques of information flow, studied in open quantum systems. We show the generic appearance of objective results in quantum measurements, provided we macroscopically coarse-grain the measuring apparatus and wait long enough. To study genericity, we employ the widely used Gaussian unitary ensemble of random matrices and the Hoeffding inequality. We derive generic objectivization time scales, given solely by the interaction strength and the systems' dimensions. Our results are manifestly universal and are a generic property of von Neumann measurements.

DOI: [10.1103/PhysRevA.96.032124](https://doi.org/10.1103/PhysRevA.96.032124)**I. INTRODUCTION**

Understanding quantum measurements has been one of the central problems of quantum theory since its beginning [1,2]. It not only provides the crucial link between the theory and experiment, the micro and macroworlds, but is at the heart of the modern quantum technologies (see, e.g., [3]). The fundamental measurement theory dates back to von Neumann [4] and since then has been further developed in various directions, e.g., the decoherence theory [5,6]. To be readable, measurement results must inevitably be encoded into macroscopic degrees of freedom and one of the crucial features expected from a good measurement process is an objective character of the results: They can be read out by arbitrary many observers and without causing any disturbance by the mere read-out. This has been realized as early as in 1929 by Mott [7]. Achieved in well-engineered measurements by a proper coupling to macroscopic degrees of freedom, it is not at all obvious if such a situation is a generic feature of a quantum measurement process with a macroscopic recording.

In a broader context of open quantum systems [5,6], this may be seen as a question about how information flows from the system to its environment. Pioneering research along this direction has been undertaken under the quantum Darwinism idea [8], arguing that in some situations (see, e.g., [9,10]) perfect information about the system can be redundantly stored in the environment and becomes effectively classical [11] and objective. The generic character of some of the quantum Darwinism features was shown in Ref. [12] and the universality of decoherence was shown on short time scales in Refs. [13–16]. A further step was recently made in Refs. [17,18] by formulating information flow and objectivity in the fundamental language of quantum states with the introduction of the so-called Spectrum Broadcast Structures (SBS's). These structures have been proven to be a useful tool allowing to obtain novel results in some of

the emblematic models of decoherence [17,19–21]. Finally, questions of genericity have traditionally been the domain of statistical mechanics and thermodynamics (see, e.g., [22–24]). Phrased in this language, we may ask to what form a generic state equilibrates during a von Neumann measurement.

In this communication we study information flow during a von Neumann measurement process with a macroscopic (in a sense of a number of degrees of freedom) measuring apparatus. Applying random matrix theory techniques [25,26], we show that generically the postmeasurement state approaches, after a coarse-graining, a form carrying almost perfect, multiple records of the measurement result, thus making the result objective. To study genericity, we use a properly structured Gaussian unitary ensemble (GUE) [25,26]. Since the seminal works of Wigner and Dyson on statistics of various experimentally observed spectra, it has been the basic choice for random Hamiltonians due to its universality and agreement with the experiment [25,26]. The apparatus is assumed to be noisy, with the initial state distributed according to some physically motivated measures of mixed states [27]. For large-dimensional measured systems, we provide estimates on the time scale of the objectivization process. Since the only assumptions we make concern the genericity measures, our results are manifestly universal and apply to the whole class of von Neumann measurements, thus showing a generic and robust character of the emergence of objectivity. It is a bit of a surprise that this property of von Neumann measurements was so far tacitly assumed (see, e.g., [28]), but never, to our best knowledge, derived.

II. MEASUREMENTS WITH COMPOUND APPARATUSES

We consider a d_S -dimensional quantum system S simultaneously measured by a collection of N measuring apparatuses or environments E_1, \dots, E_N , each of dimension d , representing a macroscopic measuring device (cf. [17]). The apparatuses are assumed to be individually coupled to the system through a general von Neumann-type interaction, so strong that the self-Hamiltonians of the system and the apparatuses can be

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neglected (the quantum measurement limit) [4]

$$\hat{H}_{\text{total}} \approx \hat{H}_{\text{int}} = \hat{A} \otimes \sum_{k=1}^N \hat{B}_k, \quad (1)$$

where \hat{A} is the measured observable (assumed nondegenerate) and the \hat{B}_k are some general measuring observables. This leads to the evolution (setting $\hbar = 1$) $\hat{U} \equiv e^{-it\hat{H}_{\text{int}}} = \sum_a |a\rangle\langle a| \otimes \bigotimes_{k=1}^N e^{-ia\hat{B}_k t}$, where $\hat{A} = \sum_{a=1}^{d_S} a|a\rangle\langle a|$. Our main object of study is a partially reduced state $\rho_{S:E_{\text{obs}}}$, with a fraction E_{uno} of size N_{uno} of unobserved subsystems traced out. This represents an inevitable loss of information during a measurement. Assuming $\rho_{SE}(0) = \rho_{0S} \otimes \bigotimes_{k=1}^N \rho_{0k}$ we obtain

$$\begin{aligned} \rho_{S:E_{\text{obs}}}(t) &= \sum_a p_a |a\rangle\langle a| \otimes \bigotimes_{k=1}^{N_{\text{obs}}} \rho_{ak}(t) + \sum_a \sum_{a' \neq a} c_{aa'} \\ &\times \left\{ \prod_{k=1}^{N_{\text{uno}}} \text{Tr}[e^{-i(a-a')\hat{B}_k t} \rho_{0k}] \right\} \\ &\times |a\rangle\langle a'| \bigotimes_{k=1}^{N_{\text{obs}}} e^{-ia\hat{B}_k t} \rho_{0k} e^{ia'\hat{B}_k t}, \end{aligned} \quad (2)$$

where $p_a \equiv \langle a|\rho_{0S}|a\rangle$, $c_{aa'} \equiv \langle a|\rho_{0S}|a'\rangle$, $\rho_{ak}(t) \equiv e^{-ia\hat{B}_k t} \rho_{0k} e^{ia\hat{B}_k t}$, $N_{\text{uno}} + N_{\text{obs}} = N$. We define the decoherence factor for the unobserved fraction E_{uno}

$$\Gamma_{aa'}^{\text{uno}}(t) \equiv \prod_{k=1}^{N_{\text{uno}}} |\text{Tr}[e^{-i(a-a')\hat{B}_k t} \rho_{0k}]|^2. \quad (3)$$

If for all $a \neq a'$: (i) $\Gamma_{aa'}^{\text{uno}}(t) = 0$, i.e., decoherence takes place, and (ii) $\rho_{ak}(t) \perp \rho_{a'k}(t)$, i.e., $\rho_{ak}(t)$ are perfectly distinguishable, then we say that $\rho_{S:E_{\text{obs}}}(t)$ is of a Spectrum Broadcast Structure (SBS) [17–19] with respect to (w.r.t.) the basis $|a\rangle$ (this context dependence is of a fundamental importance, see, e.g., [29]), defined as [30]

$$\rho_{\text{SBS}} = \sum_a p_a |a\rangle\langle a| \otimes \rho_a \otimes \dots \otimes \rho_a, \quad \rho_a \perp \rho_{a' \neq a}. \quad (4)$$

The basis $|a\rangle$ becomes then the so-called pointer basis in which the system has decohered and the result of the measurement a , appearing with the probability p_a , is stored in the measuring setup in many, perfect copies. Crucially, their readouts, through projections on the supports of $\rho_{ak}(t)$, will not disturb (on average) the joint state $\rho_{S:E_{\text{obs}}}(t)$. This leads to a form of objectivity of the measurement result: It can be read out by multiple observers without disturbing either the (decohered) system or themselves [8,17,18]. In quantum-information terms, this objectivization process is a weaker form of quantum-state broadcasting [31,32]. We can thus reformulate the original question as follows: Are SBS’s generic for the interactions (1)? To address it, we introduce an ensemble of random Hamiltonians of the form (1) and random initial conditions ρ_{0k} . We then estimate the average trace distance between the actual state (2) and an ideal SBS in the following steps: (i) calculate the averages over \hat{B}_k of the decoherence factor (3) and the so-called superfidelity bound for the states $\rho_{ak}(t)$; (ii) average them over ρ_{0k} ; (iii) coarse-grain the apparatus; (iv) further average

over \hat{A} ; and (v) use the central result of [21] to bound the average distance and show that it vanishes in the macroscopic limit. We then use the concentration inequality of Hoeffding [33], following from the classical Chernoff bound, to show genericity.

The coarse-graining is one of the crucial steps. As we will show, on the microscopic level of the individual apparatuses, the residual noise is too strong to allow a SBS formation even asymptotically. This can be overcome if we group the N_{obs} observed apparatuses into fractions scaling with N (called macrofractions) and pass to the thermodynamic limit $N \rightarrow \infty$ [17]. The number \mathcal{M} of such groups (assumed for simplicity to be equal) is irrelevant, provided their sizes $N_{\text{mac}} \equiv N_{\text{obs}}/\mathcal{M}$ satisfy $N_{\text{mac}} \sim N$. These macrofractions may be understood as reflecting some detection threshold, e.g., a minimum bunch of photons the eye can detect.

III. RANDOMIZING MEASUREMENT HAMILTONIANS

We introduce an ensemble of random measurement Hamiltonians (1) using the widely used Gaussian unitary ensemble [25,26] in the following way (cf. [34,35]): (i) \hat{B}_k are independently, identically distributed (i.i.d.) according to a GUE with a scale factor η_E ; (ii) \hat{A} is distributed according to its own GUE with a scale factor η_S . We recall that the GUE measure is defined as

$$d\mu_{\text{GUE}}(\hat{H}) = \frac{1}{Z} e^{-\frac{\eta}{2} \sum_i \lambda_i^2} \prod_{i < j} (\lambda_i - \lambda_j)^2 d\lambda d\hat{U}, \quad (5)$$

with Z the normalization, λ_i the eigenvalues, η a scale factor, and $d\hat{U}$ the Haar measure on the unitary group.

The simultaneous vanishing of the decoherence factor (3) and of the generalized overlaps [31,36] $F_{aa'} \equiv F(\rho_a, \rho_{a'}) \equiv (\text{Tr} \sqrt{\sqrt{\rho_a} \rho_{a'} \sqrt{\rho_a}})^2$ for all $a \neq a'$ has so far been used to witness a SBS formation [17,21]. This last function is, however, complicated and here we will use the so-called superfidelity bound [37] $F(\rho, \sigma) \leq G(\rho, \sigma) \equiv \text{Tr}(\rho\sigma) + \sqrt{(1 - \text{Tr}\rho^2)(1 - \text{Tr}\sigma^2)}$ (although we note that it is not tight if both states are mixed, as, e.g., for $\rho \perp \sigma$, $G(\rho, \sigma) \neq 0$), which here reads

$$G(\rho_a(t), \rho_{a'}(t)) = \text{Tr}(\rho_a(t)\rho_{a'}(t)) + S_{\text{lin}}(\rho_0) \equiv G_{aa'}(t), \quad (6)$$

where $S_{\text{lin}}(\rho_0) \equiv 1 - \text{Tr}\rho_0^2$ is the linear entropy of the initial state of an individual apparatus.

We now average Eqs. (3) and (6) over the interaction and the initial conditions. We first average over $\{\hat{B}_k\}$, fixing the levels a, a' of \hat{A} . We have

$$\langle \Gamma_{aa'}^{\text{uno}}(t) \rangle_{\{\hat{B}_k\}} = \prod_{k=1}^{N_{\text{uno}}} \langle |\text{Tr}[e^{-i(a-a')\hat{B}_k t} \rho_{0k}]|^2 \rangle_{\hat{B}_k}, \quad (7)$$

since \hat{B}_k are i.i.d. Modulo ρ_{0k} , all the factors are identical and we calculate the average over a single \hat{B}_k , dropping the index k for simplicity. Performing the Haar integration first ([38], Sec. IA) and then the eigenvalue one ([38], Sec. IIB), we obtain [39] the following.

Result 1. The GUE averages of the single environment decoherence and superfidelity factors read

$$\langle \Gamma_{aa'}(t) \rangle = \frac{1 + \text{Tr} \rho_0^2}{d+1} + \langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle \frac{2(d - \text{Tr} \rho_0^2)}{d(d^2 - 1)}, \quad (8)$$

$$\langle G_{aa'}(t) \rangle = S_{\text{lin}}(\rho_0) + \frac{1 + \text{Tr} \rho_0^2}{d+1} + \langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle \frac{2(d \text{Tr} \rho_0^2 - 1)}{d(d^2 - 1)}, \quad (9)$$

with $f_i(\mathbf{a}, \boldsymbol{\lambda}) \equiv \sum_m \sum_{n>m} \cos[(a - a')(\lambda_n - \lambda_m)t]$ and

$$\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle = p(d, \tilde{\Delta}_t) e^{-\tilde{\Delta}_t^2}, \quad (10)$$

$$p(d, \tilde{\Delta}_t) \equiv \sum_n \sum_{m>n} \left[L_n^{(0)}(\tilde{\Delta}_t^2) L_m^{(0)}(\tilde{\Delta}_t^2) - \frac{n!}{m!} \tilde{\Delta}_t^{2(m-n)} [L_n^{(m-n)}(\tilde{\Delta}_t^2)]^2 \right], \quad (11)$$

where $\tilde{\Delta}_t \equiv (a - a')t/\sqrt{\eta_E}$ and $L_n^{(m)}$ are the associated Laguerre polynomials.

The above results are exact. Although the average $\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle$ with the GUE eigenvalue distribution $P_{\text{GUE}}(\boldsymbol{\lambda})$ involves only the two-point correlation function [25] $R_2(\lambda_1, \lambda_2) \equiv d!(d-2)! \int \dots \int d\lambda_3 \dots d\lambda_d P_{\text{GUE}}(\lambda_1, \dots, \lambda_d)$ (due to the symmetry), and the large- d asymptotics of $R_2(\lambda_1, \lambda_2)$ are well known [25], they are of no use here. One can show that [25] $R_2(\lambda_1, \lambda_2) = K_d(\lambda_1, \lambda_1)K_d(\lambda_2, \lambda_2) - [K_d(\lambda_1, \lambda_2)]^2$, $K_d(\lambda_1, \lambda_2) \equiv \sum_{j=0}^{d-1} \phi_j(\lambda_1)\phi_j(\lambda_2)$, with $\phi_j(\lambda)$ the oscillator wave functions, and while the first term approaches the Wigner semicircle distribution, integrable with $f_i(\mathbf{a}, \boldsymbol{\lambda})$, the second term approaches a function of $|\lambda_1 - \lambda_2|$ only [25] and makes the integral divergent. That is the integration with $f_i(\mathbf{a}, \boldsymbol{\lambda})$ and the large- d limit are not interchangeable here.

Both Eqs. (8) and (9) depend on ρ_0 only through its purity $\text{Tr} \rho_0^2$ and we can use the known results of generic state purity to effectively get rid of the initial state dependence. Although there is no canonical choice of a measure over mixed states, there are several popular ones, e.g., the Hilbert-Schmidt and the Bures measures [27] giving

$$\langle \text{Tr} \rho_0^2 \rangle_{\text{HS}} = \frac{2d}{d^2 + 1}, \quad \langle \text{Tr} \rho_0^2 \rangle_{\text{Bu}} = \frac{5d^2 + 1}{2d(d^2 + 2)}. \quad (12)$$

Especially the Bures measure is physically important as it (i) is directly connected to quantum metrology [40] and (ii) reproduces the correct measure for pure states. In what follows we will assume that ρ_{0k} are i.i.d. with one of the above measures and are averaged over.

IV. RESIDUAL NOISE AND COARSE-GRAINING

As $p(d, \tilde{\Delta}_t)$ is an even polynomial of degree $2(2d - 3)$, Eq. (10) implies that the time-dependent part in Eqs. (8) and (9) decays for any fixed d and a gap $|a - a'| \neq 0$ with a characteristic time $\tau_{aa'} \equiv |a - a'|^{-1} \sqrt{\eta_E/(d+1)}$ ([38], Sec. II). The remaining constant terms: A common one of the order $O(1/d)$ [cf. Eq. (12)], called “white noise,” and additionally $\langle S_{\text{lin}}(\rho_0) \rangle$ in Eq. (9). This last, arising from the nontight bound (6), is intuitively understood: the noisier the

apparatus is initially, the lesser information, measured by the state distinguishability, it can accumulate. These factors, reflecting residual background fluctuations in the ensemble, pertain to a single apparatus and prevent a SBS formation. However, coming back to Eq. (3), using Eqs. (7) and (12), we actually obtain an exponential decay with N_{uno} of the collective decoherence factor:

$$0 \leq \langle \Gamma_{aa'}^{\text{uno}}(t) \rangle = \langle \Gamma_{aa'}(t) \rangle^{N_{\text{uno}}} \xrightarrow[t \gg \tau_{aa'}]{} O(t d^{-N_{\text{uno}}}), \quad (13)$$

showing that for a large local dimension d and/or large unobserved fraction N_{uno} , measurement dynamics (1) generically leads to decoherence (cf. [13]). The same step can be performed on the observed fraction as well [17]: We group the N_{obs} observed apparatuses into M groups of N_{mac} each, described by states $\rho_a^{\text{mac}}(t) \equiv \bigotimes_{k \in \text{mac}} \rho_{ak}(t)$. Due to the factorization of fidelity w.r.t. the tensor product and the i.i.d. property, the resulting superfidelity bound (6) for the group also decays [cf. (12)]:

$$0 \leq \langle F_{aa'}^{\text{mac}}(t) \rangle \leq \langle G_{aa'}(t) \rangle^{N_{\text{mac}}} \xrightarrow[t \gg \tau_{aa'}]{} O(e^{-\frac{N_{\text{mac}}}{d}}). \quad (14)$$

If both $N_{\text{uno}}, N_{\text{mac}}$ scale with N , Eqs. (13) and (14) can be made small in the macroscopic or thermodynamic limits $N \rightarrow \infty$. Crucially, increasing d alone is not enough as it damps the white noise, but $\langle S_{\text{lin}}(\rho_0) \rangle \simeq 1 - O(1/d)$ by Eq. (12).

V. GENERIC POSTMEASUREMENT STATE AND OBJECTIVITY

Results (8) and (9) still depend on the \hat{A} 's level differences $|a - a'|$. To study a completely general behavior, a further averaging of $\langle \Gamma_{aa'}^{\text{uno}}(t) \rangle, \langle G_{aa'}^{\text{mac}}(t) \rangle$ over the levels a, a' should be performed with the corresponding two-point correlation function $R_2(a, a')$ (the average is independent of the labels a, a' due to the symmetry). The resulting integrals are intractable, but from Eqs. (8) to (A52) they will eventually reach the noise-floor (see Fig. 1). Lower bounds on the relevant time scales can be obtained from a short-time analysis ([38], Sec. II), giving for the decoherence and the superfidelity, respectively,

$$\begin{aligned} \tau_{\text{dec}} &\equiv [8g^2 N_{\text{uno}} d_S (d - \langle \text{Tr} \rho_0^2 \rangle)]^{-\frac{1}{2}} d \gg 1 \frac{g^{-1}}{\sqrt{N_{\text{uno}} d_S d}}, \\ \tau_{\text{fid}} &\equiv [8g^2 N_{\text{mac}} d_S (d \langle \text{Tr} \rho_0^2 \rangle - 1)]^{-\frac{1}{2}} d \gg 1 \frac{g^{-1}}{\sqrt{N_{\text{mac}} d_S}}. \end{aligned} \quad (15)$$

Here $g^{-1} \equiv \sqrt{\eta_S \eta_E}$ is the effective interaction time scale and d_S is the system dimension. We see a characteristic separation of time scales: From Eq. (12), $\tau_{\text{fid}} \sim \sqrt{d} \tau_{\text{dec}}$ for the same macrofraction sizes. Thus, on average, it takes longer to accumulate information in the apparatus than to decohere the system [17,20]. Combining Eq. (15) with Result 1 and Eqs. (13) and (14) we arrive at (cf. [42]) the following.

Result 2. The interaction and initial state averages satisfy

$$\langle \langle \Gamma^{\text{uno}}(t) \rangle \rangle \xrightarrow[t \gg \tau_{\text{dec}}]{} O(e^{-N_{\text{uno}} \log d}), \quad (16)$$

$$\langle \langle G^{\text{mac}}(t) \rangle \rangle \xrightarrow[t \gg \tau_{\text{fid}}]{} O(e^{-\frac{N_{\text{mac}}}{d}}). \quad (17)$$

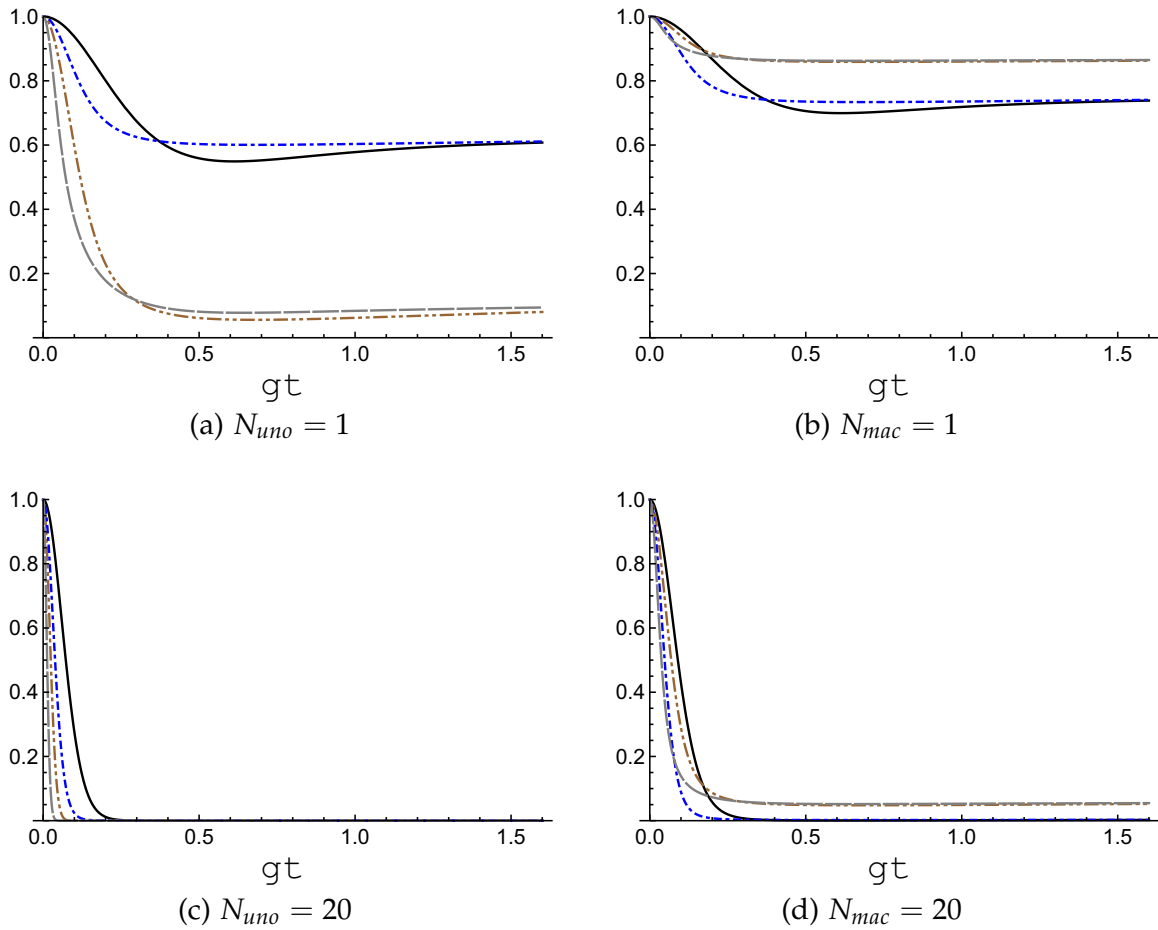


FIG. 1. Time dependence of the exact full averages of the (a), (c) decoherence factor and (b), (d) the superfidelity for different dimensions and macrofraction sizes. The two-point correlation function $R_2(a, a')$ averages of the exact solutions (8) and (9) were used, with $\langle \text{Tr} \rho_0^2 \rangle_D$ from Eq. (12). Different combinations of $\{d_s, d\}$ are plotted: $\{2, 2\}$ solid black; $\{2, 10\}$ brown dot-dot-dash; $\{10, 2\}$ blue dot-dash; $\{10, 10\}$ gray long-dash. The time is in the units of the interaction strength $g \equiv 1/\sqrt{\eta_S \eta_E}$ [41].

Next crucial step is to use the result of [21] estimating an optimal trace distance between Eq. (2) and an ideal SBS state on the coarse-grained level of macrofractions

$$\begin{aligned} \epsilon_{\text{SBS}}(t) &\equiv \frac{1}{2} \|\rho_{S:E_{\text{obs}}}(t) - \rho_{\text{SBS}}\|_{\text{Tr}} \\ &\leq \sum_a \sum_{a' \neq a} \left[|c_{aa'}| \sqrt{\Gamma_{aa'}^{\text{uno}}(t)} + \sqrt{p_a p_{a'}} \sum_{\text{mac}} \sqrt{F_{aa'}^{\text{mac}}(t)} \right]. \end{aligned} \quad (18)$$

Using $p_a, |c_{aa'}| \leq 1$, $\langle \sqrt{f} \rangle \leq \sqrt{\langle f \rangle}$ for $f \geq 0$, the superfidelity bound, and the Result 2, estimation (18) gives the following result.

Result 3. Averaged over all the von Neumann measurements (1) and the initial conditions, the optimal distance of the actual state (2) to an ideal SBS state satisfies

$$\langle \epsilon_{\text{SBS}}(t) \rangle \xrightarrow{t \gg \tau_{\text{SBS}}} O \left[d_S^2 \left(e^{-\frac{N_{\text{uno}}}{2} \log d} + \mathcal{M} e^{-\frac{N_{\text{mac}}}{2d}} \right) \right], \quad (19)$$

where τ_{SBS} is the larger of Eq. (15) and \mathcal{M} is the number of macrofractions into which the observed degrees of freedom of the apparatus are coarse-grained.

Finally, since $0 \leq \epsilon_{\text{SBS}}(t) \leq 1$ is a bounded random variable for any t , it follows from the Hoeffding inequality [33] that $P[|\epsilon_{\text{SBS}}(t) - \langle \epsilon_{\text{SBS}}(t) \rangle| \geq \delta] \leq 2e^{-2\delta^2}$ for any $\delta \geq 0$. This, together with Result 3 shows the genericity of the SBS formation for large-enough apparatuses and long-enough times.

VI. CONCLUSION

A measurement is an inevitable part of any quantum experiment and the results must inevitably be encoded into macroscopic degrees of freedom and become effectively classical for us to read. This, in particular, entails becoming objective. We studied this process using the general von Neumann measurement scheme (1) with a macroscopic measuring apparatus. A huge amount of degrees of freedom ($N \sim 10^{23}$) makes it, in practice, impossible to observe them all and to control each individual coupling. A way to model this physical situation is to introduce some randomness and ask questions about genericity. We did it in two steps. First we randomized the measurement device side (the observables and the generically noisy initial states) and showed that after including the inevitable losses and macroscopic coarse-graining,

a postmeasurement state approaches the so-called SBS form asymptotically for almost any initial conditions and couplings. The time scales of this process depends on the spectral gap of the measured observable on the system side. Afterwards, to get rid of this dependence, we went beyond a single experiment scenario, randomizing the measured observable also. An interesting aspect of that second randomization is that this may be viewed as a quite natural assumption of any quantum system interacting with many objects. Indeed it is natural to assume that it interacts with each of the objects with some fixed, yet different than with the others, way. Since there are many objects, then the averaging effect comes from that variety of the interactions and can be viewed as a self-averaging of the system plus environment complex. This leads to our central result: Almost any quantum measurement produces objective outcomes on the macroscopic level on the time scale given by the larger of Eq. (15). This is a universal, model-independent result.

We believe one can go beyond the genericity notion used here (Hoeffding inequality) and show the concentration of measure phenomenon, e.g., by combining the results for the Wigner-type matrices [43] with the methods of [22,42]. Another possible future direction is to go beyond the quantum measurement limit and consider nontrivial dynamics of the system and the measuring device. A candidate tool for such an analysis already exists in the form of dynamical SBS [19].

ACKNOWLEDGMENTS

This work was carried out at the National Quantum Information Centre in Gdańsk. We thank J. Wehr, P. Grangier, M. Horodecki, and R. Horodecki for discussions. J.K.K. and P.H. acknowledge the financial support of the John Templeton Foundation through Grant No. 56033. E.A.A. is supported by the Grant Sonata Bis 2014/14/E/ST2/00020 from the National Science Centre. P.Ć. thanks the Seventh framework programme EU Grant RAQUEL No. 323970 and the Grant PRELUDIUM 2015/17/N/ST2/04047 from the National Science Centre.

APPENDIX A: ENSEMBLE AVERAGE OVER THE APPARATUS

1. Average over the Haar distributed unitary transformations U

In this section we average the decoherence and the super-fidelity factors over the Haar measure. Due to the assumed independent identical distribution (i.i.d.) of the apparatus observables B_k , $k = 1, \dots, N$, it is enough to calculate the averages over a single observable only. This is what we shall calculate, neglecting for brevity the index k . We start with the decoherence factor and prove the following.

Theorem 4. The decoherence factor for the single copy of the environment average over the Haar distributed unitary transformations U is equal to

$$\langle \Gamma_{aa'}(t) \rangle_U = |\text{Tr } D|^2 \frac{d - \text{Tr}[\rho_0^2]}{d(d^2 - 1)} + \frac{d \text{Tr}[\rho_0^2] - 1}{d^2 - 1}, \quad (\text{A1})$$

where d is the local dimension of the environment.

We first write the decoherence factor as

$$\begin{aligned} \Gamma_{aa'}(t) &= |\text{Tr}[e^{-i(a-a')Bt} \rho_0]|^2 = \text{Tr}[e^{-i(a-a')Bt} \rho_0] \\ &\quad \times \text{Tr}[e^{-i(a-a')Bt} \rho_0]^\dagger = \text{Tr}[e^{-i(a-a')Bt} \rho_0] \\ &\quad \times \text{Tr}[(e^{-i(a-a')Bt})^\dagger \rho_0] \\ &= \text{Tr}[U D U^\dagger \rho_0 \otimes U D^\dagger U^\dagger \rho_0] \\ &= \text{Tr}[(U^\dagger \otimes U^\dagger)(\rho_0 \otimes \rho_0)(U \otimes U)(D \otimes D^\dagger)], \end{aligned} \quad (\text{A2})$$

where we diagonalized the observable B as $B = U \text{diag}[\lambda_1, \dots, \lambda_d] U^\dagger$ and defined

$$D \equiv \text{diag}[e^{-i\Delta_t \lambda_1}, \dots, e^{-i\Delta_t \lambda_d}], \quad \Delta_t \equiv (a - a')t. \quad (\text{A3})$$

We also used $\text{Tr } A \text{Tr } B = \text{Tr}(A \otimes B)$ in the second line and the following fact in the first step.

Fact 1. For any operator X the following is true

$$|\text{Tr } X|^2 = \text{Tr } X \text{Tr } X^\dagger, \quad (\text{A4})$$

where \dagger stands for the Hermitian conjugation.

Proof. Please note that

$$|\text{Tr } X|^2 = \text{Tr } X \overline{\text{Tr } X} = \text{Tr } X \overline{\text{Tr } X^T} = \text{Tr } X \text{Tr } X^\dagger, \quad (\text{A5})$$

where \overline{X} stands for the complex conjugation of X and we used $\text{Tr } X = \text{Tr } X^T$, and $\overline{X^T} = X^\dagger$.

We will also need two more well-known facts.

Fact 2: For any operators A, B , and the *SWAP* operator \mathbb{V} , we have that

$$\text{Tr}[\mathbb{V}A \otimes B] = \text{Tr}(AB). \quad (\text{A6})$$

Proof. Let us write the *SWAP* operator as

$$\mathbb{V} = \sum_{ij} |ij\rangle \langle ji|. \quad (\text{A7})$$

Inserting Eq. (A7) into Eq. (A6) we have that

$$\begin{aligned} \text{Tr}[\mathbb{V}A \otimes B] &= \text{Tr} \left(\sum_{ij} |ij\rangle \langle ji| A \otimes B \right) \\ &= \text{Tr} \left(\sum_{ij} \langle ji| A \otimes B |ij\rangle \right) \\ &= \sum_{ijkl} \langle ji| A |kl\rangle \langle kl| B |ij\rangle = \text{Tr}(AB). \end{aligned} \quad (\text{A8})$$

Fact 3. For any Hermitian operator X from \mathbb{C}^d to \mathbb{C}^d , it holds

$$\begin{aligned} &\int dU U \otimes U X \otimes X U^\dagger \otimes U^\dagger \\ &= \frac{2}{d(d+1)} \text{Tr}[\Pi_{\text{sym}} X \otimes X] \Pi_{\text{sym}} \\ &\quad + \frac{2}{d(d-1)} \text{Tr}[\Pi_{\text{asym}} X \otimes X] \Pi_{\text{asym}}, \end{aligned} \quad (\text{A9})$$

where Π_{sym} and Π_{asym} are the orthogonal projectors onto the symmetric and antisymmetric subspaces, respectively,

equal to

$$\Pi_{\text{sym}} \equiv \frac{\mathbb{I} + \mathbb{V}}{2}, \quad \Pi_{\text{asym}} \equiv \frac{\mathbb{I} - \mathbb{V}}{2}, \quad (\text{A10})$$

where \mathbb{V} is the SWAP operator.

We now integrate Eq. (A2) over $U \otimes U$. Using linearity of the trace we pull the integral inside the trace

$$\begin{aligned} \langle \Gamma_{aa'}(t) \rangle_U &= \int dU \text{Tr}(U^\dagger \otimes U^\dagger \rho_0 \otimes \rho_0 U \otimes U D \otimes D^\dagger) \\ &= \text{Tr} \left[\left(\int dU U \otimes U \rho_0 \otimes \rho_0 U^\dagger \otimes U^\dagger \right) D \otimes D^\dagger \right]. \end{aligned} \quad (\text{A11})$$

We then use Fact 3 with $X \equiv \rho_0$. We can easily calculate $\text{Tr}[\Pi_{\text{sym}}\rho_0 \otimes \rho_0]$ and $\text{Tr}[\Pi_{\text{asym}}\rho_0 \otimes \rho_0]$ using Facts 1 and 2 and obtain

$$\begin{aligned} \text{Tr}[\Pi_{\text{sym}}\rho_0 \otimes \rho_0] \Pi_{\text{sym}} &= \frac{1}{2} \text{Tr}[(\mathbb{I} + \mathbb{V})(\rho_0 \otimes \rho_0)] \frac{\mathbb{I} + \mathbb{V}}{2} \\ &= \frac{1 + \text{Tr}[\rho_0^2]}{2} \frac{\mathbb{I} + \mathbb{V}}{2}, \end{aligned} \quad (\text{A12})$$

and for the antisymmetric projector

$$\begin{aligned} \text{Tr}[\Pi_{\text{asym}}\rho_0 \otimes \rho_0] \Pi_{\text{asym}} &= \frac{1}{2} \text{Tr}[(\mathbb{I} - \mathbb{V})(\rho_0 \otimes \rho_0)] \frac{\mathbb{I} - \mathbb{V}}{2} \\ &= \frac{1 - \text{Tr}[\rho_0^2]}{2} \frac{\mathbb{I} - \mathbb{V}}{2}. \end{aligned} \quad (\text{A13})$$

We then again use Facts 1 and 2 to calculate the remaining traces $\text{Tr}[(\mathbb{I} \pm \mathbb{V})D \otimes D^\dagger]$, keeping in mind that D is Hermitian and that $\text{Tr} D^2 = d$. This finally gives

$$\langle \Gamma_{aa'}(t) \rangle_U = |\text{Tr} \hat{D}|^2 \frac{d - \text{Tr}[\rho_0^2]}{d(d^2 - 1)} + \frac{d \text{Tr}[\rho_0^2] - 1}{d^2 - 1}, \quad (\text{A14})$$

proving our Theorem. ■

Using the same technique, one can also calculate the average of the superfidelity factor. The only nontrivial part is the Hilbert-Schmidt product between the apparatus states $\rho_a(t)$ and $\rho_{a'}(t)$. Using the same notation as in Eq. (A2) we obtain

$$\begin{aligned} \text{Tr}(\rho_a(t)\rho_{a'}(t)) &= \text{Tr}[\rho_0 e^{i(a-a')Bt} \rho_0 e^{-i(a-a')Bt}] \\ &= \text{Tr}[U^\dagger \rho_0 U D^\dagger U^\dagger \rho_0 U D] \\ &= \text{Tr}[\mathbb{V}(U^\dagger \otimes U^\dagger)(\rho_0 \otimes \rho_0) \\ &\quad \times (U \otimes U)(D^\dagger \otimes D)], \end{aligned} \quad (\text{A15})$$

where in the last step we used Fact 2. We note that the only difference between the above Hilbert-Schmidt factor and the decoherence factor (A2) is the presence of the SWAP operator \mathbb{V} . Repeating the same steps as above gives

$$\langle G_{aa'}(t) \rangle_U = S_{\text{lin}}(\rho_0) + \frac{d - \text{Tr} \rho_0^2}{d^2 - 1} + |\text{Tr} D|^2 \frac{d \text{Tr} \rho_0^2 - 1}{d(d^2 - 1)}. \quad (\text{A16})$$

Finally, we evaluate $|\text{Tr} D|^2$ from its definition in Eq. (A3)

$$\begin{aligned} |\text{Tr} D|^2 &\equiv d + 2f_t(\mathbf{a}, \boldsymbol{\lambda}), \\ f_t(\mathbf{a}, \boldsymbol{\lambda}) &\equiv \sum_m \sum_{n>m} \cos[\Delta_t(\lambda_n - \lambda_m)] \\ &= \sum_m \sum_{n>m} \cos[(a - a')(\lambda_n - \lambda_m)t], \end{aligned} \quad (\text{A17})$$

which is a function of the eigenvalues \mathbf{a} of the observable A and the eigenvalues $\boldsymbol{\lambda}$ of B .

2. Averaging over the eigenvalues

After averaging over the unitary group in Sec. A1, we perform the average over the GUE eigenvalue distribution

$$P_{\text{GUE}}(\boldsymbol{\lambda}) = \frac{1}{Z} e^{-\frac{1}{2}\eta_E \sum_m \lambda_m^2} \prod_{i<j} |\lambda_j - \lambda_i|^2. \quad (\text{A18})$$

Here η_E is the eigenvalue scale of the observable B and Z is a normalization constant (the GUE partition function). The task then is to find the following average:

$$\begin{aligned} \langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle &= \left\langle \sum_i \sum_{j>i} \cos[\Delta_t(\lambda_i - \lambda_j)] \right\rangle \\ &= \sum_i \sum_{j>i} \int d\boldsymbol{\lambda} P_{\text{GUE}}(\boldsymbol{\lambda}) \cos[\Delta_t(\lambda_i - \lambda_j)]. \end{aligned} \quad (\text{A19})$$

Quite surprisingly, this average can be performed explicitly using the standard methods of dealing with GUE [25]. We first introduce the harmonic oscillator wave functions

$$\phi_n(x) \equiv \frac{1}{\sqrt{\sqrt{2\pi}n!}} e^{-\frac{x^2}{4}} \text{He}_n(x). \quad (\text{A20})$$

Notice that we define the wave functions using the so-called ‘‘probabilist’’ Hermite polynomials

$$\text{He}_n(x) \equiv (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}.$$

That is, they are orthogonal with respect to the weight function $\exp[-x^2/2]$, and are related to the physicist’s polynomials $H_n(x)$ via $\text{He}_n(x) = 2^{-n/2} H_n(x/\sqrt{2})$. Of course we still have $\int dx \phi_n(x)\phi_m(x) = \delta_{nm}$. Then the GUE eigenvalue distribution takes on a very compact and elegant form, after rescaling $\lambda_k \equiv \frac{\zeta_k}{\sqrt{\eta_E}}$ [25] (6.2.4):

$$P_{\text{GUE}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda} = \frac{1}{d!} \det[\phi_{j-1}(\zeta_i)]^2 d\boldsymbol{\zeta}, \quad (\text{A21})$$

where $i, j = 1, \dots, d$.

a. Exploiting the symmetry

A crucial step is the realization that this average has an index permutation symmetry. Let $\sigma \in S_d$ be a permutation, then

$$P_{\text{GUE}}(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(d)}) = P_{\text{GUE}}(\lambda_1, \dots, \lambda_d). \quad (\text{A22})$$

Analogously we have (keeping the eigenvalues \mathbf{a} fixed)

$$f_i(\mathbf{a}, \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(d)}) = \sum_{i < j} \cos[\Delta_i(\lambda_{\sigma(i)} - \lambda_{\sigma(j)})] \\ = \sum_{i < j} \cos[\Delta_i(\lambda_i - \lambda_j)] = f_i(\mathbf{a}, \boldsymbol{\lambda}). \tag{A23}$$

This is because in both expressions the eigenvalue functions are symmetric and all pairs of indices are taken (i.e., the product or sum is over all $i < j$). Equivalently, we can recall that $\text{Tr } D = \sum_i e^{-i\Delta_i \lambda_i}$ and from Eq. (A17) $f_i(\mathbf{a}, \boldsymbol{\lambda}) = 1/2(|\text{Tr } D|^2 - d)$, which is clearly symmetric under the permutations. Hence, the calculation of the average $f_i(\mathbf{a}, \boldsymbol{\lambda})$ reduces to a single term

$$\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle = \frac{d(d-1)}{2} \int d\boldsymbol{\lambda} P_{\text{GUE}}(\boldsymbol{\lambda}) \cos[\Delta_i(\lambda_1 - \lambda_2)]. \tag{A24}$$

As the integrand only depends on two variables, we can take the marginal distribution, which is essentially the 2-point correlation function, defined as [25] (6.1.2)

$$R_2(\lambda_1, \lambda_2) \equiv \frac{d!}{(d-2)!} \int d\lambda_3, \dots, d\lambda_d P_{\text{GUE}}(\lambda_1, \dots, \lambda_d). \tag{A25}$$

Hence, we reduced the problem to the integral

$$\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle = \frac{1}{2} \int d\lambda_1 d\lambda_2 R_2(\lambda_1, \lambda_2) \cos[\Delta_i(\lambda_1 - \lambda_2)]. \tag{A26}$$

b. The crucial integral

To calculate (A26) we need the following integral

$$J_{n,m}(\alpha) \equiv \int dx \phi_n(x) \phi_m(x) e^{i\alpha x}. \tag{A27}$$

In fact, Eq. (A27) may be interpreted as a special case of a matrix element of the displacement operator $D(\beta) = \exp[\beta \hat{a}^\dagger - \beta^* \hat{a}]$ in the Fock basis $\{|n\rangle\}_{n \in \mathbb{N}}$. We recover our integral setting $\beta \equiv i\alpha$, $\alpha \in \mathbb{R}$. This turns out to be a well-known quantity in quantum optics (see, e.g., [44]), but for completeness we present its calculation below. Without a loss of generality, we assume $m \geq n$. First, we express the wave functions through the (probabilist) Hermite polynomials as in Eq. (A20) and use the generating function for $\text{He}_n(x)$ with parameters r, s to perform the integral

$$J_{n,m}(\alpha) = \frac{1}{\sqrt{2\pi n! m!}} \int dx \text{He}_n(x) \text{He}_m(x) e^{-\frac{1}{2}x^2 + i\alpha x} \tag{A28}$$

$$= \frac{1}{\sqrt{2\pi n! m!}} \left. \frac{\partial^n}{\partial r^n} \right|_{r=0} \left. \frac{\partial^m}{\partial s^m} \right|_{s=0} \\ \times \int dx e^{(i\alpha + r + s)x - \frac{1}{2}(r^2 + s^2 + x^2)} \tag{A29}$$

$$= \frac{e^{-\frac{1}{2}\alpha^2}}{\sqrt{n! m!}} \left. \frac{\partial^n}{\partial r^n} \right|_{r=0} \left. \frac{\partial^m}{\partial s^m} \right|_{s=0} e^{i\alpha r + i\alpha s + rs} \tag{A30}$$

$$= \frac{e^{-\frac{1}{2}\alpha^2}}{\sqrt{n! m!}} \left. \frac{\partial^n}{\partial r^n} \right|_{r=0} (i\alpha + r)^m e^{i\alpha r}. \tag{A31}$$

Then we use the binomial formula for the derivatives $\frac{d^n}{dx^n} f(x)g(x) = \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k} f(x)}{dx^{n-k}} \frac{d^k g(x)}{dx^k}$. Since $m \geq n$ we do not run into any unexpected problems and obtain

$$J_{n,m}(\alpha) = \frac{e^{-\frac{1}{2}\alpha^2}}{\sqrt{n! m!}} \sum_{k=0}^n \binom{n}{k} \binom{m}{k} k! (i\alpha)^{n+m-2k}. \tag{A32}$$

This may be nicely expressed in terms of the associated Laguerre polynomials as (taking $\alpha \in \mathbb{R}$)

$$J_{n,m}(\alpha) = e^{-\frac{1}{2}\alpha^2} \sqrt{\frac{n!}{m!}} (i\alpha)^{m-n} L_n^{(m-n)}(\alpha^2), \tag{A33}$$

where

$$L_n^{(m)}(x) \equiv \sum_{k=0}^n \binom{n+m}{n-k} \frac{(-x)^k}{k!} \tag{A34}$$

[we adopt the common standardization for the Laguerre polynomials that the leading coefficient is equal to $(-1)^n/n!$]. Equation (A27) can also be expressed more compactly in terms of the so-called two-dimensional (2D) Laguerre functions introduced in Ref. [45]

$$\langle m | D(\alpha) | n \rangle = (-1)^n \sqrt{\pi} l_{m,n}(\alpha, \alpha^*) \tag{A35}$$

for a general complex displacement α . The 2D Laguerre functions are defined as [45]

$$l_{m,n}(z, z^*) \equiv \frac{1}{\sqrt{\pi}} e^{-\frac{zz^*}{2}} \frac{1}{\sqrt{m! n!}} \\ \times \sum_{j=0}^m \binom{m}{j} \binom{n}{j} j! (-1)^j z^{m-j} z^{*n-j}. \tag{A36}$$

c. Putting the results together

We return to calculating the integral (A26). We use Eq. (A21), rescale the variables, and introduce a more friendly notation $(x, y) \equiv (\zeta_1, \zeta_2)$:

$$\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle = \frac{1}{2} \int dx dy R_2(x, y) \cos[\tilde{\Delta}_i(x - y)], \tag{A37}$$

where

$$\tilde{\Delta}_i \equiv \frac{(a - a')t}{\sqrt{\eta_E}}. \tag{A38}$$

Now, Dyson's Theorem will let us calculate the 2-point correlation function [25] Theorem 5.14, (6.2.6-7)

$$R_2(x, y) = K(x, x)K(y, y) - K(x, y)^2, \tag{A39}$$

where the kernel is defined through the oscillator wave functions (A20) as

$$K(x, y) \equiv \sum_{j=0}^{d-1} \phi_j(x) \phi_j(y). \tag{A40}$$

Hence, we can express our integral as the following sum:

$$\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle = \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} \int dx dy (\phi_n(x)^2 \phi_m(y)^2 \\ - \phi_n(x) \phi_m(x) \phi_n(y) \phi_m(y)) \cos[\tilde{\Delta}_i(x - y)]. \tag{A41}$$

By expressing the cosine function in exponential form, the integrals become separable and we obtain

$$\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle = \frac{1}{4} \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} (A_{n,m} + \tilde{A}_{n,m} - 2B_{n,m}), \quad (\text{A42})$$

where we introduce the auxiliary functions

$$A_{n,m} \equiv J_{n,n}(\tilde{\Delta}_t) J_{m,m}(-\tilde{\Delta}_t), \quad (\text{A43})$$

$$\tilde{A}_{n,m} \equiv J_{n,n}(-\tilde{\Delta}_t) J_{m,m}(\tilde{\Delta}_t), \quad (\text{A44})$$

$$B_{n,m} \equiv J_{n,m}(\tilde{\Delta}_t) J_{n,m}(-\tilde{\Delta}_t). \quad (\text{A45})$$

Now, we are able to separate the n, m summation into three parts $n < m$, $n = m$, and $n > m$. From the definition of the auxiliary functions, we easily see that the diagonal summation

$n = m$ vanishes. The remaining sums $n < m$ and $n > m$ become the same since $A_{m,n} = \tilde{A}_{n,m}$ and $B_{m,n} = B_{n,m}$. Hence, for convenience sake we will calculate the sum $m > n$ only. We use the explicit result (A32) for the $J_{n,m}$ and obtain

$$A_{n,m} = e^{-\tilde{\Delta}_t^2} \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \frac{(-1)^{n+m-k-l} \tilde{\Delta}_t^{2(n+m-k-l)}}{(n-k)!(m-l)!}. \quad (\text{A46})$$

And by doing the same for \tilde{A} , we indeed realize that $A_{n,m} = \tilde{A}_{n,m}$. For $B_{n,m}$, in turn, we obtain

$$B_{n,m} = e^{-\tilde{\Delta}_t^2} \sum_{k=0}^n \sum_{l=0}^n \binom{n}{k} \binom{m}{l} \frac{(-1)^{k+l} \tilde{\Delta}_t^{2(n+m-k-l)}}{(m-k)!(n-l)!}. \quad (\text{A47})$$

Hence, putting it all together (with a factor of 2 since now we only sum $m > n$), we arrive at

$$\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle = e^{-\tilde{\Delta}_t^2} \sum_{n=0}^{d-2} \sum_{m=n+1}^{d-1} \left[\sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \frac{(-1)^{n+m-k-l} \tilde{\Delta}_t^{2(n+m-k-l)}}{(n-k)!(m-l)!} - \sum_{k=0}^n \sum_{l=0}^n \binom{n}{k} \binom{m}{l} \frac{(-1)^{k+l} \tilde{\Delta}_t^{2(n+m-k-l)}}{(m-k)!(n-l)!} \right]. \quad (\text{A48})$$

This result may also be rewritten using the associated Laguerre polynomials and Eq. (A33)

$$\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle = e^{-\tilde{\Delta}_t^2} \sum_{n=0}^{d-2} \sum_{m=n+1}^{d-1} \left[L_n^{(0)}(\tilde{\Delta}_t^2) L_m^{(0)}(\tilde{\Delta}_t^2) - \frac{n!}{m!} \tilde{\Delta}_t^{2(m-n)} [L_n^{(m-n)}(\tilde{\Delta}_t^2)]^2 \right]. \quad (\text{A49})$$

One of the sums can be performed using the following identity for the associated Laguerre polynomials:

$$\sum_{m=0}^M L_m^{(\alpha)}(x) = L_M^{(\alpha+1)}(x), \quad (\text{A50})$$

giving

$$\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle = e^{-\tilde{\Delta}_t^2} \left[L_{d-1}^{(1)}(\tilde{\Delta}_t^2) L_{d-2}^{(1)}(\tilde{\Delta}_t^2) - \sum_{n=0}^{d-2} L_n^{(0)}(\tilde{\Delta}_t^2) L_n^{(1)}(\tilde{\Delta}_t^2) - \sum_{n=0}^{d-2} \sum_{m=n+1}^{d-1} \frac{n!}{m!} \tilde{\Delta}_t^{2(m-n)} [L_n^{(m-n)}(\tilde{\Delta}_t^2)]^2 \right]. \quad (\text{A51})$$

However, we keep Eq. (A49) in the main text since it is more compact.

Please note that in the main text we do not directly use $\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle$, but rather separate its Gaussian and polynomial parts, i.e.,

$$\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle \equiv e^{-\tilde{\Delta}_t^2} p(d, \tilde{\Delta}_t). \quad (\text{A52})$$

APPENDIX B: SHORT TIME ANALYSIS

In this Appendix we perform the final averaging over the system observable A . In the previous section, we used the i.i.d. property of the apparatus ensemble to reduce the big, compound averages to single copy ones. Here, however, we cannot do so, as ultimately we are interested in the macroscopic quantities $\Gamma_{aa'}^{N_{\text{uno}}}$, $G_{aa'}^{\text{mac}}$. Thus we need

$$\langle\langle X^f(t) \rangle\rangle_{aa'} \equiv \int d\mathbf{a} P_{\text{GUE}}(\mathbf{a}) \langle X_{aa'}(t) \rangle^{N_f}, \quad (\text{B1})$$

where $X_{aa'} = \Gamma_{aa'}$ or $G_{aa'}$ and $f = \text{uno}$ or mac , respectively. We note that both $\langle \Gamma_{aa'}(t) \rangle$, $\langle G_{aa'}(t) \rangle$ depend on A only through the eigenvalue differences $|a - a'|$. Thus, the A -averaging reduces to averaging over the eigenvalues only with its own

GUE eigenvalue distribution

$$P_{\text{GUE}}(\mathbf{a}) \equiv P_{\text{GUE}}(a_1, \dots, a_{d_S}) \equiv \frac{1}{Z_S} e^{-\frac{1}{2} \eta_S \sum_i a_i^2} \prod_{i < j} |a_j - a_i|^2, \quad (\text{B2})$$

where d_S is the system dimension and η_S is the eigenvalue scale of the system observable A . From the permutational symmetry of the GUE distribution (best seen through the Vandermonde determinant), the integral in Eq. (B1) reduces to the integration with the same 2-point correlation function (A25), but now defined for the distribution (B2) and thus all the averages for different pairs aa' are the same and equal to

$$\langle\langle X^f(t) \rangle\rangle = \left[\frac{d_S!}{(d_S - 2)!} \right]^{-1} \int d\mathbf{a} d\mathbf{a}' R_2(a, a') \langle X_{aa'}(t) \rangle^{N_f}. \quad (\text{B3})$$

The resulting integral is too complicated to be performed analytically and actually this is, in fact, not needed as we see from Eq. (A52) that $\langle f_t(\mathbf{a}, \boldsymbol{\lambda}) \rangle$ will eventually decay so that both factors will approach their noise-floor values (cf. Result 1 from the main text). What we are interested in are

the relevant time scales. We can estimate lower bounds on those time scales from the decay times of Eq. (A52). We perform this analysis in the following steps: (i) assume a short-time limit; (ii) approximate the polynomial $p(d, \tilde{\Delta})$ of Eq. (A52); (iii) approximate the N_f power; (iv) using Eqs. (A39), (A40), and (A20) estimate the fastest decaying term in Eq. (B3). This will then give lower bounds on the desired times of the asymptotic approach: These times are for sure greater than the initial decay times.

First, we assume $\tilde{\Delta}_t \ll 1$, or $t \ll \sqrt{\eta_E}/|a - a'|$. The maximum of $|a - a'|$ is of the order of $\sqrt{d_S/\eta_S}$ from the Wigner semicircle law, defining the short-time limit

$$t \ll \sqrt{\frac{\eta_E \eta_S}{d_S}} \equiv \frac{1}{g\sqrt{d_S}}, \quad (\text{B4})$$

where $g \equiv 1/\sqrt{\eta_E \eta_S}$ is the effective interaction strength.

We now explicitly calculate the coefficients of the lowest-order terms in $p(d, \tilde{\Delta})$ directly from Eq. (A48). One immediately sees that the polynomial is even so the lowest terms are the constant and the quadratic ones. The constant term occurs when $k + l = m + n$, but this can only occur in the first summand of the polynomial when $k = n$ and $l = m$, so that we have

$$\sum_{n < m} [1] = \frac{d(d-1)}{2}. \quad (\text{B5})$$

The quadratic term occurs when the indices fulfill the condition $k + l + 1 = m + n$. On the first term, this can occur if $(k = n, l = m - 1)$ or if $(k = n - 1, l = m)$. On the second term this can occur only when the m index is $m + 1$ and the inner indices are $k = l = n$. Thus we obtain

$$\begin{aligned} & \sum_{n < m} [m(-1)\tilde{\Delta}_t^2] + \sum_{n < m} [n(-1)\tilde{\Delta}_t^2] - \sum_n [(n+1)\tilde{\Delta}_t^2] \\ &= -\frac{d^2(d-1)}{2} \tilde{\Delta}_t^2. \end{aligned} \quad (\text{B6})$$

Thus, for short times we have

$$p(d, \tilde{\Delta}_t) = \frac{d(d-1)}{2} (1 - d\tilde{\Delta}_t^2) + O(\tilde{\Delta}_t^4). \quad (\text{B7})$$

To proceed further, we upper bound the above expression by the Gaussian function $1 - d\tilde{\Delta}_t^2 \leq e^{-d\tilde{\Delta}_t^2}$ resulting from Eq. (A52) in the short-time bound

$$\begin{aligned} \langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle &\lesssim e^{-(d+1)\tilde{\Delta}_t^2} \equiv e^{-(t/\tau_{aa'})^2}, \\ \tau_{aa'} &\equiv \frac{\sqrt{\eta_E}}{\sqrt{d+1}|a-a'|}. \end{aligned} \quad (\text{B8})$$

We then use the following approximation of the power

$$(\alpha + \beta e^{-x})^{N_f} \approx (\alpha + \beta)^{N_f} e^{-N_f \frac{\beta}{\alpha + \beta} x} \quad (\text{B9})$$

for $x \ll 1$. We apply it to the single-copy averaged factors

$$\langle \Gamma_{aa'}(t) \rangle = \frac{1 + \text{Tr} \rho_0^2}{d+1} + \langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle \frac{2(d - \text{Tr} \rho_0^2)}{d(d^2 - 1)}, \quad (\text{B10})$$

$$\langle G_{aa'}(t) \rangle = S_{\text{lin}}(\rho_0) + \frac{1 + \text{Tr} \rho_0^2}{d+1} + \langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle \frac{2(d \text{Tr} \rho_0^2 - 1)}{d(d^2 - 1)}, \quad (\text{B11})$$

identifying β with the constant terms and α with the multiplicative one. After a simple algebra, this leads to the following short time approximations of the macroscopic factors:

$$\langle \Gamma_{aa'}^{\text{uno}}(t) \rangle = \langle \Gamma_{aa'}(t) \rangle^{N_{\text{uno}}} \approx \exp[-N_{\text{uno}} \tilde{\Delta}_t^2 (d - \langle \text{Tr} \rho_0^2 \rangle)], \quad (\text{B12})$$

$$\langle G_{aa'}^{\text{mac}}(t) \rangle = \langle G_{aa'}(t) \rangle^{N_{\text{mac}}} \approx \exp[-N_{\text{mac}} \tilde{\Delta}_t^2 (d \langle \text{Tr} \rho_0^2 \rangle - 1)]. \quad (\text{B13})$$

For $\langle \text{Tr} \rho_0^2 \rangle$ we can use either the Hilbert-Schmidt or Bures measure from the main text. It is interesting to note that in any case $\langle \text{Tr} \rho_0^2 \rangle \propto 1/d$, so that Eq. (B12) shows a dependence on both N_{uno} and d while Eq. (B13) shows a dependence mainly on N_{mac} .

We are now ready to estimate the lower bound on the decay time of the integral (B3). Substituting Eq. (B12) or (B13) and using Eqs. (A39), (A40), and (A20), we are left with a sum of integrals of the following structure:

$$\begin{aligned} & \int d a d a' \text{He}_m(a) \text{He}_n(a) \text{He}_j(a') \text{He}_l(a') e^{-\mu_t (a-a')^2 - \frac{1}{2}(a^2+a'^2)} \\ & \sim \int d a d a' a^r a'^s e^{-\mu_t (a-a')^2 - \frac{1}{2}(a^2+a'^2)}, \end{aligned} \quad (\text{B14})$$

where the powers satisfy $0 \leq r, s \leq 2(d_S - 1)$ [cf. Eq. (A40)] and $\mu_t \equiv N_{\text{uno}}(gt)^2(d - \langle \text{Tr} \rho_0^2 \rangle)$ for the decoherence factor and $\mu_t \equiv N_{\text{mac}}(gt)^2(d \langle \text{Tr} \rho_0^2 \rangle - 1)$ for the superfidelity [cf. Eqs. (B12) and (B13)]. To separate the integrals, we diagonalize the quadratic form in the exponent. Its eigenvalues read $(1 + 4\mu_t)/2$ and $1/2$. Denoting by O the diagonalizing $SO(2)$ transformation, we pass to the new variables $(x, y)^T \equiv O(a, a')^T$ in which the integrals (B14) separate

$$\begin{aligned} & \int d a d a' a^r a'^s e^{-\mu_t (a-a')^2 - \frac{1}{2}(a^2+a'^2)} \\ & \sim \left(\int d x x^\gamma e^{-\frac{1+4\mu_t}{2} x^2} \right) \left(\int d y y^\delta e^{-\frac{1}{2} y^2} \right), \end{aligned} \quad (\text{B15})$$

where $0 \leq \gamma, \delta \leq 2(d_S - 1)$, $\gamma + \delta = r + s$. The integrals on the right-hand side are now elementary and depend on time as $1/(1 + 4\mu_t)^{\gamma+\epsilon}$, where $\epsilon = 1$ or $1/2$ depending on the parity of the power γ . Putting all together, the integral (B3) has the following time dependence [we neglect possible time dependent coefficients, originating from the matrix O , as they are given by the sine and cosine functions and hence $\sim O(1)$]

$$\begin{aligned} \langle \langle X^f(t) \rangle \rangle &\sim \frac{1}{\sqrt{1 + 4\mu_t}} + \frac{1}{1 + 4\mu_t} + \frac{1}{(1 + 4\mu_t)^{3/2}} + \dots \\ &+ \frac{1}{(1 + 4\mu_t)^{2(d_S-1)+\epsilon}}. \end{aligned} \quad (\text{B16})$$

The fastest decaying is the last term, which for $\mu_t \ll 1$ or $t \ll O(\frac{1}{g^2 N_f})$ can be approximated by an exponential

$$\frac{1}{(1 + 4\mu_t)^{2(d_S-1)+\epsilon}} \approx \exp[-8d_S c(d) N_f (gt)^2], \quad (\text{B17})$$

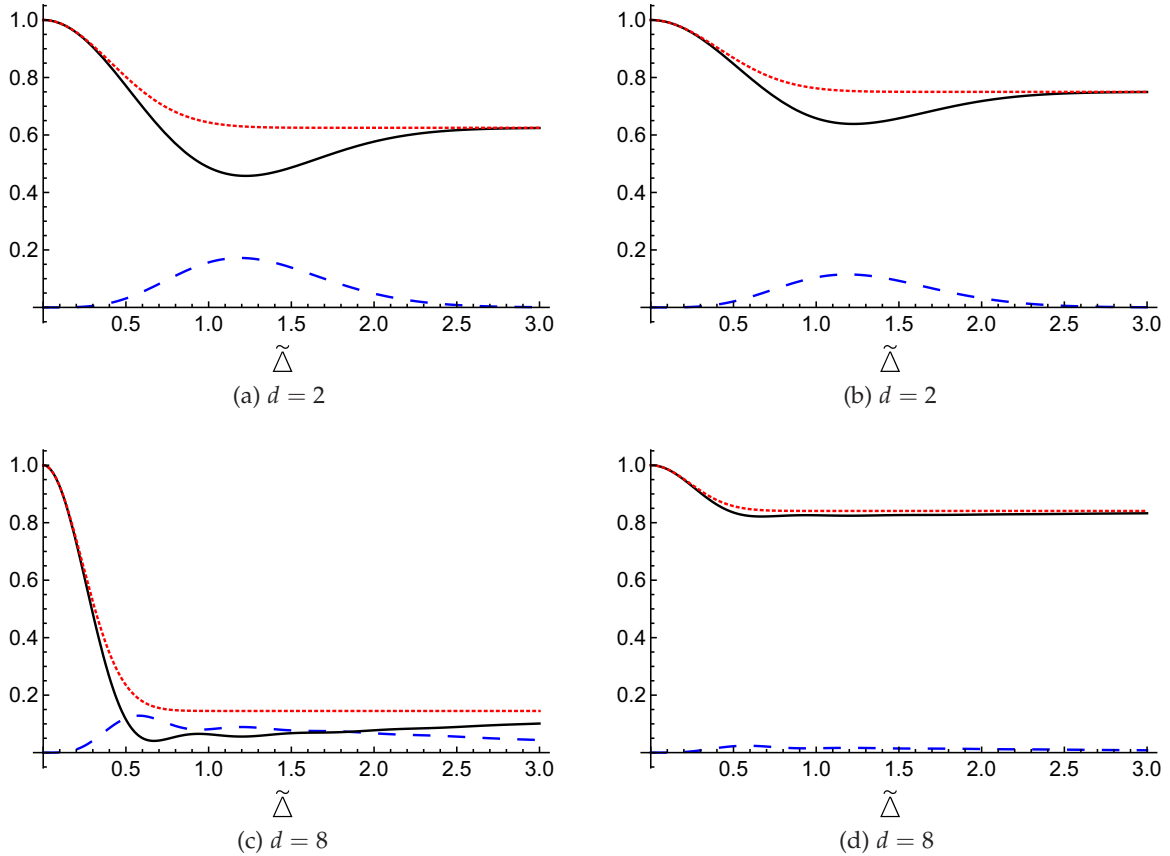


FIG. 2. In figures (a) and (c) the dotted (red) curve represents the Ansatz $\tilde{\Gamma}(t)$, the solid (black) line represents the exact decoherence factor $\langle \Gamma(t) \rangle$, and the dashed (blue) curve the difference $\tilde{\Gamma}(t) - \langle \Gamma(t) \rangle$. Similarly, in (b) and (d) the dotted (red) curve represents the Ansatz $\tilde{G}(t)$, the solid (black) line represents the exact averaged superfidelity $\langle G(t) \rangle$, and the dashed (blue) curve the difference $\tilde{G}(t) - \langle G(t) \rangle$. The time scale is given in units of $\tilde{\Delta} = (a - a')/\sqrt{\eta E}$, the Bures average purity is used throughout, and the exact integration is done assuming $d_S = 2$.

where we neglected the factor $-2 + \epsilon = -1$ or $-3/2$ in the power as compared to $2d_S$ and $c(d) \equiv d - \langle \text{Tr} \rho_0^2 \rangle$ for the decoherence factor and $c(d) \equiv d \langle \text{Tr} \rho_0^2 \rangle - 1$ for the superfidelity. This finally gives the following lower bounds on the decay times

$$\tau_{dec} \equiv [8g^2 N_{\text{uno}} d_S (d - \langle \text{Tr} \rho_0^2 \rangle)]^{-1/2} \sim [8g^2 N_{\text{uno}} d_S d]^{-1/2}, \quad (\text{B18})$$

$$\tau_{fid} \equiv [8g^2 N_{\text{mac}} d_S (d \langle \text{Tr} \rho_0^2 \rangle - 1)]^{-1/2} \sim [8g^2 N_{\text{mac}} d_S]^{-1/2}, \quad (\text{B19})$$

where the simplified estimates are in the limit of a large local dimension of the apparatus d . The above times are clearly within the short-time approximation range (B4). It is interesting to note that the above separation of physical time scales of decoherence and information accumulation is, on the mathematical level, a consequence of a simple symmetry difference in the initial formulas (A2) and (A15). This last equation has an additional SWAP operator \mathbb{V} under the trace.

As mentioned, the above time scales are only lower bounds for the actual times of the noise floor approach. Perhaps they can be tightened using different analytical tools, but for the purpose of this work we will use the above τ_{dec}, τ_{fid} .

APPENDIX C: UPPERBOUNDING THE EXACT EXPRESSION FOR $\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle$

This Appendix is complimentary to the main line of reasoning and is not necessary for understanding it. The exact expression (A48) is algebraically complicated and a simplified expression, reproducing short- and long-time behavior and preferably upper bounding it would be desirable. Here we attempt a construction of such an upper bound. For short times, we may neglect in the average $\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle = \rho(d, \tilde{\Delta}_t) e^{-\tilde{\Delta}_t^2}$ all terms except the first two giving $\langle f_i(\mathbf{a}, \boldsymbol{\lambda}) \rangle \lesssim [d(d-1)/2] e^{-(d+1)\tilde{\Delta}_t^2}$ [see Eq. (B8)]. On the other hand, for large times the highest-order term $\tilde{\Delta}_t^{2(d-3)}$ dominates and it always comes with a negative coefficient as can be seen from Eq. (A48). These observations suggest the following Ansatz:

$$\tilde{\Gamma}_{aa'}(t) \equiv \frac{1 + \langle \text{Tr} \rho_0^2 \rangle}{d+1} + \frac{d - \langle \text{Tr} \rho_0^2 \rangle}{d+1} e^{-(d+1)\tilde{\Delta}_t^2}, \quad (\text{C1})$$

$$\tilde{G}_{aa'}(t) \equiv \langle S_{\text{lin}}(\rho_0) \rangle + \frac{1 + \langle \text{Tr} \rho_0^2 \rangle}{d+1} + \frac{d \langle \text{Tr} \rho_0^2 \rangle - 1}{d+1} e^{-(d+1)\tilde{\Delta}_t^2}, \quad (\text{C2})$$

for the decoherence and the superfidelity factors, respectively. By construction, the expressions (C1) and (C2) reproduce the correct short-time behavior as $\langle \Gamma_{aa'}(t) \rangle = \tilde{\Gamma}_{aa'}(t) + O(\tilde{\Delta}_t^4)$,

and similarly $\langle G_{aa'}(t) \rangle$, for $t \ll \tau_{aa'}$, where $\tau_{aa'}$ is the characteristic time of the Gaussian decay in Eqs. (C1) and (C2)

$$\tau_{aa'} \equiv \frac{\sqrt{\eta E}}{\sqrt{d+1}|a-a'|}. \quad (\text{C3})$$

They are also upper bounds since $1-x \leq e^{-x}$ [cf. (B7)]. Similarly, for long times $t \gg \tau_{aa'}$, Eqs. (C1) and (C2) reproduce the correct white noise factors and also upper bound the exact averages since $p(d, \tilde{\Delta}_t)$ is then negative. These two facts together suggest that Eqs. (C1) and (C2) may be upper bounds for all times t and dimensions d . Unfortunately, we are unable to prove it analytically, but numerical evidence

for $d \leq 20$ suggests that it is indeed so. In Fig. 2 we present some sample plots of both the exact expressions and Eqs. (C1) and (C2) together with the errors. The price to pay for working with the simplified expressions (C1) and (C2) is that one loses the important physical information on non-Markovianity, reflected by the nonmonotonic behavior of the exact averages for low ratios N/d . Also the exact functions approach their asymptotic limits from below, signaling recovery of coherences or loss of information in the environment, while Eqs. (C1) and (C2) approach them from above. However, if one is solely interested in an SBS formation, an upper bound decaying to the correct noise level is just enough.

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