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Weakly connected Roman domination in graphs

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Abstract

A *Roman dominating function* on a graph $G = (V, E)$ is defined to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A dominating set $D \subseteq V$ is a *weakly connected dominating set* of G if the graph $(V, E \cap (D \times V))$ is connected. We define a *weakly connected Roman dominating function* on a graph G to be a Roman dominating function such that the set $\{u \in V : f(u) \in \{1, 2\}\}$ is a weakly connected dominating set of G. The weight of a weakly connected Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a weakly connected Roman dominating function on a graph G is called the *weakly connected Roman domination number* of G and is denoted by $\gamma_R^{wc}(G)$. In this paper, we initiate the study of this parameter.

Keywords: Roman domination number, weakly connected set, weakly connected Roman domination number, trees.

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1. Introduction

Cockayne et al. in [7] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f\colon V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real–valued function, $f\colon V \to \mathbb{R}$, the *weight* of f is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G) = \min\{w(f) : f \text{ is an RDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$–function. Roman domination in graphs has been studied, for example, in [7, 9, 13].

As it is mention in [14], this definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman Empire!" [16]. Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered *unsecured* if no legions are stationed there (i.e., $f(v) = 0$) and *secured* otherwise (i.e., if $f(v) \in \{1, 2\}$). An unsecured location (vertex v) can be secured by sending a legion to v from an adjacent location (an adjacent vertex u). In the fourth century A.D. emperor Constantine the Great decreed that a legion cannot be sent from a secured location to an unsecured location if doing so leaves that location unsecured. Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight $\gamma_R(G)$ corresponds to such an optimal assignment of legions to locations.

In order to generalize or improve some properties of the Roman domination in its standard form, some variants of Roman domination have been introduced and studied. Those variants are often related to modifying the conditions in which the vertices are dominated, or to adding extra properties to the Roman domination property itself. For instance we remark here variants like the following ones: total Roman domination (see [3, 5]), mixed Roman domination (see [2]) or strong Roman domination (see [4]).

In this paper we explore the idea of strengthening security of the Roman Empire by providing a better communication in emergency between the legions, while still having substantial costs of maintaining legions as low as possible.

Two legions at different location (vertices u and v) can *contact directly* if there is at most one unsecured location between them and the distance between u and v is at most 2. Moreover, u and v can *contact undirectly* if there is a sequence of secured vertices $(u = u_1, u_2, \ldots, u_k = v)$ such that u_i and u_{i+1} can contact directly for $i = 1, 2, \ldots, k-1$. The Roman Empire is *communicated* if any two legions at different locations can contact directly or undirectly.

Let $G = (V, E)$ be a graph and let $f\colon V \to \{0, 1, 2\}$ be a function. Let V_0, V_1, and V_2 be the sets of vertices assigned with the values 0, 1, and 2, respectively, under f. Note that there is a one to one correspondence between the functions $f\colon V \to \{0, 1, 2\}$ and the ordered triple (V_0, V_1, V_2) of V. Thus we will write $f = (V_0, V_1, V_2)$.

Denote $|V(G)| = n(G)$. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v in G and the closed neighbourhood is $N_G[v] = N_G(u) \cup \{v\}$. The *degree* $d_G(v)$ of v is the number of edges incident to v in G, $d_G(v) = |N_G(v)|$. Let $L(G)$ be the set of all leaves of G, that is the set of vertices with degree 1, and let $n_1(G)$ be the cardinality of $L(G)$. A vertex v is called a *support vertex* if v is a neighbour of a leaf. Denote by $S(G)$ the set of all support vertices in G and let $n_S(G)$ be the cardinality of $S(G)$. A *strong support vertex* is a vertex adjacent to at least two leaves. A vertex adjacent to exactly one leaf is a *weak support vertex*.

A set $D \subseteq V(G)$ is a *dominating set of* G if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the *domination number of* G and is denoted by $\gamma(G)$. A minimum dominating set of a graph G is called a $\gamma(G)$-set.

From now on, G will be assumed to be connected. The *subgraph weakly induced by a set* $D \subseteq V(G)$ is the graph $\langle D\rangle_w = (N[D], E_w)$, where E_w consists of the set of all edges of G having at least one vertex in D. A set $D \subseteq V(G)$ is a *weakly connected dominating set* (WCDS) of G if D is dominating and $\langle D\rangle_w$ is connected. The *weakly connected domination number* of G, denoted $\gamma_{wc}(G)$, is the minimum cardinality of a WCDS. A minimum WCDS of a graph G is called a $\gamma_{wc}(G)$-set. The weakly connected domination number was introduced in 1997 by Dunbar et al. [10] and studied for example in [8], [15] and [17].

We call the function f a *weakly connected Roman dominating function* in G (WCRDF) if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_2$ and the subgraph $\langle V_1 \cup V_2 \rangle_w$ weakly induced by $V_1 \cup V_2$ is connected in G. The weight $w(f)$ of f is $|V_1| + 2|V_2|$. The *weakly connected Roman domination number*, denoted $\gamma_R^{wc}(G)$, is the minimum weight of a WCRDF in G; that is, $\gamma_R^{wc}(G) = \min\{w(f)\colon f \text{ is a WCRDF in } G\}$. A WCRDF of weight $\gamma_R^{wc}(G)$ is called a $\gamma_R^{wc}(G)$–function.

This definition of a WCRDF is motivated as follows. Using the notation introduced earlier, we define a location of a legion to be *uncommunicated* if there exists another location of a legion such that the legions cannot contact directly nor undirectly. If the locations are uncommunicated, they cannot safely inform the other locations nor ask them for help in case of urgent emergency. When all locations of legions are communicated, Emperor Constantine the Great can defend the Roman Empire more efficiently: he can supervise whole Empire and send orders to his legions in reasonable time. Such a placement of legions corresponds to a WCRDF and a minimum such placement of legions corresponds to a minimum WCRDF. Hence this concept of weakly connected Roman domination is an attractive alternative to Emperor Constantines notion of Roman domination.

For a vertex $v \in V$, we denote by $f[v]$ the set $\{f(u)\colon u \in N[v]\}$ for notational convenience. For any unexplained terms and symbols see [12].

In [1] Ahangar et al. introduced the concept of outer-independent Roman domination as follows: a function $f\colon V(G) \to \{0,1,2\}$ is an *outer-independent Roman dominating function* (OIRDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and $\{v\colon f(v) = 0\}$ is an independent set. The *outer-independent Roman domination number* $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G.

Clearly, any outer-independent Roman dominating function on a connected graph G is an WCRDF of G, so

$$\gamma_{oiR}(G) \geq \gamma_R^{wc}(G).$$

On the other hand, for any tree T, it is easy to see that any WCRDF of T is an OIRDF of T and this implies that

$$\gamma_{oiR}(T) \leq \gamma_R^{wc}(T).$$

Therefore, for any tree T

$$\gamma_{oiR}(T) = \gamma_R^{wc}(T). \tag{1}$$

2. Preliminary results

In this section we study basic properties of weakly connected Roman domination number of graphs.

Proposition 1. *If G is a connected graph, then*

$$\gamma_{wc}(G) \leq \gamma_R^{wc}(G) \leq 2\gamma_{wc}(G).$$

PROOF. Let $f = (V_0, V_1, V_2)$ be $\gamma_R^{wc}(G)$–function. Then $V_1 \cup V_2$ is a WCDS of G. Hence $\gamma_{wc}(G) \leq \gamma_R^{wc}(G)$.

If D_w is a $\gamma_{wc}(G)$-set, then the function

$$f(u) = \begin{cases} 2 & \text{for } u \in D_w \\ 0 & \text{otherwise} \end{cases}$$

is a WCRDF in G. Thus $\gamma_R^{wc}(G) \leq 2\gamma_{wc}(G)$.

Proposition 2. *For any connected graph G of order n, $\gamma_{wc}(G) = \gamma_R^{wc}(G)$ if and only if $G = K_1$.*

PROOF. It is obvious that if $G = K_1$, then $\gamma_{wc}(G) = \gamma_R^{wc}(G)$.

Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^{wc}(G)$–function. Then $\gamma_{wc}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_R^{wc}(G)$. Since $\gamma_{wc}(G) = \gamma_R^{wc}(G)$, we obtain $|V_2| = 0$ and hence $|V_0| = 0$. Therefore, $\gamma_R^{wc}(G) = |V_1| = n$. This implies that $\gamma_{wc}(G) = n$, which, in turn, implies that $G = K_1$.

Proposition 3. *For any connected graph G of order n,*

$$\gamma_R^{wc}(G) \leq n.$$

The equality $\gamma_R^{wc}(G) = n$ holds if and only if $G \in \{K_1, K_2\}$.

PROOF. Let $G = (V, E)$ be a connected graph. Then $f = (\emptyset, V, \emptyset)$ is a WCRDF in G and hence $\gamma_R^{wc}(G) \leq n$.

If $G = K_1$ or $G = K_2$, then clearly $\gamma_R^{wc}(G) = n$. Thus suppose $G \notin \{K_1, K_2\}$ and $\gamma_R^{wc}(G) = n$. If $u \in V$ is a vertex of degree at least 2 and $x, y \in N(u)$, then $f = (\{x, y\}, V - \{u, x, y\}, \{u\})$ is a WCRDF in G of weight smaller than $\gamma_R^{wc}(G)$, which is impossible.

Corollary 4. *If $\gamma_R^{wc}(G) < n$ and $f = (V_0, V_1, V_2)$ is a $\gamma_R^{wc}(G)$–function, then $|V_0| > 0$ and $|V_2| > 0$.*

3. Complexity results

In this section, we show that the problem of computing $\gamma_R^{wc}(G)$–function is NP-hard. We will state the corresponding decision problem in the standard form (see [11]) and we indicate the polynomial time reduction used to prove that it is NP-complete. Details are omited.

WEAKLY CONNECTED ROMAN DOMINATING FUNCTION (WCRDF)

Instance: A connected graph and a positive integer k.

Question: Does G have a weakly connected Roman dominating function of weight at most k?

A *split graph* is a graph in which the vertex set can be partitioned into a clique and an independent set.

Theorem 5. *WCRDF is NP-complete, even for split graphs and even for bipartite graphs.*

PROOF. *(Outline)* It is obvious that WCRDF is a member of NP, since we can, in polynomial time, guess at a function $f\colon V(G) \to \{0,1,2\}$ and verify that f has weight at most k and is a WCRDF.

The reduction is from EXACT COVER BY 3-SETS (X3C). Given an instance $X = \{x_1, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, \ldots, C_m\}$ of X3C, where $C_j \subseteq X$ and $|C_j| = 3$ for $1 \le j \le m$, construct a split graph G with vertices for each $x_i \in X$, and with edges $x_i C_j$ for all $x_i \in C_j$ and edges so that $\langle\{C_1, \ldots, C_m\}\rangle = K_m$. Let $k = 2q$. It is not hard to show that $\mathcal{C}$ contains an exact cover if and only if G has a weakly connected Roman dominating function of weight at most k.

Similarly, construct a bipartite graph in the same way, except that rather than adding all the edges between vertices of $\mathcal{C}$, add four new vertices, y_0, y_1, y_2, y_3 and edges y_0y_1, y_0y_2, y_0y_3 and y_0C_j for all j. Set $k = 2q + 2$.

4. Lower bound on the weakly connected Roman domination number of a tree without strong support vertices

In this section we prove a lower bound for the weakly connected Roman domination number of a tree without strong support vertices in terms of the order of a graph. We start with a result for general graphs.

Lemma 6. *Let G be a graph and let $P = (v_1, v_2, v_3, v_4)$ be an induced path in G such that $d(v_1) = 1$, $d(v_2) = d(v_3) = d(v_4) = 2$. Denote $G' = G - P$. Then*

$$\gamma_R^{wc}(G) = \gamma_R^{wc}(G') + 3. \tag{2}$$

PROOF. Let $f' = (V_0, V_1, V_2)$ be a $\gamma_R^{wc}(G')$–function. Then $(V_0 \cup \{v_1, v_3\}, V_1 \cup \{v_4\}, V_2 \cup \{v_2\})$ is a WCRDF of G. Hence, $\gamma_R^{wc}(G) \le \gamma_R^{wc}(G') + 3$.

On the other hand, let $f = (V_0, V_1, V_2)$ be a $\gamma_R^{wc}(G)$–function. Let $v_5 \ne v_3$ be a neighbour of v_4. If $f(v_5) \in \{1, 2\}$, we may assume that $f(v_2) = f(v_4) = 0$, $f(v_1) = 1$ and $f(v_3) = 2$. Then $(V_0 - \{v_2, v_4\}, V_1 - \{v_1\}, V_2 - \{v_3\})$ is a WCRDF of G'. If $f(v_5) = 0$ and $f(v_4) = 2$, we may assume that $f(v_1) = f(v_3) = 0$, $f(v_2) = 2$ and then $(V_0 - \{v_1, v_3, v_5\}, V_1 \cup \{v_5\}, V_2 - \{v_2, v_4\})$ is a WCRDF of G'. If $f(v_5) = 0$ and $f(v_4) = 1$, we may assume that $f(v_1) = f(v_3) = 0$, $f(v_2) = 2$

and then $(V_0 - \{v_1, v_3\}, V_1 - \{v_4\}, V_2 - \{v_2\})$ is a WCRDF of G'. Notice that the situation when $f(v_4) = f(v_5) = 0$ is impossible. In all situations we obtain a WCRDF of G' of weight smaller than the weight of f by three. Therefore, $\gamma_R^{wc}(G') \leq \gamma_R^{wc}(G) - 3$. Hence the equality (2) follows.

Let T be a tree and let $f = (V_0, V_1, V_2)$ be a $\gamma_R^{wc}(T)$–function. If $v \in V(T)$ is a strong support vertex, then without loss of generality we may assume that $v \in V_2$ and each leaf neighbour of v belongs to V_0. If $v \in V(T)$ is a weak support vertex and x is the leaf adjacent to v, then without loss of generality we may assume that either $v \in V_2$ and $x \in V_0$ or $v \in V_0$ and $x \in V_1$.

Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be the following three operations defined on a tree T. Let f be a $\gamma_R^{wc}(T)$– function and let $v \in V(T)$.

Operation $\mathcal{T}_1$. If $f(v) = 0$ and v is not a support vertex, then add a vertex x and the edge vx.

Operation $\mathcal{T}_2$. If $f(v) = 2$, add a path (x, y) and the edge vx.

Operation $\mathcal{T}_3$. If $f(v) \in \{1, 2\}$, add a path (x, y, z) and the edge vx.

Let $\mathcal{T}$ be the minimum family of trees obtained from the path P_2 by a finite sequence of Operations $\mathcal{T}_2$ and at most one either Operation $\mathcal{T}_1$ or $\mathcal{T}_3$.

Theorem 7. *Let T be a tree of order n without a strong support vertex. Then*

$$\gamma_R^{wc}(T) \geq \left\lceil \frac{n}{2} \right\rceil + 1, \tag{3}$$

with equality if and only if T belongs to the family $\mathcal{T}$.

Proof. First we prove that if T is a tree without a strong support vertex, then equation (8) is true and if equality in (8) holds, then T belongs to the family $\mathcal{T}$. If $\mathrm{diam}(T) = 1$, then $T = P_2$ and the statement is clearly true. If $\mathrm{diam}(T) = 2$, then T is a star and the central vertex is a strong support vertex, which is impossible. If $\mathrm{diam}(T) = 3$, then, since T is a tree without a strong support vertex, $T = P_4$ and the statement holds, since P_4 can be obtained from P_2 by Operation $\mathcal{T}_2$.

Hence assume $\operatorname{diam}(T) \geq 4$. We proceed by induction on n. Assume for each tree T' without a strong support vertiex and with $n(T') < n$ the inequality (8) holds for T' and in case of equality in (8), $T' \in \mathcal{T}$. Let $(v_1, v_2, \ldots, v_k)$ be a longest path in T. Then $d(v_2) = 2$. We consider a few cases depending on the structure of T.

Case 1: $d(v_3) > 2$. Then without loss of generality we let f be a minimum WCRDF of T such that $f(v_3) = 2$, the weight assigned to every neighbour of v_3, except possibly v_4, is 0, and the weight assigned to every leaf vertex at distance 2 from v_3 is 1. Let $T' = T - \{v_1, v_2\}$. Since T is without strong support vertices and $d(v_3) > 2$, T' is also a tree without strong support vertices and hence equation (8) holds for T'. Moreover, the function f restricted on T' is a WCRDF of T'. Hence,

$$\gamma_R^{wc}(T) \geq 1 + \gamma_R^{wc}(T') \geq 1 + \left\lceil \frac{n-2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil + 1. \tag{4}$$

Hence the inequality (8) holds for T.

If $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$, then we have equalities throughout the inequality chain (4). Particulary, $\gamma_R^{wc}(T') = \lceil \frac{n(T')}{2} \rceil + 1$. By the induction, $T' \in \mathcal{T}$ and f restricted on T' is a $\gamma_R^{wc}(T')$–function. Hence, for some minimum WCRDF f' of T' is $f'(v_3) = 2$. Therefore T may be obtained from T' by Operation $\mathcal{T}_2$ and we conclude that $T \in \mathcal{T}$.

Case 2: $d(v_3) = 2$ and $f(v_1) = 1$ for some minimum WCRDF f of T. Then $f(v_2) = 0$ and $f(v_3) = 2$. Consider $T' = T - v_1$. Since $n(T') < n$ and T' is without a strong support vertex, we apply the induction hypothesis to T'. Moreover, the function f restricted on T' is a WCRDF of T'. Therefore,

$$\gamma_R^{wc}(T) \geq 1 + \gamma_R^{wc}(T') \geq \left\lceil \frac{n(T')}{2} \right\rceil + 2 = \left\lceil \frac{n+1}{2} \right\rceil + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{5}$$

Hence the inequality (8) holds for T.

If $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$, then we have equalities throughout the inequality chain (5) and $n(T')$ is even. Particulary, $\gamma_R^{wc}(T') = \lceil \frac{n(T')}{2} \rceil + 1$. By the induction, $T' \in \mathcal{T}$ and f restricted on T' is a minimum WCRDF of T'. Hence, for some $\gamma_R^{wc}(T')$-function f' is $f'(v_2) = 0$. Therefore T may be obtained from T' by Operation $\mathcal{T}_1$ and we conclude that $T \in \mathcal{T}$.

Case 3: $d(v_3) = 2$ and $f(v_1) = 0$ for each minimum WCRDF of T. Let f be a minimum WCRDF of T. Then $f(v_1) = 0$, $f(v_2) = 2$ and without loss of generality we assume $f(v_3) = 0$ and $f(v_4) \in \{1, 2\}$. Assume additionally $d(v_4) > 2$ or v_5 is not a support vertex. Let $T' = T - \{v_1, v_2, v_3\}$. Then T' is a tree without a strong support vertex and with less vertices than T. Moreover, f restricted on T' is a WCRDF of T'. Therefore by the induction, the inequality (8) is true for T'. Hence,

$$\gamma_R^{wc}(T) \geq 2 + \gamma_R^{wc}(T') \geq \left\lceil \frac{n(T')}{2} \right\rceil + 3 = \left\lceil \frac{n+1}{2} \right\rceil + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1. \quad (6)$$

Hence in this situation the inequality (8) holds for T.

If $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$, then we have equalities throughout the inequality chain (6). Particulary, $\gamma_R^{wc}(T') = \lceil \frac{n(T')}{2} \rceil + 1$. By the induction, $T' \in \mathcal{T}$ and f restricted on T' is a minimum WCRDF function. Hence, for some minimum WCRDF f' of T' is $f'(v_4) \in \{1, 2\}$. Therefore T may be obtained from T' by Operation $\mathcal{T}_3$ and we conclude that $T \in \mathcal{T}$.

Assume now $d(v_4) = 2$ and v_5 is a support vertex. Without loss of generality we may assume $f(v_4) = 1$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Then T' is a tree without a strong support vertex and with less vertices than T. Therefore by the induction, the inequality (8) is true for T'. Moreover, f restricted on T' is a WCRDF of T'. Hence,

$$\gamma_R^{wc}(T) \geq 3 + \gamma_R^{wc}(T') \geq \left\lceil \frac{n(T')}{2} \right\rceil + 4 = \left\lceil \frac{n}{2} \right\rceil + 2 > \left\lceil \frac{n}{2} \right\rceil + 1. \quad (7)$$

Hence in this situation the inequality (8) holds for T.

If $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$, then we can not have equalities in the inequality chain (7), so this case is impossible.

Notice that Operations $\mathcal{T}_1$ and $\mathcal{T}_3$ may be performed on a tree $T \in \mathcal{T}$ only when

$n(T)$ is even and both of these operations change the parity of the number of vertices of a tree. Therefore these operations may be performed at most once.

This is the end of the proof for inequality (8) and for the case of equality in (8).

Now we prove that if $T \in \mathcal{T}$, then $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$. We proceed by induction on the number $s(T)$ of operations required to construct the tree T. If $s(T) = 0$, then $T = P_2$ and clearly $\gamma_R^{wc}(P_2) = 2 = \lceil \frac{n}{2} \rceil + 1$.

Assume now that $T \in \mathcal{T}$ is a tree with $s(T) = k$ for some positive integer $k > 1$ and for each tree $T' \in \mathcal{T}$ with $s(T') < k$ is equality in (8). Then T can be obtained from a tree T' belonging to $\mathcal{T}$ by operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$. We now consider three possibilities depending on whether T is obtained from T' by operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.

Case 1. T is obtained from $T' \in \mathcal{T}$ by Operation $\mathcal{T}_1$. Let f' be a minimum WCRDF in T'. Suppose T is obtained from T' by adding a vertex x and the edge xv, where $v \in V(T')$ is not a support vertex and $f'(v) = 0$. Since the Operation $\mathcal{T}_1$ is performed, T' is obtained by applying only Operations $\mathcal{T}_2$ and hence $|V(T')|$ is even and $n = |V(T')| + 1$. We can extend f' to a WCRDF of T by assigning the weight 1 to x. For this reason,

$$\gamma_R^{wc}(T) \leq |f'| + 1 = \frac{|V(T')|}{2} + 2 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Since the inequality (8) is true for T, we conclude that $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$.

Case 2. T is obtained from $T' \in \mathcal{T}$ by Operation $\mathcal{T}_2$. Let f' be a minimum WCRDF in T'. Suppose T is obtained from T' by adding a path (x, y) and the edge xv, where $v \in V(T')$ and $f'(v) = 2$. We can extend f' to a WCRDF of T by assigning the weight 1 to y and the weight 0 to x. For this reason,

$$\gamma_R^{wc}(T) \leq |f'| + 1 = \left\lceil \frac{|V(T')|}{2} \right\rceil + 2 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Since $\gamma_R^{wc}(T) > \gamma_R^{wc}(T')$, we conclude that $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$.

Case 3. T is obtained from $T' \in \mathcal{T}$ by Operation $\mathcal{T}_3$. Let f' be a minimum WCRDF in T'. Suppose T is obtained from T' by adding a path (x, y, z) and the edge xv, where $v \in V(T')$ and $f'(v) \in \{1, 2\}$. Since the Operation $\mathcal{T}_3$ is performed, T' is obtained by applying only Operations $\mathcal{T}_2$ and hence $|V(T')|$ is even. We can extend f' to a WCRDF of T by assigning the weight 2 to y and the weight 0 to x and z. For this reason,

$$\gamma_R^{wc}(T) \leq |f'| + 2 = \frac{|V(T')|}{2} + 3 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Since the inequality (8) is true for T, we conclude that $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$.

Thus if $T \in \mathcal{T}$, then $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$.

The proof is complete.

Since the weakly connected Roman domination number and the outer-independent Roman domination number are equal for trees, we have the following

Corollary 8. *Let T be a tree of order n without a strong support vertex. Then*

$$\gamma_{oiR}(T) \geq \left\lceil \frac{n}{2} \right\rceil + 1, \tag{8}$$

with equality if and only if T belongs to the family $\mathcal{T}$.

5. Upper bound on the weakly connected Roman number of a tree

In this section we present an upper bound for the weakly connected Roman domination number of a tree in terms of the order of a tree T.

Let $\mathcal{F}$ be a family of all trees T whose vertex set can be partitioned into sets, each set inducing a path P_6, such that the subgraph induced by the two central vertices of these P_6's is connected. We call the subtree induced by these central vertices the *underlying subtree* of the resulting tree T, and is called each such path P_6 a *base path* of the tree T.

A graph G is a γ_{wc}*–excellent graph* if each vertex of G is contained in some $\gamma_{wc}(G)$–set.

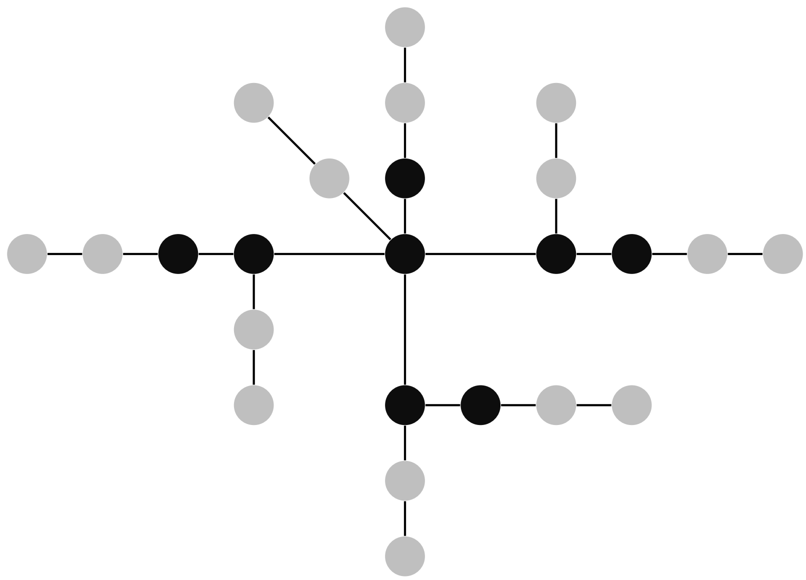

Figure 1: A tree in $\mathcal{F}$ with underlying tree denoted black

Domke et al. [8] have defined the class $\mathcal{E}$ to be the class of trees obtained from P_2 by a finite sequence of the following operation: attach to any vertex a P_2. They have proved the followiong result.

Theorem 9 (Domke et al. [8]). *A nontrivial tree T is γ_{wc}–excellent if and only if T belongs to the family $\mathcal{E}$.*

A set S of vertices of $G = (V, E)$ is an *independent set* if no two vertices of S are adjacent. The *independence number of* G, denoted $\beta(G)$, is the maximum cardinality among all independent sets of vertices of G.

Theorem 10 (Domke et al. [8]). *A nontrivial tree T of order n is γ_{wc}–excellent if and only if*

$$\beta(T) = \frac{n}{2}.$$

The following result appears in [10].

Theorem 11 (Dunbar et al. [10]). *If T is a nontrivial tree of order n, then*

$$\gamma_{wc}(T) = n - \beta(T).$$

Therefore, if a tree T of order n belongs to the family $\mathcal{E}$, then $\gamma_{wc}(T) = \frac{n}{2}$.

Our next lemma gives some properties of trees in $\mathcal{F}$.

Lemma 12. *If T is a tree of order n that belongs to the family $\mathcal{F}$, then*

$$\gamma_R^{wc}(T) = \frac{5}{6}n. \tag{9}$$

Additionally,

1. *if $v \in V$ is a support vertex, then there exists a $\gamma_R^{wc}(T)$–function that assigns to v value 2;*
2. *if $v \in V$ is a leaf, then there exists a $\gamma_R^{wc}(T)$–function that assigns to v value 2.*

Proof. Let $T \in \mathcal{F}$ have order n and let the underlying subtree of T have order k. Then $n = 3k$ where $k \geq 1$. Let f be a $\gamma_R^{wc}(T)$–function and let $P = (v_1, v_2, \ldots, v_6)$ be an arbitrary base path in T. Hence $d(v_1) = d(v_6) = 1$ and $d(v_2) = d(v_5) = 2$. Vertices v_3 and v_4 belong to the underlying subtree of T. The sum of weights given to v_1 and v_2 by f must be at least 2, unless the weight assigned by f to v_3 is 2, to v_1 is 1 and to v_2 is 0. Moreover, if the sum of weights given to v_1 and v_2 is 2 and the sum of weights given to v_5 and v_6 is also 2, then the sum of weights given to v_3 and v_4 is at least 1 to ensure f is a WCRDF of T. This implies that the sum of the weights assigned by f to the vertices of the base path P is at least 5. Since there are at least k vertex disjoint base paths in T, each of which receives a total weight at least 5, the weight of f is $\mathrm{w}(f) \geq 5k$. Since f is an arbitrary $\gamma_R^{wc}(T)$–function, this implies that $\gamma_R^{wc}(T) \geq \frac{5}{6}n$.

Conversely, it is no problem to observe, that the underlying tree of T belongs to the family $\mathcal{E}$. Hence, by Theorems 9, 10 and 11, the weakly connected domination number of underlying tree of T is equal to $\frac{k}{2}$. Hence the function f that assigns the weight 2 to every support vertex of T, the weight 0 to every leaf and the weight 1 to each vertex of a minimum weakly connected dominating set of the underlying subtree of T is a WCRDF of T of weight $\frac{5}{2}k$, which proves statement 1. Therefore, $\gamma_R^{wc}(T) \le \mathrm{w}(f) = \frac{5}{2}k = \frac{5}{6}n$, which proves the equality (9).

Let v_1 be a leaf of $T \in \mathcal{F}$ and let $(v_1, v_2, v_3, v_4, v_5, v_6)$ be a base path of T. In what follows we construct a $\gamma_R^{wc}(T)$–function which assigns to v value 2. Since the underlying tree of T belongs to the family $\mathcal{E}$, it is γ_{wc}–excellent. Thus there exists a γ_{wc}–set of the underlying tree of T containing v_3. Let f be the function that assigns the weight 2 to v_1 and to every support vertex of T except of v_2, the weight 0 to v_2 and every leaf except of v_1 and the weight 1 to each vertex of a minimum weakly connected dominating set of the underlying subtree of T that contains v_3. Then f is a WCRDF of T of weight $\gamma_R^{wc}(T) = \frac{5}{6}n$, which proves statement 2.

Theorem 13. *If T is a tree of order $n \ge 3$, then*

$$\gamma_R^{wc}(T) \le \frac{5}{6}n,$$

with equality if and only if $T \in \mathcal{F}$.

Proof. We proceed by induction on the order $n \ge 3$ of a tree T. If $n = 3$, then $T = P_3$ and $\gamma_R^{wc}(T) = 2 < \frac{5}{6}n$. This establishes the base case.

Let $n \ge 4$ and assume that if T' is a tree of order n', where $3 \le n' < n$, then $\gamma_R^{wc}(T') \le \frac{5}{6}n'$ with equality if and only if $T' \in \mathcal{F}$.

If T is a star, then the function that assigns the weight 2 to the central vertex and the weight 0 to every leaf of the star is a WCRDF of T of weight 2, and so $\gamma_R^{wc}(T) = 2 < \frac{5}{6}n$. Hence we may assume that $\mathrm{diam}(T) \ge 3$.

If $T = P_4$, then $\gamma_R^{wc}(T) = 3 < \frac{5}{6}n$. If T is a double star which is not P_4, then the function that assigns the weight 2 to the two central vertices and the weight 0 to every leaf of the double star is a WCRDF of T of weight 4, and so $\gamma_R^{wc}(T) = 4 < \frac{5}{6}n$. Hence we may assume that $\mathrm{diam}(T) \ge 4$.

Let v_1 and r be two vertices at maximum distance apart in T. Necessarily, v_1 and r are leaves and $d(v_1, r) = \mathrm{diam}(T)$. We now root the tree T at the vertex r. Let v_2 be the parent of v_1, v_3 parent of v_2, v_4 parent of v_3 and v_5 parent of v_4. We note that if $\mathrm{diam}(T) = 4$, then $r = v_5$.

Suppose that $d_T(v_2) \geq 3$. Let T' be the tree obtained from T by deleting v_2 and its children. Let T' have order n', and so $n' \leq n-3$. Since $\mathrm{diam}(T) \geq 4$, we note that $n' \geq 3$. Applying the inductive hypothesis to the tree T', $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n-3)$. Lef f' be a $\gamma_R^{wc}(T')$–function. We can extend f' to the WCRDF of T by assigning the weight 2 to v_2 and the weight 0 to the children of v_2. The resulting function f has weight $\mathrm{w}(f) = \mathrm{w}(f') + 2$. Hence, $\gamma_R^{wc}(T) \leq \mathrm{w}(f) = \mathrm{w}(f') + 2 \leq \frac{5}{6}(n-3) + 2 < \frac{5}{6}n$.

Therefore we may assume that every child of v_3 in T is a leaf or has degree 2, for otherwise the desired result follows. By symmetry, we assume that every support vertex on a longest path of T is of degree 2.

If $\mathrm{diam}(T) = 4$, then T is a spider graph, that is a tree with $\mathrm{diam}(T) = 4$, $d_T(v_3) \geq 3$, $d_T(v_2) = d_T(v_4) = 2$ and all other vertices with degree at most 2. Denote by k_2 the number of neighbours of v_3 of degree 2. Note that $k_2 \geq 2$ and $n \geq 2k_2 + 1$. Then the function that assigns the weight 2 to v_3, the weight 1 to each leaf at distance 2 from v_3 and the weight 0 to every other vertex of the spider is a WCRDF of T of weight $2 + k_2$, and so $\gamma_R^{wc}(T) = 2 + k_2 < \frac{13}{6} + k_2 \leq \frac{5}{6} + \frac{4}{6}k_2 + k_2 = \frac{5}{6}(1 + 2k_2) \leq \frac{5}{6}n$. Hence we may assume that $\mathrm{diam}(T) \geq 5$.

Let t_1 be the number of children of v_3 of degree 1 and let t_2 be the number of children of v_3 of degree 2. Then $t_2 \geq 1$. Suppose that $t_1 + t_2 \geq 2$. Let T' be the tree obtained from T by deleting v_3 and its descendants. Let T' have order n', and so $n' = n - 2t_2 - t_1 - 1$. Since $\mathrm{diam}(T) \geq 5$, we note that $n' \geq 3$. Applying the inductive hypothesis to the tree T', $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n - 2t_2 - t_1 - 1)$. Lef f' be a $\gamma_R^{wc}(T')$–function. We can extend f' to the WCRDF of T by assigning the weight 2 to v_3, the weight 1 to each descendant at distance 2 from v_3 and the weight 0 to the children of v_3. The resulting function f has weight $\mathrm{w}(f) = \mathrm{w}(f') + 2 + t_2$. Hence, $\gamma_R^{wc}(T) \leq \mathrm{w}(f) = \mathrm{w}(f') + 2 + t_2 \leq \frac{5}{6}(n - 2t_2 - t_1 - 1) + 2 + t_2 = \frac{1}{6}(5n - 5t_1 - 4t_2 + 7)$ and since we supposed $t_1 + t_2 \geq 2$, we obtain $\gamma_R^{wc}(T) < \frac{5}{6}n$.

Therefore we may assume that $t_1 + t_2 = 1$. Since v_3 is on a longest path of T, we conclude that $t_1 = 0$ and $t_2 = 1$, which implies that $d_T(v_3) = 2$, for otherwise the desired result follows.

Suppose now that $d_T(v_4) = 2$. Let T' be the tree obtained from T by deleting v_4 and its descendants, that is v_1, v_2, v_3 and v_4. Let T' have order n', and so $n' = n - 4$. If $n' \geq 3$, then applying the inductive hypothesis to the tree T', $\gamma_R^{wc}(T') \leq \frac{5}{6}n' = \frac{5}{6}(n-4)$. Moreover, Lemma 6 implies that $\gamma_R^{wc}(T) = \gamma_R^{wc}(T') + 3$. Hence, $\gamma_R^{wc}(T) \leq \frac{5}{6}(n-4) + 3 < \frac{5}{6}n$. If $n' \leq 2$, then since $\mathrm{diam}(T) \geq 5$, $n' = 2$ and thus $T = P_6$. In this case $\gamma_R^{wc}(T) = \frac{5}{6}n$ and clearly $P_6 \in \mathcal{F}$.

Therefore in what follows we may assume that $d_T(v_4) \geq 3$.

Suppose that a child of v_4, say x, is a strong support vertex. Let T' be the tree obtained from T by deleting x and the children x. Let T' have order n', and so $n' \leq n - 3$. Since $\mathrm{diam}(T) \geq 5$, we note that $n' \geq 3$. Applying the inductive hypothesis to the tree T', $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n-3)$. Lef f' be a $\gamma_R^{wc}(T')$–function. We can extend f' to a WCRDF of T by assigning the weight 2 to x and the weight 0 to the children of x. The resulting function f has weight $\mathrm{w}(f) = \mathrm{w}(f') + 2$. Hence, $\gamma_R^{wc}(T) \leq \mathrm{w}(f) = \mathrm{w}(f') + 2 \leq \frac{5}{6}(n-3) + 2 < \frac{5}{6}n$.

Therefore we may assume that every child of v_4 is of degree 1 or 2, for otherwise the desired result follows.

Let t_1 be the number of children of v_4 of degree 1, let t_2 be the number of children of v_4 which are support vertices and let t_3 be the number of children of v_4 which are not support vertices. Then $d_T(v_4) = t_1 + t_2 + t_3 + 1$, $t_3 \geq 1$ and $t_1 + t_2 + t_3 \geq 2$.

Suppose $t_2 = 0$. Then $t_1 + t_3 \geq 2$. Let T' be the tree obtained from T by deleting v_4 and its descendants. Let T' have order n', and so $n' = n - (1 + t_1 + 3t_3)$. Since $\mathrm{diam}(T) \geq 5$, we note that $n' \geq 2$.

If $n' = 2$, then $n = 3 + t_1 + 3t_3$ and $V(T') = \{v_5, r\}$. Let f be a WCRDF of T which assigns the weight 2 to v_4 and to all support descendants of v_4, the weight 0 to the remaining descendants of v_4 and to v_5, and the weight 1 to r. The resulting function f has weight $\mathrm{w}(f) = 3 + 2t_3$. Hence, $\gamma_R^{wc}(T) \leq \mathrm{w}(f) = 3 + 2t_3$. If $t_1 \geq 1$, then $\gamma_R^{wc}(T) \leq 2 + \frac{1}{2}t_1 + \frac{5}{2}t_3 < \frac{5}{6}(3 + t_1 + 3t_3) = \frac{5}{6}n$. If $t_1 = 0$, then $t_3 \geq 2$ and $\gamma_R^{wc}(T) \leq 2 + \frac{5}{2}t_3 < \frac{5}{6}(3 + 3t_3) = \frac{5}{6}n$.

If $n' \geq 3$, then by applying the inductive hypothesis to the tree T', $\gamma_R^{wc}(T') \leq$

$\frac{5}{6}n' \le \frac{5}{6}(n-1-t_1-3t_3)$. Lef f' be a $\gamma_R^{wc}(T')$–function. If $t_1 = 0$, then $t_3 \ge 2$ and we can extend f' to a WCRDF of T by assigning the weight 1 to v_4, the weight 2 to all support descendants of v_4, and the weight 0 to the remaining descendants of v_4. The resulting function f has weight $\mathrm{w}(f) = \mathrm{w}(f') + 1 + 2t_3$. Hence, $\gamma_R^{wc}(T) \le \mathrm{w}(f) = \mathrm{w}(f') + 1 + 2t_3 \le \frac{5}{6}(n-1-3t_3) + 1 + 2t_3 < \frac{5}{6}n$. If $t_1 \ge 1$, then we can extend f' to a WCRDF of T by assigning the weight 2 to v_4 and to all support descendants of v_4, and the weight 0 to the remaining descendants of v_4. The resulting function f has weight $\mathrm{w}(f) = \mathrm{w}(f') + 2 + 2t_3$. Hence, $\gamma_R^{wc}(T) \le \mathrm{w}(f) = \mathrm{w}(f') + 2 + 2t_3 \le \frac{5}{6}(n-1-t_1-3t_3) + 2 + 2t_3 < \frac{5}{6}n$.

Therefore we may assume that $t_2 \ge 1$, for otherwise the desired result follows.

Suppose $t_1 \ge 1$. Then $t_2 \ge 1$ and $t_3 \ge 1$. Let T' be the tree obtained from T by deleting v_4 and its descendants. Let T' have order n', and so $n' = n - (1 + t_1 + 2t_2 + 3t_3)$. Since $\mathrm{diam}(T) \ge 5$, we note that $n' \ge 2$.

If $n' = 2$, then $n = 3 + t_1 + 2t_2 + 3t_3$ and $V(T') = \{v_5, r\}$. Let f be a WCRDF of T which assigns the weight 2 to v_4 and to all support descendants of v_4 at distace 2 from v_4, the weight 1 to r and the leaf descendants of v_4 at distace 2 from v_4, and the weight 0 to the remaining descendants of v_4 and to v_5. The resulting function f has weight $\mathrm{w}(f) = 3 + t_2 + 2t_3$. Hence, $\gamma_R^{wc}(T) \le \mathrm{w}(f) = 3 + t_2 + 2t_3 \le 2 + \frac{3}{2}t_2 + \frac{5}{2}t_3 < \frac{5}{6}(3 + t_1 + 2t_2 + 3t_3) = \frac{5}{6}n$.

If $n' \ge 3$, then by applying the inductive hypothesis to the tree T', $\gamma_R^{wc}(T') \le \frac{5}{6}n' \le \frac{5}{6}(n-1-t_1-2t_2-3t_3)$. Lef f' be a $\gamma_R^{wc}(T')$–function. We can extend f' to a WCRDF of T by assigning the weight 2 to v_4 and to all support descendants of v_4 at distance 2 from v_4, the weight 1 to the leaf descendants of v_4 at distace 2 from v_4, and the weight 0 to the remaining descendants of v_4. The resulting function f has weight $\mathrm{w}(f) = \mathrm{w}(f') + 2 + t_2 + 2t_3$. Hence,

$$\begin{aligned} \gamma_R^{wc}(T) &\le \mathrm{w}(f) = \mathrm{w}(f') + 2 + 2t_3 \\ &\le \frac{5}{6}(n-1-t_1-2t_2-3t_3) + 2 + t_2 + 2t_3 \\ &= \frac{1}{6}(5n + 7 - 5t_1 - 4t_2 - 3t_3). \end{aligned} \tag{10}$$

Since $t_1 \ge 1$, $t_2 \ge 1$, and $t_3 \ge 1$, equation (10) implies that $\gamma_R^{wc}(T) < \frac{5}{6}n$. Therefore we may assume that $t_1 = 0$, for otherwise the desired result follows.

Similarly, if $t_2 \geq 2$ or $t_3 \geq 2$, equation (10) again implies that $\gamma_R^{wc}(T) < \frac{5}{6}n$. Therefore we may assume that $t_2 = t_3 = 1$, for otherwise the desired result follows.

Denote by x the child of v_4 which is a support vertex different from v_5 and let y be the child of x. Then $(v_1, v_2, v_3, v_4, x, y)$ induce a path P_6 in T. Let T' be the tree obtained from T by deleting v_4 and its descendants. Let T' have order n', and so $n' = n - 6$. Since $\mathrm{diam}(T) \geq 5$, we note that $n' \geq 2$. If $n' = 2$, then $n = 8$ and $V(T') = \{v_5, r\}$. Let f be a WCRDF of T which assigns the weight 2 to v_4 and v_2, the weight 1 to r and y, and the weight 0 to the remaining vertices of T. The resulting function f has weight $\mathrm{w}(f) = 6$. Hence, $\gamma_R^{wc}(T) \leq \mathrm{w}(f) = 6 < \frac{5}{6}n$.

Hence $n' \geq 3$. By (10), if $\mathrm{w}(f') < \frac{5}{6}n'$, then $\gamma_R^{wc}(T) < \frac{5}{6}n$ and the result follows. Hence assume $\gamma_R^{wc}(T') = \frac{5}{6}n'$. Then by the induction hypothesis, $T' \in \mathcal{F}$. Now it suffices to show, that v_5 belongs to the underlying subtree of T'. Suppose to the contrary, that v_5 is a support vertex or a leaf in T'.

Consider first the situation when v_5 is a support vertex. Denote by z_1 the leaf neighbour of v_5 and by z_2 the neighbour of v_5 belonging to the underlying subtree of T'. Since the underlying subtree of T' belongs to the family $\mathcal{E}$, Theorem 9 and Lemma 12 imply that there exists a $\gamma_R^{wc}(T')$–function f' such that the weight assigned to z_2 is 1, the weight assigned to v_5 is 2 *tu poprawi* v_5 and the weight assigned to z_1 is 0. We can extend f' to a WCRDF of T by assigning the weight 2 to v_2 and v_4, the weight 1 to y and the weight 0 to v_1, v_3 and x. Additionally, we change the weight of v_5 to 0 and the weight of z_1 to 1. The resulting function f is a WCRDF of T and has weight $\mathrm{w}(f) = \mathrm{w}(f') + 5 - 1 = \frac{5}{6}n' + 4 = \frac{5}{6}(n-6) + 4 < \frac{5}{6}n$. Therefore v_5 is not a support vertex.

Suppose now v_5 is a leaf. Denote by z the neighbour of v_5 in T'. Then Lemma 12 implies that there exists a $\gamma_R^{wc}(T')$–function f' such that the weight assigned to v_5 is 2 and hence we can assume that the weight assigned to z is 0. We can extend f' to a WCRDF of T by assigning the weight 2 to v_2 and v_4, the weight 1 to y and the weight 0 to v_1, v_3 and x. Additionally, we change the weight of v_5 to 0 and the weight of z to 1. The resulting function f is a WCRDF of T and has weight $\mathrm{w}(f) = \mathrm{w}(f') + 5 - 1 = \frac{5}{6}n' + 4 = \frac{5}{6}(n-6) + 4 < \frac{5}{6}n$. Therefore v_5 is not a leaf.

We conclude that v_5 belongs to the underlying subtree of T'. For this reason, T belongs to the family $\mathcal{F}$, which completes the proof.

Since the weakly connected Roman domination number and the outer-independent Roman domination number are equal for trees, we have the following

Corollary 14. *If T is a tree of order $n \geq 3$, then*

$$\gamma_{oiR}(T) \leq \frac{5}{6}n,$$

with equality if and only if $T \in \mathcal{F}$.

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