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# MOUNTAIN PASS SOLUTIONS TO EULER-LAGRANGE EQUATIONS WITH GENERAL ANISOTROPIC OPERATOR

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Abstract. Using the Mountain Pass Theorem we show that the problem

$$\begin{cases} \frac{d}{dt}\mathcal{L}_v(t,u(t),\dot{u}(t)) = \mathcal{L}_x(t,u(t),\dot{u}(t)) & \text{for a.e. } t \in [a,b] \\ u(a)=u(b)=0 \end{cases}$$

has a solution in anisotropic Orlicz-Sobolev space. We consider Lagrangian $\mathcal{L} = F(t,x,v) + V(t,x) + \langle f(t), x\rangle$ with growth conditions determined by anisotropic G-function and some geometric conditions of Ambrosetti-Rabinowitz type.

## 1. Introduction

We consider the second order boundary value problem:

$$\text{(ELT)} \qquad \begin{cases} \frac{d}{dt}\mathcal{L}_v(t,u(t),\dot{u}(t)) = \mathcal{L}_x(t,u(t),\dot{u}(t)) & \text{for a.e. } t \in [a,b] \\ u(a)=u(b)=0 \end{cases}$$

where $\mathcal{L}\colon [a,b]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ is given by

$$\mathcal{L}(t,x,v) = F(t,x,v) + V(t,x) + \langle f(t), x\rangle .$$

Using the Mountain Pass Theorem we show that the problem (ELT) has a solution in anisotropic Orlicz-Sobolev space.

Recently, existence of periodic solution to the equation

$$\frac{d}{dt}\nabla G(\dot{u}(t)) = \nabla V(t,u(t)) + f(t)$$

was established by Authors in [1] via the Mountain Pass Theorem. In this paper we consider more general differential operator:

$$\frac{d}{dt}F_v(t,u,\dot{u}).$$

We assume that $F$ is convex in the last variable and that the growth of $F$ and its derivatives is determined by underlying G-function. We also assume that $F$ and $V$ satisfies some geometric conditions of Ambrosetti-Rabinowitz type.

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If $F(v) = \frac{1}{p}|v|^p$ then the equation (ELT) reduces to (one-dimensional or ordinary) $p$-laplacian equation

$$\frac{d}{dt}(|\dot{u}|^{p-2}\,\dot{u}) = \nabla V(t,u) + f(t),\ u(a) = u(b) = 0.$$

Existence of solutions to the above problem has been studied by many authors in many different contexts. See for example,to mention only a few, [2, 3, 4, 5, 6, 7] and references therein. One can also consider more general case $F(v) = \phi(|v|)$, where $\phi$ is convex and nonnegative. In all the above cases $F$ does not depend on $v$ directly but rather on its norm $|v|$ and the growth of $F$ is the same in all directions, i.e. $F$ has isotropic growth.

The novelty of this article lies in fact that $F$ can be dependent not only on $\dot{u}$ but also on $t$ and $u$. Moreover we consider anisotropic case, i.e. $F(t,x,\cdot)$ depends on all components of $v$ not only on $|v|$ and has different growth in different directions. To the best author knowledge there is no results on existence in our setting.

We obtain solution to the problem (ELT) by applying the Mountain Pass Theorem. To do this we first need to show that corresponding action functional satisfies the Palais-Smale condition. First we prove that a Palais-Smale sequence $\{u_n\}$ is bounded, the proof is rather standard and involves Ambrosetti-Rabinowitz condition and assumption $F(t,x,v) \geq \Lambda G(v)$. Then we need to show that $\{u_n\}$ has a convergent subsequence. We show that

$$\lim_{n\to\infty}\int_I \langle F_v(t,u_n,\dot{u}_n,\dot{u}-\dot{u}_n\rangle\; dt = 0,$$

where $u$ is a weak limit of $\{u_n\}$, which in turn implies that

$$\lim_{n\to\infty}\int_I F(t,u_n,\dot{u}_n)\,dt = \int_I F(t,u,\dot{u})\,dt.$$

The proof of this fact is based on convexity of $F$ and embedding $\mathbf{W}^1\,\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$. Next, using convexity of $F$ and condition $F(t,x,v) \geq \Lambda G(v)$, we obtain that $\{\dot{u}_n\}$ converges strongly. This reasoning shows that action functional satisfies so called $(S_+)$ condition (see for example [8]).

This result seems to be of independent interest and the methods presented in this paper can be also applied in other problems (e.g. in the case of periodic problem).

Our work was partially inspired by the paper of de Napoli and Mariani [9]. They consider elliptic PDE

$$-div(a(x,\nabla u)) = f(x,u)$$

with Dirichlet conditions. To show that corresponding functional satisfies the Palais-Smale condition they also prove that $(S_+)$ condition is satisfied. However, they use stronger condition, namely they assume uniform convexity of functional.

As in [1] we consider two cases: $G$ satisfying $\Delta_2$, $\nabla_2$ at infinity and globally. It turns out that in both cases the mountain pass geometry of action functional is strongly dependent on two factors: the embedding constant for $\mathbf{W}^1\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ and on Simonenko indices $p_G$ and $q_G$ (see Lemmas 4.4 and 4.5).

Similar observation can be found in [10, 11, 12] where the existence of elliptic systems via the Mountain Pass Theorem is considered. In [11] authors deal with an anisotropic problem. The isotropic case is considered in [10, 12].

## 2. Orlicz-Sobolev spaces

In this section we briefly recall the notion of anisotropic Orlicz-Sobolev spaces. For more details we refer the reader to [13, 1] and references therein. We assume that

(G) $G\colon \mathbb{R}^N \to [0,\infty)$ is a continuously differentiable G-function (i.e. $G$ is convex, even, $G(0)=0$ and $G(x)/|x| \to \infty$ as $|x|\to\infty$ ) satisfying $\Delta_2$ and $\nabla_2$ conditions (at infinity).

Typical examples of such $G$ are: $G(x)=|x|^p$, $G(x_1,x_2)=|x_1|^{p_1}+|x_2|^{p_2}$ and $G(x)=|x|^p\log(1+|x|)$, $1<p_i<\infty$, $1<p<\infty$.

Let $I=[a,b]$. The Orlicz space associated with $G$ is defined to be

$$\mathbf{L}^G=\mathbf{L}^G(I,\mathbb{R}^N)=\left\{u\colon I\to\mathbb{R}^N\colon \int_I G(u)\,dt<\infty\right\}.$$

The space $\mathbf{L}^G$ equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G}=\inf\left\{\lambda>0\colon \int_I G\left(\frac{u}{\lambda}\right)\,dt\le 1\right\}$$

is a separable, reflexive Banach space. We have two important inequalities:

a) the Fenchel inequality

$$\langle u,v\rangle \le G(u)+G^*(v),\ \text{for every } u,\ v\in\mathbb{R}^N,$$

b) the Hölder inequality

$$\int_I \langle u,v\rangle\ dt \le 2\|u\|_{\mathbf{L}^G}\|v\|_{\mathbf{L}^{G^\star}},\ \text{for every } u\in\mathbf{L}^G \text{ and } v\in\mathbf{L}^{G^\star},$$

where $G^*$ is a convex conjugate of $G$. Functional $R_G(u)=\int_I G(u)\,dt$ is called modular. Note that if $G(x)=|x|^p$ then $\mathbf{L}^G=\mathbf{L}^p$ and $R_G(u)=\|u\|^p_{\mathbf{L}^p}$. In general case, relation between modular and the Luxemburg norm is more complicated.

The Simonenko indices for G-function are defined by

$$p_G=\inf_{|x|>0}\frac{\langle x,\nabla G(x)\rangle}{G(x)},\quad q_G=\sup_{|x|>0}\frac{\langle x,\nabla G(x)\rangle}{G(x)}.$$

It is obvious that $p_G\le q_G$. Moreover, since $G$ satisfies $\Delta_2$ and $\nabla_2$, $1<p_G$ and $q_G<\infty$. If $G(x)=\frac{1}{p}|x|^p$ then $p_G=q_G=p$. The following results are crucial to Lemma 4.5

**Proposition 2.1.** *Assume that $G$ satisfies $\Delta_2$ and $\nabla_2$ globally.*

*a) If* $\|u\|_{\mathbf{L}^G} \le 1$*, then* $\|u\|_{\mathbf{L}^G}^{q_G} \le R_G(u)$.
*b) If* $\|u\|_{\mathbf{L}^G} > 1$*, then* $\|u\|_{\mathbf{L}^G}^{p_G} \le R_G(u)$.

The proof can be found in [1, Appendix A]. More information about indices for isotropic case can be found in [14, 10]. When $G$ satisfies $\Delta_2$ and $\nabla_2$ only at infinity we have weaker estimates

**Proposition 2.2.** *If* $\|u\|_{\mathbf{L}^G} > 1$ *then* $R_G(u) \ge \|u\|_{\mathbf{L}^G}$*. If* $\|u\|_{\mathbf{L}^G} \le 1$ *then* $R_G(u) \le \|u\|_{\mathbf{L}^G}$*.*

For relations between Luxemburg norm and modular for anisotropic spaces we refer the reader to [13, Examples 3.8 and 3.9]. We will also use the following simple observations

**Lemma 2.3.**

$$\lim_{\|u\|_{\mathbf{L}^G}\to\infty} \frac{R_G(u)}{\|u\|_{\mathbf{L}^G}} = \infty.$$

**Lemma 2.4.** *Let* $\{u_n\} \subset \mathbf{L}^G$*. Then* $\{u_n\}$ *is bounded if and only if* $\{R_G(u_n)\}$ *is bounded.*

The anisotropic Orlicz-Sobolev space is defined to be

$$\mathbf{W}^1\,\mathbf{L}^G = \mathbf{W}^1\,\mathbf{L}^G(I,\mathbb{R}^N) = \{u \in \mathbf{L}^G \colon \dot{u} \in \mathbf{L}^G\},$$

with usual norm

$$\|u\|_{\mathbf{W}^1\,\mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}.$$

It is known that elements of $\mathbf{W}^1\,\mathbf{L}^G$ are absolutely continuous functions. An important role in our considerations plays an embedding constant for $\mathbf{W}^1\,\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$. We denote this constant by $C_{\infty,G}$. Let $A_G \colon \mathbb{R}^N \to [0,\infty)$ be the greatest convex minorant of $G$ (see [15]), then

$$\|u\|_{\mathbf{L}^\infty} \le \max\{1,|I|\} A_G^{-1}\left(\frac{1}{|I|}\right) \|u\|_{\mathbf{W}^1\,\mathbf{L}^G}.$$

We introduce the following subspace of $\mathbf{W}^1\,\mathbf{L}^G$:

$$\mathbf{W}_0^1\,\mathbf{L}^G = \{u \in \mathbf{W}^1\,\mathbf{L}^G \colon u = 0 \text{ on } \partial I\}.$$

It is proved in [13, Theorem 4.5] that for every $u \in \mathbf{W}_0^1\,\mathbf{L}^G$ the following form of Poincaré inequality holds

$$\|u\|_{\mathbf{L}^G} \le |I|\,\|\dot{u}\|_{\mathbf{L}^G}. \tag{1}$$

It follows that one can introduce an equivalent norm on $\mathbf{W}_0^1\,\mathbf{L}^G$:

$$\|u\|_{\mathbf{W}_0^1\,\mathbf{L}^G} = \|\dot{u}\|_{\mathbf{L}^G}.$$

## 3. Main results

Let $G$ satisfies assumption (G) and let $I = [a, b]$. We consider Lagrangian $\mathcal{L}\colon I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$\mathcal{L}(t, x, v) = F(t, x, v) + V(t, x) + \langle f(t), x\rangle .$$

We assume that $F\colon I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $V\colon I \times \mathbb{R}^N \to \mathbb{R}$ are of class $C^1$ and satisfy

($F_1$) $F(t, x, \cdot)$ is convex for all $(t, x) \in I \times \mathbb{R}^N$,

($F_2$) there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $b \in \mathbf{L}^1(I, \mathbb{R}_+)$ such that for all $(t, x, v) \in I \times \mathbb{R}^N \times \mathbb{R}^N$:

$$|F(t, x, v)| \le a(|x|)\,(b(t) + G(v)), \tag{2}$$

$$|F_x(t, x, v)| \le a(|x|)\,(b(t) + G(v)), \tag{3}$$

$$G^*(F_v(t, x, v)) \le a(|x|)\,(b(t) + G^*(\nabla G(v))), \tag{4}$$

($F_3$) There exist $\theta_F > 0$ such that for all $(t, x, v) \in I \times \mathbb{R}^N \times \mathbb{R}^N$:

$$\langle F_x(t, x, v), x\rangle + \langle F_v(t, x, v), v\rangle \le \theta_F\, F(t, x, v),$$

($F_4$) there exists $\Lambda > 0$ such that for all $(t, x, v) \in I \times \mathbb{R}^N \times \mathbb{R}^N$:

$$F(t, x, v) \ge \Lambda\, G(v),$$

($F_5$) $F(t, x, 0) = 0$ for all $(t, x) \in I \times \mathbb{R}^N$,

($V_1$) there exist $\theta_V > 1$, $\theta_V > \theta_F$ and $r_0 > 0$ such that for all $t \in I$

$$\langle \nabla V(t, x), x\rangle \le \theta_V V(t, x),\ \ |x| \ge r_0,$$

($V_2$)

$$\int_I V(t, 0)\, dt = 0,$$

($V_3$) there exist $\rho_0 > 0$ and $g \in \mathbf{L}^1(I, \mathbb{R})$ such that for all $t \in I$

$$V(t, x) \ge -g(t),\ \ |x| \le \rho_0,$$

($V_4$)

$$V(t, x) < 0,\ \ t \in I,\ \ |x| \ge r_0,$$

(f) $f \in \mathbf{L}^{G^\star}(I, \mathbb{R}^N)$.

Now we can state our main theorems.

**Theorem 3.1.** *Assume that $\rho_0 \ge C_{\infty,G}$ and*

$$\int_I g(t)\, dt < (\Lambda - 2|I|\, \|f\|_{\mathbf{L}^{G^\star}})\frac{\rho_0}{C_{\infty,G}}. \tag{A}$$

*Then* (ELT) *has at least one nontrivial solution.*

Assumption $\rho_0 \ge C_{\infty,G}$ can be relaxed if we assume that $G$ satisfies $\Delta_2$ and $\nabla_2$ globally. In this case we also have weaker assumptions on $V$.

**Theorem 3.2.** *Assume that $G$ satisfies $\Delta_2$ and $\nabla_2$ globally and*

$$\text{(B)} \qquad \int_I g(t)\,dt + 2|I|\,\|f\|_{\mathbf{L}^{G^\star}} \frac{\rho_0}{C_{\infty,G}} < \Lambda \begin{cases} \left(\frac{\rho_0}{C_{\infty,G}}\right)^{q_G}, & \rho_0 \le C_{\infty,G} \\ \left(\frac{\rho_0}{C_{\infty,G}}\right)^{p_G}, & \rho_0 > C_{\infty,G} \end{cases}$$

*Then* (ELT) *has at least one nontrivial solution.*

One can show that, in fact, every solution of (ELT) is of class $\mathbf{W}^{1,\infty}$ (see [1, Proposition 3.5]).

3.1. **Some remarks on assumptions.** Assumptions ($F_3$) and ($V_1$) are Ambrosetti-Rabinowitz type conditions. It follows that $F$ and $V$ are subhomogeneous respectively everywhere and for large arguments (cf. [9]).

**Lemma 3.3.** *For every $\lambda > 1$*

*a)*

$$F(t, \lambda x, \lambda v) \le \lambda^{\theta_F} F(t, x, v) \quad \textit{for all } (t, x, v) \in I \times \mathbb{R}^N \times \mathbb{R}^N$$

*b)*

$$V(t, \lambda x) \le \lambda^{\theta_V} V(t, x) \quad \textit{for all } t \in I,\ |x| \ge r_0$$

*Proof.* Let $(t, x, v) \in I \times \mathbb{R}^N \times \mathbb{R}^N$ and $\lambda > 1$, then

$$\log\left(\frac{F(t,\lambda x,\lambda v)}{F(t,x,v)}\right) = \int_1^\lambda \frac{d}{d\lambda} \log F(t, \lambda x, \lambda v)\, d\lambda =$$
$$= \int_1^\lambda \frac{\langle F_x(t,\lambda x,\lambda v), x\rangle + \langle F_v(t,\lambda x,\lambda v), v\rangle}{F(t,\lambda x,\lambda v)}\, d\lambda \le \int_1^\lambda \frac{\theta_F}{\lambda} d\lambda = \log \lambda^{\theta_F}.$$

by ($F_3$) and the result follows. The proof of b) is similar □

**Remark 3.4.** *Note that if $F(t, x, v) = G(v)$ then assumption ($F_3$) implies that $G$ satisfies $\Delta_2$ condition globally.*

## 4. Proof of the main theorems

Define action functional $\mathcal{J} \colon \mathbf{W}^1_0\,\mathbf{L}^G(I, \mathbb{R}^N) \to \mathbb{R}$ by

$$(\mathcal{J}) \qquad \mathcal{J}(u) = \int_I F(t, u, \dot u) + V(t, u) + \langle f, u\rangle\ dt$$

Under above assumptions, $\mathcal{J}$ is well defined and of class $C^1$. Furthermore, its derivative is given by

($\mathcal{J}'$)

$$\mathcal{J}'(u)\varphi = \int_I \langle F_x(t,u,\dot u), \varphi\rangle\ dt + \int_I \langle F_v(t,u,\dot u), \dot\varphi\rangle\ dt + \int_I \langle \nabla V(t,u), \varphi\rangle + \langle f, \varphi\rangle\ dt$$

See [13, Theorem 5.7] for more details. It is standard to prove that critical points of $\mathcal{J}|_{\mathbf{W}^1_0\mathbf{L}^G}$ are solutions of (ELT).

Our proof is based on the well-known Mountain Pass Theorem (see [16]).

**Theorem 4.1.** *Let $X$ be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies the following conditions:*

*a) $I$ satisfies Palais-Smale condition,*
*b) $I(0) = 0$,*
*c) there exist $\rho > 0$, $e \in X$ such that $\|e\|_X > \rho$ and $I(e) < 0$.*
*d) there exists $\alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$,*

*Then $I$ possesses a critical value $c \geq \alpha$ given by $c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s))$, where $\Gamma = \{g \in C([0,1], X) : \ g(0) = 0,\ g(1) = e\}$.*

We divide the proof into sequence of lemmas.

4.1. **The Palais-Smale condition.** Now we show that $\mathcal{J}$ satisfies the Palais-Smale condition. We divide the proof into two steps. First we show that every (PS)-sequence is bounded and then that it contains a convergent subsequence.

The first part of the proof is standard. Let us note that assumptions ($F_3$), ($F_4$) and ($V_1$) are crucial. The second part is more involved, let us outline it. First we show that $u_n \rightharpoonup u$ and embedding $\mathbf{W}^1\,\mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ imply that

$$\int_I \langle F_v(t, u_n, \dot{u}_n), \dot{u} - \dot{u}_n\rangle \; dt \to 0.$$

Then we show that

$$\int_I F(t, u_n, \dot{u}_n)\, dt \to \int_I F(t, u, \dot{u})\, dt.$$

In the case of p-Laplacian equation (i.e. $F(t, x, v) = \frac{1}{p}|v|^p$), the last condition implies that $\dot{u}_n \to \dot{u}$. The same is true if $F(t, u, v) = G(v)$, since in this case $R_G(\dot{u}_n) \to R_G(\dot{u})$. The last condition implies desired convergence for $\{\dot{u}_n\}$ (see [13, Lemma 3.16] and [1, p. 593]).

In our case this argument does not apply directly because convergence of above integrals does not imply that $R_G(\dot{u}_n) \to R_G(\dot{u})$. However, we can extend the reasoning presented in the proof of [13, Lemma 3.16] to our general integrand and show that

$$\int_I F\left(t, u_n, \frac{\dot{u}_n - \dot{u}}{2}\right)\, dt \to 0$$

and then apply condition ($F_4$) to show that $R_G(\dot{u}_n - \dot{u}) \to 0$ and hence $\dot{u}_n \to \dot{u}$ in $\mathbf{L}^G$ by [13, Lemma 3.13].

**Lemma 4.2.** *Functional $\mathcal{J}$ satisfies the Palais-Smale condition*

*Proof.* Fix $u \in \mathbf{W}^1_0\,\mathbf{L}^G$. From assumptions ($F_3$) and ($F_4$) we obtain

$$\int_I \theta_V F(t, u, \dot{u}) - \langle F_x(t, u, \dot{u}), u\rangle - \langle F_v(t, u, \dot{u}), \dot{u}\rangle \; dt \geq \tag{5}$$

$$\geq (\theta_V - \theta_F) \int_I F(t, u, \dot{u})\, dt \geq C_1 \int_I G(\dot{u})\, dt,$$

where $C_1 = \Lambda(\theta_V - \theta_F) > 0$, since $\theta_V > \theta_F$. Set $M = \sup\{|\theta_V V(t,x) - \langle \nabla V(t,x), x\rangle| : t \in I, |x| \le r_0\}$, then by ($V_1$) we obtain

(6)
$$\int_I \theta_V V(t,u) - \langle \nabla V(t,u), u\rangle \, dt \ge \int_{\{|u(t)|>r_0\}} \theta_V V(t,u) - \langle \nabla V(t,u), u\rangle \, dt - |I|M \ge -|I|M.$$

We also have, by Hölder's inequality, (1) and (f), that

$$(\theta_V - 1)\int_I \langle f(t), u\rangle \, dt \ge -2(\theta_V - 1)\|f\|_{\mathbf{L}^{G^\star}}\|u\|_{\mathbf{L}^G} \ge -C_2\|\dot{u}\|_{\mathbf{L}^G}, \tag{7}$$

where $C_2 = 2|I|(\theta_V - 1)\|f\|_{\mathbf{L}^{G^\star}} > 0$. From ($\mathcal{J}$) and ($\mathcal{J}'$) we get

$$\begin{aligned}\theta_V \mathcal{J}(u) - \mathcal{J}'(u)u = &\int_I \theta_V F(t,u,\dot{u}) - \langle F_x(t,u,\dot{u}), u\rangle - \langle F_v(t,u,\dot{u}), \dot{u}\rangle \, dt + \\ &+ \int_I \theta_V V(t,u) - \langle \nabla V(t,u), u\rangle \, dt + (\theta_V - 1)\int_I \langle f(t), u\rangle \, dt.\end{aligned}$$

Using (5), (6) and 7 we obtain

$$C_1 \int_I G(\dot{u}) \, dt \le \theta_V |\mathcal{J}(u)| + \|\mathcal{J}'(u)\|\|u\|_{\mathbf{W}^1_0\mathbf{L}^G} + C_2\|\dot{u}\|_{\mathbf{L}^G} + |I|M.$$

Let $\{u_n\} \subset \mathbf{W}^1_0\mathbf{L}^G$ be a Palais-Smale sequence, i.e. $\{\mathcal{J}(u_n)\}$ is bounded and $\mathcal{J}'(u_n) \to 0$. If $\{u_n\}$ is not bounded, we may assume that $\|u_n\|_{\mathbf{W}^1_0\mathbf{L}^G} \to \infty$. Then dividing by $\|u_n\|_{\mathbf{W}^1_0\mathbf{L}^G} = \|\dot{u}_n\|_{\mathbf{L}^G}$ we get

$$\frac{C_1}{\|\dot{u}_n\|_{\mathbf{L}^G}} \int_I G(\dot{u}_n) \, dt \le \frac{\theta_V |\mathcal{J}(u_n)|}{\|\dot{u}_n\|_{\mathbf{L}^G}} + \|\mathcal{J}'(u_n)\| + C_2 + \frac{|I|M}{\|\dot{u}_n\|_{\mathbf{L}^G}}.$$

Letting $n \to \infty$, we obtain a contradiction with Lemma 2.3, thus $\{u_n\}$ is bounded.

Next we show that $\{u_n\}$ has a convergent subsequence. Passing to a subsequence if necessary, we may assume that $u_n \to u$ in $\mathbf{L}^\infty$, $\{\dot{u}_n\}$ bounded in $\mathbf{L}^G$, $\dot{u}_n \to \dot{u}$ a.e. and $u_n \to u$ a.e.

Since $\mathcal{J}'(u_n) \to 0$ and $\{u_n - u\}$ is bounded in $\mathbf{W}^1_0\mathbf{L}^G$, we conclude that

$$\lim_{n\to\infty} \big\langle \mathcal{J}'(u_n), u_n - u\big\rangle = 0,$$

from the other hand,

$$\lim_{n\to\infty} \int_I \langle \nabla V(t,u_n) + f(t), u_n - u\rangle \, dt = 0.$$

Thus, by ($\mathcal{J}'$) we have that

$$\lim_{n\to\infty} \int_I \langle F_x(t,u_n,\dot{u}_n), u_n - u\rangle \, dt + \int_I \langle F_v(t,u_n,\dot{u}_n), \dot{u}_n - \dot{u}\rangle \, dt = 0$$

Define nondecreasing function $\alpha(s) = \sup_{\tau\in[0,s]} a(\tau)$. Since $\{u_n\}$ is bounded in $\mathbf{W}^1_0\mathbf{L}^G$, there exists $C_3 > 0$ such that

$$a(|u_n(t)|) \le \alpha(\|u_n\|_{\mathbf{L}^\infty}) \le C_3$$

and there exists $C_4 > 0$ such that

$$\int_I G(\dot{u}_n)\,dt \leq C_4.$$

It follows from (3) and the above that $\|F_x(\cdot, u_n, \dot{u}_n)\|_{\mathbf{L}^1}$ is uniformly bounded. Since $u_n \to u$ in $\mathbf{L}^\infty$, we get

$$\left|\int_I \langle F_x(t, u_n, \dot{u}_n), u_n - u\rangle\;dt\right| \leq \|F_x(\cdot, u_n, \dot{u}_n)\|_{\mathbf{L}^1}\|u_n - u\|_{\mathbf{L}^\infty} \to 0.$$

and consequently

$$\lim_{n\to\infty}\int_I \langle F_v(t, u_n, \dot{u}_n), \dot{u}_n - \dot{u}\rangle\;dt = 0. \tag{8}$$

By continuity of $F$ we have that $F(t, u_n(t), \pm\dot{u}(t)) \to F(t, u(t), \pm\dot{u}(t))$ a.e. From (2) and $\dot{u} \in \mathbf{L}^G$ we get

$$|F(t, u_n(t), \pm\dot{u}(t))| \leq C_3(b(t) + G(\pm\dot{u}(t)) \in \mathbf{L}^1\,.$$

Hence

$$\lim_{n\to\infty}\int_I F(t, u_n, \pm\dot{u})\,dt = \int_I F(t, u, \pm\dot{u})\,dt. \tag{9}$$

From the other hand, convexity of $F(t, x, \cdot)$, (8) and (9) yields

$$\limsup_{n\to\infty}\int_I F(t, u_n, \dot{u}_n)\,dt \leq \lim_{n\to\infty}\int_I F(t, u_n, \dot{u}) + \langle F_v(t, u_n, \dot{u}_n), \dot{u}_n - \dot{u}\rangle\;dt = \int_I F(t, u, \dot{u})\,dt.$$

Since $F(t, u_n(t), \dot{u}_n(t)) \geq 0$ and $F(t, u_n(t), \dot{u}_n(t)) \to F(t, u(t), \dot{u}(t))$ a.e., we have

$$\int_I F(t, u, \dot{u})\,dt \leq \liminf_{n\to\infty}\int_I F(t, u_n, \dot{u}_n)\,dt$$

by Fatou's Theorem. Finally,

$$\lim_{n\to\infty}\int_I F(t, u_n, \dot{u}_n)\,dt = \int_I F(t, u, \dot{u})\,dt. \tag{10}$$

Now we are in position to show that $\dot{u}_n \to \dot{u}$ in $\mathbf{L}^G$. The following is a modification of [13, Lemma 3.16]. Convexity of $F(t, x, \cdot)$ yields

$$\frac{F(t, u_n(t), \dot{u}_n(t)) + F(t, u_n(t), -\dot{u}(t))}{2} - F\left(t, u_n(t), \frac{\dot{u}_n(t) - \dot{u}(t)}{2}\right) \geq 0.$$

By continuity of $F$, $\dot{u}_n \to \dot{u}$ a.e. and ($F_5$) we obtain

$$\lim_{n\to\infty}\frac{F(t, u_n(t), \dot{u}_n(t)) + F(t, u_n(t), -\dot{u}(t))}{2} - F\left(t, u_n(t), \frac{\dot{u}_n(t) - \dot{u}(t)}{2}\right) = \\ = \frac{F(t, u(t), \dot{u}(t)) + F(t, u(t), -\dot{u}(t))}{2} \text{ a.e.}$$

Thus, by Fatou's Lemma,

$$\int_I \frac{F(t, u, \dot{u}) + F(t, u, -\dot{u})}{2}\,dt \leq \liminf_{n\to\infty}\int_I \frac{F(t, u_n, \dot{u}_n) + F(t, u_n, -\dot{u})}{2} - F\left(t, u_n, \frac{\dot{u}_n - \dot{u}}{2}\right)\,dt$$

Taking into account (9) and (10) we have

$$\lim_{n\to\infty}\int_I \frac{F(t,u_n,\dot{u}_n)+F(t,u_n,-\dot{u})}{2} = \int_I \frac{F(t,u,\dot{u})+F(t,u,-\dot{u})}{2}\,dt$$

and consequently

$$\int_I \frac{F(t,u,\dot{u})+F(t,u,-\dot{u})}{2}\,dt \leq \int_I \frac{F(t,u,\dot{u})+F(t,u,-\dot{u})}{2}\,dt - \limsup \int_I F\left(t,u_n,\frac{\dot{u}_n-\dot{u}}{2}\right)\,dt$$

It follows that

$$\lim_{n\to\infty}\int_I F\left(t,u_n,\frac{\dot{u}_n-\dot{u}}{2}\right)\,dt = 0$$

From ellipticity condition ($F_4$) we get

$$\lim_{n\to\infty}\int_I G\left(\frac{\dot{u}_n-\dot{u}}{2}\right)\,dt \leq \lim_{n\to\infty}\frac{1}{\Lambda}\int_I F\left(t,u_n,\frac{\dot{u}_n-\dot{u}}{2}\right)\,dt = 0$$

Thus $\dot{u}_n \to \dot{u}$ in $\mathbf{L}^G$ by [13, Theorem 3.13].

□

4.2. **Mountain Pass geometry.** Now we show that $\mathcal{J}$ has a mountain pass geometry. It follows immediately from ($\mathcal{J}$), ($F_5$) and ($V_2$) that

**Lemma 4.3.** $\mathcal{J}(0) = 0$

We next prove that $\mathcal{J}$ is negative at some point outside $B_\rho(0)$, where $\rho = \frac{\rho_0}{C_{\infty,G}}$.

**Lemma 4.4.** *There exists $e \in \mathbf{W}_0^1\mathbf{L}^G$ such that $\|e\|_{\mathbf{W}^1\mathbf{L}^G} > \rho$ and $\mathcal{J}(e) < 0$.*

*Proof.* Choose $u_0 \in \mathbf{W}_0^1\mathbf{L}^G$ such that $|\{t \in I\colon |u_0(t)| \geq r_0\}| > 0$. Set $M = \sup\{|V(t,x)|\colon t\in I, |x|\leq r_0\}$. For any $\lambda > 1$ we have

$$\mathcal{J}(\lambda u_0) = \int_I F(t,\lambda u_0,\lambda\dot{u}_0)\,dt + \int_I V(t,\lambda u_0)\,dt + \int_I \langle f,\lambda u_0\rangle\,dt \leq$$
$$\leq \lambda^{\theta_F}\int_I F(t,u_0,\dot{u}_0)\,dt + \lambda^{\theta_V}\int_{\{|u_0(t)|\geq r_0\}} V(t,u_0)\,dt + M|I| + \lambda\int_I \langle f,u_0\rangle\,dt$$

by Lemma 3.3. Since $V(t,x)$ is negative for $|x|\geq r_0$ and $\theta_V > 1$, $\theta_F < \theta_V$,

$$\lim_{\lambda\to\infty}\mathcal{J}(\lambda u_0) = -\infty$$

Thus, choosing $\lambda_0$ large enough, we can set $e = \lambda_0 u_0$. □

**Lemma 4.5.** *Assume that either A or B holds. Then*

$$\inf_{\|u\|_{\mathbf{W}_0^1\mathbf{L}^G}=\rho}\mathcal{J}(u) > 0$$

*Proof.* Let $\|u\|_{\mathbf{W}_0^1\mathbf{L}^G} = \rho$. Then

$$|u(t)| \leq C_{\infty,G}\|u\|_{\mathbf{W}_0^1\mathbf{L}^G} = \rho_0, \text{ for all } t \in I.$$

Using ($F_4$), ($V_3$) and Hölder's inequality we have

$$\mathcal{J}(u) \geq \Lambda \int_I G(\dot{u})\,dt - \int_I g(t)\,dt - 2\|f\|_{\mathbf{L}^{G^\star}}\|u\|_{\mathbf{L}^G}.$$

Assume that (A) holds. Since $\rho \geq 1$, using Proposition 2.2 and (1), we have

$$\mathcal{J}(u) \geq \Lambda\rho - \int_I g(t)\,dt - 2|I|\,\|f\|_{\mathbf{L}^{G^\star}}\rho > 0$$

by assumption (A).

Assume that (B) holds. If $\rho > 1$ then by Proposition 2.1

$$\mathcal{J}(u) \geq \Lambda\rho^{p_G} - \int_I g(t)\,dt - 2|I|\,\|f\|_{\mathbf{L}^{G^\star}}\rho$$

Similarly, if $\rho \leq 1$ then

$$\mathcal{J}(u) \geq \Lambda\rho^{q_G} - \int_I g(t)\,dt - 2|I|\,\|f\|_{\mathbf{L}^{G^\star}}\rho$$

From (B) it follows that in both cases $\mathcal{J}(u) > 0$. □

## 5. Some examples

In this section we provide some examples illustrating our assumptions. Due to computational difficulties we restrict ourselves to isotropic case. First we note that the function $F$ need not be polynomial.

**Example 5.1.** *Let $G_1(v) = \frac{1}{2}|v|^2$ and set*

$$F_1(v) = 16|v|^2 + \cos(4|v|) - 1$$

*Then $F_1$ is $C^1$, convex and satisfies*

$$\begin{aligned} 16\,G_1(v) &\leq F_1(v) \leq 32\,(1 + G_1(v)), \\ G_1^*(F_{1,v}(v)) &\leq 1032\,(1 + G_1^*(\nabla G_1(v))), \\ \langle F_{1,v}(v), v\rangle &\leq 3\,F_1(v). \end{aligned}$$

Next example shows that $F$ can change its growth on intermediate sets.

**Example 5.2.** *Let $G_2(v) = \frac{1}{4}|v|^4$ and define*

$$F_2(v) = \begin{cases} |v|^4 & |v| \leq 1 \\ 2|v|^2 - 1 & 1 < |v| \leq 2 \\ |v|^4 - 6|v|^2 + 15 & |v| > 2 \end{cases}$$

*Then $F_2$ is $C^1$, convex and*

$$\begin{aligned} G_2(v) \le F_2(v) &\le 4\,(1+G_2(v)), \\ G_2^*(F_{2,v}(v)) &\le 7(1+G_2^*(\nabla G_2(v))), \\ \langle F_{2,v}(v), v\rangle &\le 6\,F_2(v). \end{aligned}$$

*Thus assumptions $(F_1)$–$(F_5)$ are satisfied.*

In the next example $F$ depends on all variables $(t, x, v)$.

**Example 5.3.** *Let $G_3(v) = \frac{1}{4}|v|^4$ and define*

$$F_3(t, x, v) = |v|^4(2 + |x|^{\frac{9}{2}} - \sin t)$$

*Then $F_3$ is $C^1$, convex and*

$$\begin{aligned} 4\,G_3(v) \le F_3(t,x,v) &\le 4\,a(|x|)\,(1+G_3(v)), \\ |F_{3,x}(t,x,v)| &\le 9\,a(|x|)\,(1+G_3(v)), \\ G_3^*(F_{3,v}(t,x,v)) &\le 7\,a(|x|)^{\frac{4}{3}}\,(1+G_3^*(\nabla G_3(v))), \\ \langle F_{3,x}(t,x,v), x\rangle + \langle F_{3,v}(t,x,v), v\rangle &\le 9\,F_3(t,x,v), \end{aligned}$$

*where $a(|x|) = F_3(\pi/2, x, 1)$. Thus assumptions $(F_1)$–$(F_5)$ are satisfied.*

Now we provide example with G-function not satisfying $\Delta_2$ globally.

**Example 5.4.** *Define*

$$G_4(v) = \begin{cases} e|v|^2\exp(1 - \frac{1}{|v|}) & |v| \le 1 \\ \frac{4}{9}|v|^4 - \frac{5}{3}|v|^3 + 4|v|^2 - \frac{16}{9}|v| & 1 \le |v| \le 4 \\ \frac{1}{4}|v|^4 & 4 \le |v| \end{cases}$$

*This function satisfies (G) but does not satisfies $\Delta_2$ near zero. If we set $F_4(v) = F_2(v)$ then*

$$\begin{aligned} \frac{1}{2}G_4(v) \le F_4(v) &\le 4(1+G_4(v)) \\ |F_{4,v}(v)| &\le 4(1+|\nabla G_4(v)|) \end{aligned}$$

*In this example it is not possible to give explicit formula for $G^*$. Nevertheless, $G_4^*$ is monotone in the sense that*

$$|v_1| \le |v_2| \implies G_4^*(v_1) \le G_4^*(v_2).$$

*Thus assumption $(F_2)$ is satisfied.*

Finally, we give an example of $F$ and $V$ satisfying assumptions given in theorem 3.2.

**Example 5.5.** *Set $G(v) = \frac{1}{4}|v|^4$, $F(v) = F_2(v)$ and let $|I| = 1$. Recall that in this case $\Lambda = 1$ and $\theta_F = 6$. Furthermore, since $A_G(t) = G(t)$, we get $A_G^{-1}(s) = \sqrt{2}|s|^{1/4}$ and*

$$C_{\infty,G} = \max\{1, |I|\}\sqrt{2}|I|^{-1/4} = \sqrt{2}$$

*Define*

$$V(x) = -\frac{x^8}{1000} + \frac{x^4}{10} - \frac{x}{10} + \cos(x) - 1$$

*Then setting $r_0 \geq 4$ we have $V(x) < 0$, for $|x| > r_0$, and $\theta_F < \theta_V = 8$.*

*Moreover, setting $\rho_0 = 7/4$ we have $V(x) > -\frac{1}{2}$. Hence inequality* (B) *holds for every $f$ satisfying $\|f\|_{\mathbf{L}^{G^\star}} < 0.7$.*

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