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A New Approach to Stability Evaluation of Digital Filters

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***Abstract*—In this paper, a new numerical method of evaluating digital filter stability is presented. This approach is based on novel root-finding algorithms at the complex plane using the Delaunay triangulation and Cauchy's Argument Principle. The presented algorithm locates unstable zeros of the characteristic equation with their multiplicities. The proposed method is generic and can be applied to a vast range of systems. Verification of this method is presented with benchmarks that include integer-order and fractional-order digital filters.**

***Keywords*—Digital filters, Stability analysis, Digital signal processing.**

I. Introduction

The stability is a fundamental property required in the design process of any digital filter. Hence, this topic is widely investigated in control engineering and signal processing for many years. The stability condition for a discrete-time linear time-invariant (LTI) system, e.g., a digital filter, can be formulated as follows: The system is (asymptotically) stable if and only if all zeros of the characteristic equation are within a unit circle at the complex z-plane [1], [2], [3]. This condition is also applicable to digital filters modeled by the transfer function. In such a case, the denominator of the transfer function represents the characteristic equation of a system.

For integer-order systems, the characteristic equation is a polynomial of degree equal to the system order. The evaluation of the system stability can employ either searching for roots of a polynomial or tests based on polynomial coefficients only, e.g., refer to Schur-Cohn and Jury tests [3]. However, such methods are not applicable to systems whose characteristic equation is not a polynomial, e.g., fractional-order systems and their connections.

Therefore, we decided to develop a general numerical test for stability evaluation of discrete-time systems [4]. In this contribution, additional results of its application to digital filters are presented that extend the results in [4]. The developed stability test is based on innovative root-finding techniques at the complex plane employing the Delaunay triangulation and Cauchy's Argument Principle [5], [6]. It allows an exploration of various digital systems by analyzing characteristic equation ($f(z) = 0$), also containing singular points and branch cuts. The developed numerical test returns values of unstable zeros of the characteristic equation with their multiplicities if the system is unstable. It allows to evaluate how far from the stability margin the considered system is. Then, the tuning of system parameters can be employed, targeting the system stability.

II. Stability of Digital Filters

In this section, we provide the fundamental equations and definitions used throughout the rest of this paper.

A. Integer Order Digital Filters

Consider a discrete-time LTI system (digital filter) of integer order that is defined using the transfer function in the z-domain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Num(z)}{Den(z)} \tag{1}$$

where X is the input signal, Y is the output signal, $Num(z)/Den(z)$ is a rational function and z is the $\mathcal{Z}$-transform variable [2]. The characteristic equation of the system (1) is based on a polynomial of the z variable

$$f(z) = Den(z) = \prod_{i=1}^{L}(z - p_i) = 0. \tag{2}$$

B. Fractional Order Digital Filters

Consider a discrete-time LTI system (digital filter) of fractional order that is defined using the transfer function in the z-domain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Num(u)}{Den(u)} \tag{3}$$

where $u = z(1 - z^{-1})^{\alpha}$ [8]. The characteristic equation of the system (1) is not based on a polynomial of the z variable, i.e.

$$f(z) = Den\left[z(1 - z^{-1})^{\alpha}\right] = 0. \tag{4}$$

Systems (1), (3) are stable if and only if all of the zeros of the characteristic equation at the complex z-plane lay inside the unit circle [1], [2], [3], [7], [8], [9]. If for a given system at least a single zero of its characteristic equation, i.e., $f(z) = 0$, lays outside the unit circle at the complex z-plane, then this system is unstable. We can apply the following transformation to map the outer region of the complex z-plane to the inner region of the unit circle:

$$z = w^{-1}. \tag{5}$$

This is the inversion of the variable that maps the complex z-plane to the w-plane. It is worth extracting singularities before computations to avoid numerical-precision issues. For instance, if $f(w^{-1})$ has a pole p of multiplicity K, then the following equation is further analyzed:

$$F(w) = (w - p)^K f(w^{-1}) = 0. \tag{6}$$

The method allows one to evaluate stability of systems whose characteristic equations are not based on polynomials. The root-finding technique applied here allows the characteristic equation to include singularities and branch cuts in general [5], [6].

III. Numerical Test for Stability Evaluation

The system described by the characteristic equation (6) is stable only if there are no zero of $F(w)$ located inside the unit circle. The goal of the algorithm is to find any such a zero. This is achieved by iteratively applying the Delaunay triangulation to the inner area of the unit circle in the complex plane with new points added near located zeros. Finally, candidate triangles are evaluated using Cauchy's Argument Principle [5], [6].

Let us assume that zeros are located with precision Δr. The algorithm can be summarized as follows:

1) Initialize vector of initial vertices $\underline{V} = \{v_1, v_2, ..., v_n\}$. When connected, these must form a closed contour that envelops the unit circle in the complex plane.
2) Apply the Delaunay triangulation to vector $\underline{V}$. The output is a triangular mesh that spans the entirety of the unit circle, $T = \{t_1, t_2, ..., t_n\}$.
3) For each triangle t in T, for each of its edges of length greater than Δr, compute the complex function values at the ends of the edge. If the real or imaginary part of the complex number changes the sign, then add a new point in the middle of this edge.
4) If any new point has been added, then come back to step 2.
5) From the final set T, select triangles with edge length equal to or lower than Δr. These are the candidate triangles.
6) For each candidate triangle, apply Cauchy's Argument Principle and sum the phase change of function values along the triangle edges.
7) If the sum of phase changes of function values is a positive value equal to $2\pi q$, then inside the given triangle a zero with multiplicity q is located.

If any zero is located, then the system characterized by (6) is unstable. The algorithm utilizes the fact that close to the location of a zero, function values will fall into different quadrants of the complex plane. The signs of the real and imaginary parts of the complex function $F(w)$ are used to determine the quadrant it belongs to:

I - $Re[F(w)] > 0, Im[F(w)] > 0$
II - $Re[F(w)] < 0, Im[F(w)] > 0$
III - $Re[F(w)] < 0, Im[F(w)] < 0$
IV - $Re[F(w)] > 0, Im[F(w)] < 0.$

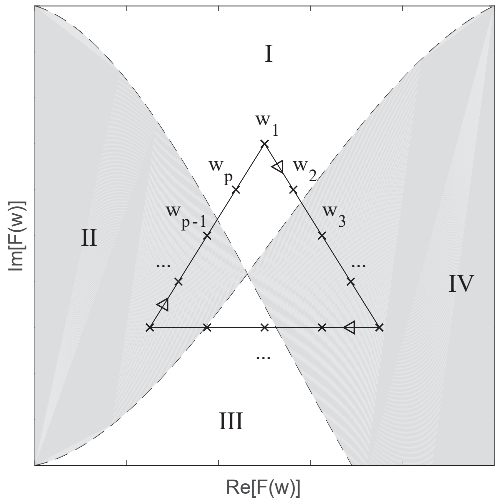


Fig. 1. Contour integral over the contour C that is a circumference of a triangle t in the complex w-plane. Dashed lines represent either $Re[F(w)] = 0$ or $Im[F(w)] = 0$. I-IV represent the quadrants of $F(w)$. Zero is located inside the triangle area at the point where the dashed lines cross.

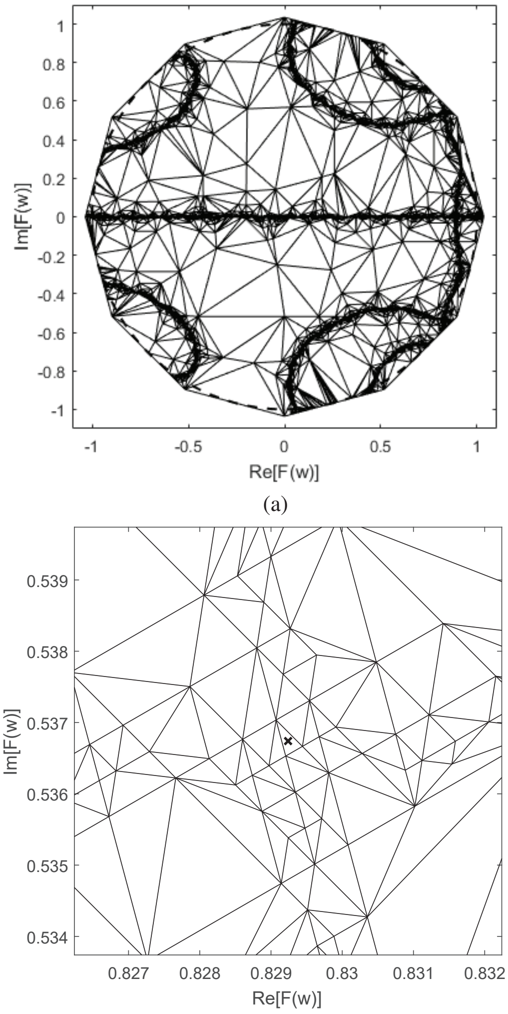
In each iteration, the triangular mesh obtained from the Delaunay triangulation becomes denser close to zeros of the characteristic equation as new vertices are added inside the edges of the triangles that envelop zeros. Once required precision of computations is achieved, i.e., edges of triangles are shorter or equal Δr, the verification step commences. Candidate triangles are selected by calculating the length of triangle edges and selecting these triangles whose any edge length is less or equal Δr. By applying Cauchy's Argument Principle to the contour C which is the circumference of a candidate triangle, the algorithm evaluates if a zero is in fact located inside this triangle. This step requires calculating the contour integral

$$q = \frac{1}{2\pi j} \oint_C \frac{F'(w)}{F(w)} dw = \frac{1}{2\pi} \sum_{p=1}^{P} Arg\left[\frac{F(w_{p+1})}{F(w_p)}\right] \tag{7}$$

where C is the circumference of the triangle. The integral (7) can be calculated as the sum of phase changes of function values along contour C (refer to Fig. 1). If $q > 0$, then a zero with multiplicity q is located inside the area constrained by contour C and the system is unstable.

IV. Numerical Results

The code is developed in Matlab [10] environment which returns general information if the system is stable. Furthermore, if the system is unstable it returns either (i) a value of the first-found unstable zero of (6) and its multiplicity or (ii) values of all unstable zeros of (6) in the unit circle with their multiplicities. The operation mode (i) is faster than (ii) but returns less information about the system stability.



(a)

(b)

Fig. 2. Algorithm convergence for (8) with $\Delta r = 0.001$. (a) The Delaunay triangulation over the unit circle. (b) The close-up view of the triangle with a single unstable zero of the conjugate zero pair, i.e., $w_0 = 0.82924 + 0.53674j$.

A. Integer Order Digital Filter

Let us consider the digital filter described by the transfer function

$$H(z) = \frac{G}{A(z)} \tag{8}$$

where G denotes the gain factor and $A(z)$ is given by

$$\begin{aligned} A(z) &= 1.0000 \\ &- 2.5400z^{-1} + 3.0429z^{-2} - 2.9211z^{-3} + 3.7088z^{-4} \\ &- 3.9740z^{-5} + 3.0221z^{-6} - 2.3163z^{-7} + 1.9791z^{-8} \\ &- 1.1265z^{-9} + 0.3855z^{-10} - 0.2189z^{-11} + 0.1171z^{-12}. \end{aligned} \tag{9}$$

This could be the transfer function of a 12th order infinite-impulse-response filter which models a vocal tract in the linear predictive analysis of speech signals [11]. These type of filters may be implemented as multiple conjugate pairs of zeros to properly map the human vocal tract. For the filter to be stable, all of $z^{12}A(z)$ zeros must lay inside the unit circle. The considered digital filter (8) is unstable as its characteristic equation has a pair of conjugate zeros at $z_{0,0^*} = 0.85 \pm 0.55j$ with module equal to 1.0124. By applying (5) to the characteristic equation $z^{12}A(z) = 0$, one obtains

$$\begin{aligned} A(w^{-1}) &= 1.0000 \\ &- 2.5400w + 3.0429w^2 - 2.9211w^3 + 3.7088w^4 \\ &- 3.9740w^5 + 3.0221w^6 - 2.3163w^7 + 1.9791w^8 \\ &- 1.1265w^9 + 0.3855w^{10} - 0.2189w^{11} + 0.1171w^{12} = 0. \end{aligned} \tag{10}$$

Since the outer region of the complex plane is mapped into the unit circle, the stability condition is that no zero of (10) should be inside the unit circle in the w-plane. Fig. 2 shows the convergence of the proposed algorithm for (10). A conjugate pair of zeros of $A(w^{-1})$ is located at $w_{0,0^*} = 0.8293 \pm 0.5366j$. After application of the proposed method, one obtains a pair of conjugate zeros at $w_{0,0^*} = 0.82924 \pm 0.53674j$ (module is equal to 0.9878), which gives the numerical error of zero localization equal to 0.00015. This indicates that the considered system is unstable.

B. Fractional Order Digital Filter

Let us consider the digital fractional order Butterworth filter given by [12]

$$H(z) = \frac{1}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha+\beta} + a\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha} + c} \tag{11}$$

where α and β denote fractional-order parameters, a and c denote the design parameters and T denotes the sampling period. Applying the transformation (5) to the filter described by (11), the characteristic equation $F(w) = 0$ of such a system in the complex w-plane is given by

$$F(w) = s^{\alpha+\beta} + as^{\alpha} + c \tag{12}$$

where

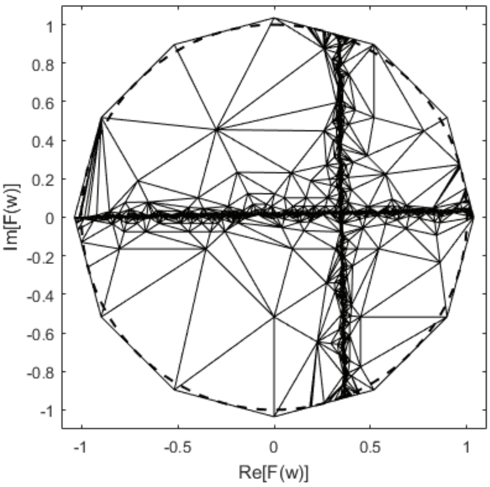
$$s = \frac{2}{T}\frac{1-w}{1+w}. \tag{13}$$

The function $F(w)$ includes the pole $w = -1$ which can be extracted. Therefore, the stability evaluation for the digital filter (11) can be executed by considering the equation

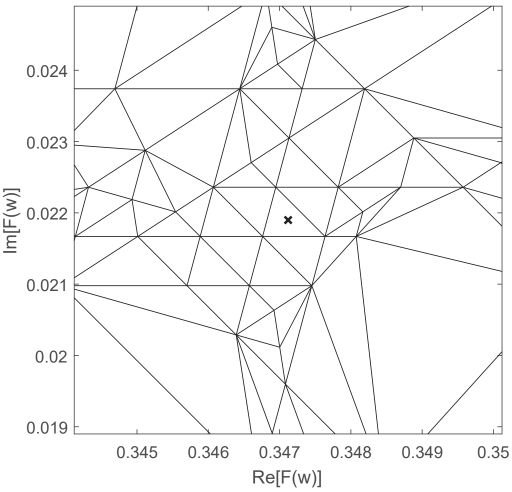
$$(1+w)^{\alpha+\beta}F(w) = 0. \tag{14}$$

Hence, numerical precision issues resulting from the overflow of arithmetic operations can be eliminated.

Let us take $\alpha = \beta = 0.5$, $a = 1$, $c = -1000 + 50j$ and $T = 0.001$. For such parameters, the digital filter (11) is unstable (refer to Fig. 3). As seen, the algorithm converges and the fine meshing tracks the function values as they switch signs of the real and imaginary parts. Finally, the unstable zero is located at $w_0 = 0.34712 + 0.021899j$.



(a)



(b)

Fig. 3. Algorithm convergence for (11) with $\alpha = 0.5$, $\beta = 0.5$, $a = 1$, $c = -1000 + 50j$, $\Delta r = 0.001$. (a) The Delaunay triangulation over the unit circle. (b) The close-up view of the triangle with one unstable zero, i.e., $w_0 = 0.34712 + 0.021899j$.

V. Conclusion

A new generic numerical test for evaluation of stability of digital filters is developed. It employs innovative root-finding algorithms at the complex plane that are based on the Delaunay triangulation and Cauchy's Argument Principle. The test is verified in a vast range of filters including those of integer and fractional order. The results obtained facilitate research and design in the area of the digital signal processing.

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