A Measurable Selector in Kadison's Carpenter's Theorem

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Abstract. We show the existence of a measurable selector in Carpenter's Theorem due to Kadison. This solves a problem posed by Jasper and the first author in an earlier work. As an application we obtain a characterization of all possible spectral functions of shift-invariant subspaces of $L^2(\mathbb{R}^d)$ and Carpenter's Theorem for type I_∞ von Neumann algebras.

1 Introduction

Kadison [30, 31] gave a complete characterization of the diagonals of orthogonal projections on a separable infinite-dimensional Hilbert space H.

Theorem 1.1 (Kadison) *Let $(d_i)_{i\in\mathbb{N}}$ be a sequence in $[0,1]$. Define*

$$a = \sum_{d_i \leq 1/2} d_i \quad and \quad b = \sum_{d_i > 1/2} (1 - d_i).$$

There exists a projection P with diagonal $(d_i)_{i\in\mathbb{N}}$ if and only if one of the following holds:

- *$a, b < \infty$ and $a - b \in \mathbb{Z}$,*
- *$a = \infty$ or $b = \infty$.*

Kadison [30, 31] referred to the necessity part of Theorem 1.1 as the Pythagorean Theorem and the sufficiency as Carpenter's Theorem. It has been studied by a number of authors [1, 5, 15, 16, 32]. Kadison's Theorem can be generalized to the setting of von Neumann algebras. A general version of the Schur–Horn problem asks for a characterization of possible diagonals of an operator based on its spectral data. In the setting of von Neumann algebras the notion of a diagonal of an operator is replaced by the conditional expectation onto a maximal abelian self-adjoint subalgebra (MASA). The general Schur–Horn problem can then be formulated in the following way; see [34].

Problem 1.2 Let T be an operator in a von Neumann algebra $\mathfrak{M}$, and let $\mathcal{A}$ be a MASA in $\mathfrak{M}$ with corresponding conditional expectation $E_{\mathcal{A}}\colon \mathfrak{M} \to \mathcal{A}$.

Characterize the elements of the set

(1.1) $$D_{\mathcal{A}}(T) := \big\{ E_{\mathcal{A}}(U^*TU) : U \text{ is a unitary in } \mathfrak{M} \big\}.$$

This research area was initiated by Arveson and Kadison [7, 31] who have asked for a characterization of $D_{\mathcal{A}}(T)$ when T is a projection, or more generally a self-adjoint operator, in a von Neumann factor of type II$_1$. Problem 1.2 was investigated by a number of authors [2–4, 10, 24] and settled for II$_1$ factors by Ravichandran [38]. The same problem when T is a normal operator was studied by Arveson [6], by Massey and Ravichandran [37], and by Kennedy and Skoufranis [34], who have shown an approximate Carpenter's Theorem for the closure of $D_{\mathcal{A}}(T)$ in von Neumann factors of type I$_\infty$, II, and III. In the setting of von Neumann factor I$_\infty$, when $\mathfrak{M} = \mathcal{B}(H)$, Problem 1.2 was studied by Kaftal, Loreaux, and Weiss [33, 35] when T is a positive compact operator and by Jasper and the first author when T has finite spectrum [17, 29].

The goal of this paper is to show the existence of a measurable selector in Kadison's Carpenter Theorem. This problem was explicitly posed by Jasper and the first author in [16], although it was motivated by the earlier work of Rzeszotnik and the first author on spectral functions of shift-invariant spaces [19, 20].

Definition 1.3 Let X be a measurable space and H be a separable Hilbert space. We say that $P \colon X \to \mathcal{B}(H)$ is a *measurable projection*, if

  (i) $P(x)$ is an orthogonal projection for all $x \in X$, and
  (ii) P is weakly operator measurable, *i.e.*,

$$x \longmapsto \langle P(x)u, v \rangle \quad \text{is measurable for all } u, v \in H.$$

Equivalently, all entries of $P(x)$ with respect to a fixed orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of H are measurable.

The *diagonal* of P is a sequence of measurable functions $(\langle P(x)e_i, e_i \rangle)_{i \in \mathbb{N}}$.

Our main result extends Kadison's Theorem 1.1 by characterizing diagonals of measurable projections.

Theorem 1.4 *Let X be a measurable space. Let $f_i \colon X \to [0,1]$, $i \in \mathbb{N}$, be measurable functions. Define functions $a, b \colon X \to [0, \infty]$ by*
(1.2)
$$a(x) = \sum_{i \in \mathbb{N}: f_i(x) \le \frac{1}{2}} f_i(x) \quad and \quad b(x) = \sum_{i \in \mathbb{N}: f_i(x) > \frac{1}{2}} (1 - f_i(x)) \quad for\ x \in X.$$

There exists a measurable projection $P \colon X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$ if and only if for every $x \in X$, we have either

  (i) $a(x) = \infty$ *or* $b(x) = \infty$, *or*
  (ii) $a(x), b(x) < \infty$ *and* $a(x) - b(x) \in \mathbb{Z}$.

The necessity part of Theorem 1.4 follows immediately from Theorem 1.1. Thus, the sufficiency is the difficult part of this theorem. Theorem 1.4 also

yields a characterization of spectral functions of shift-invariant subspaces of $L^2(\mathbb{R}^n)$, which were introduced and studied by Rzeszotnik and the first author [19, 20]. In fact, this was the main motivation for studying this problem. In addition, Theorem 1.4 solves Problem 1.2 for von Neumann algebras of type I_∞ when T is a projection.

The proof of Theorem 1.4 splits into two natural cases: the nonsummable case (i) and the summable case (ii), which are shown in Sections 2 and 3, respectively. The nonsummable case of Theorem 1.4 is based on an algorithmic technique for finding a projection with all diagonal entries in $[0, 1/2]$ except possibly one term. This technique was introduced by Jasper and the first author in [16, Section 4]. It is related to the spectral tetris construction of tight frames introduced by Casazza, Fickus, Mixon, Wang, and Zhou in [21]. The proof of case (i) also relies heavily on techniques of measurable permutations.

The proof of the summable case (ii) employs a measurable variant of the finite-dimensional Schur–Horn theorem which was shown by Benac, Massey, and Stojanoff in [8]. The key role is played by a decoupling procedure that splits a desired diagonal sequence into three parts, modifying at most one entry in each group. The resulting sequences correspond either to infinite-dimensional rank one projections or finite-dimensional projections. Then the measurable variant of the Schur–Horn theorem enables us to recover a projection with the original diagonal.

Throughout the paper, X denotes a measurable space. For a function $f: X \to \mathbb{R}$ and $a \in \mathbb{R}$, the set $\{f = a\}$ stands for $\{x \in X : f(x) = a\} = f^{-1}(a)$; similarly, we define $\{f < a\}, \{f \leq a\}$, etc.

2 Nonsummable Case

In this section, we prove the nonsummable case of Theorem 1.4.

Theorem 2.1 *Let $f_i: X \to [0, 1]$, $i \in \mathbb{N}$, be measurable functions. Define functions $a, b: X \to [0, \infty]$ by (1.2). If $a(x) = \infty$ or $b(x) = \infty$ for every $x \in X$, then there is a measurable projection $P: X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.*

We will use the following two elementary lemmas repeatedly.

Lemma 2.2 *Let $X_i \subset X$, $i \in \mathbb{N}$, be measurable, Z be a topological space, and let $g_i: X_i \to Z$ be measurable functions. If X_i, $i \in \mathbb{N}$, are pairwise disjoint, then the function $g = \bigcup g_i: \bigcup X_i \to Z$ is measurable.*

Lemma 2.3 *Let $f_i: X \to \mathbb{R}$, $i \in \mathbb{N}$, and $h: X \to \mathbb{N}$ be measurable. Define a function*

$$f_h: X \longrightarrow \mathbb{R}, \quad f_h(x) = f_{h(x)}(x), \ x \in X.$$

Then f_h is measurable.

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Proof For any $k \in \mathbb{N}$, the set $\{h = k\}$ is measurable, and on this set, the function $f_h = f_k$ is measurable. ∎

We first prove Theorem 2.1 in a very special case given by Theorem 2.8. This is precisely the setting of the algorithm in [16, Section 4]. To do this we need to introduce some auxiliary functions.

Definition 2.4 Let $f_i \colon X \to [0,1]$, $i \in \mathbb{N}$, be measurable functions and let $N \in \mathbb{N} \cup \{\infty\}$ be such that

$$(2.1) \qquad \sum_i f_i(x) = N \quad \text{for all } x \in X.$$

For any $n < N$, $n \in \mathbb{N}$, we define $K_n \colon X \to \mathbb{N}$ by

$$K_n(x) = \min\{i : S_i(x) \geq n\}, \qquad \text{where } S_i = \sum_{j=1}^{i} f_j.$$

To emphasize the dependence on $(f_i)_{i \in \mathbb{N}}$, we will use the notation $K_n^{(f_i)}$.

The assumption (2.4) guarantees that a function K_n is defined on the whole X when $n < N$. In the next two results, we will assume that the sequence of functions $(f_i)_{i \in \mathbb{N}}$ is as in Definition 2.4.

Lemma 2.5 *For any $n < N$, the function K_n is measurable.*

Proof For all $k \in \mathbb{N}$ the function S_k is measurable. Fix n and k. The set

$$\{K_n = k\} = \{S_k \geq n\} \cap \{S_{k-1} < n\}$$

is measurable. The function K_n is constant (equal to k) on $\{K_n = k\}$. By Lemma 2.2, we are done. ∎

The following is a corollary of Lemmas 2.3 and 2.5.

Corollary 2.6 *For any $n < N$, the functions*

$$f_{K_n} \qquad and \qquad \sum_{i=1}^{K_n} f_i \; = \; S_{K_n}$$

are measurable.

We will use the the following elementary lemma from [16].

Lemma 2.7 ([16, Lemma 4.1]) *Let d_1, d_2, $\sigma \in [0,1]$, $\max\{d_1, d_2\} \leq \sigma \leq d_1 + d_2$ and $2\sigma > d_1 + d_2$. The number*

$$a = \frac{\sigma(\sigma - d_2)}{2\sigma - d_1 - d_2}$$

satisfies a, $\sigma - a$, $d_1 - a$, $d_2 - \sigma + a \in [0,1]$ and

$$(2.2) \qquad a(d_1 - a) = (\sigma - a)(d_2 - \sigma + a).$$

Theorem 2.8 *Let $f_i\colon X \to [0,1]$, $i \in \mathbb{N}$, be measurable functions. Let $N \in \mathbb{N} \cup \{\infty\}$. Suppose that for all $x \in X$,*

(i) $\sum_i f_i(x) = N$,

(ii) $f_i(x) \in [0, \frac{1}{2}]$ *for $i > 1$, and*

(iii) $f_{K_n-1}(x) \geq f_{K_n}(x)$ *for all $1 \leq n < N$.*

Then there is a measurable projection $P\colon X \to \mathcal{B}(H)$ such that $P(x)$ has diagonal $(f_i(x))_{i \in \mathbb{N}}$ for all $x \in X$.

Proof The proof is a repetition of the algorithm in [16, Theorem 4.3]. Consequently, we only need to verify that this construction yields a measurable projection as in Definition 1.3.

Without loss of generality, we can assume that $f_1(x) < 1$ for all $x \in X$, and hence $K_1 \geq 2$. Indeed, assume momentarily that Theorem 2.8 is already shown under this assumption. Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of the space H. Let $H_1 = \mathrm{span}\{e_1\}$ and $H_2 = \overline{\mathrm{span}}\{e_i : i \geq 2\}$. Applying Theorem 2.8 to the sequence of functions $(f_i)_{i \geq 2}$ on the set $\{x \in X : f_1(x) = 1\}$ yields a measurable projection P_2 with values in $\mathcal{B}(H_2)$ and with diagonal $(f_i(x))_{i \geq 2}$. Then $P = P_1 \oplus P_2$, where P_1 is the identity on H_1, is a measurable projection on the set $\{x \in X : f_1(x) = 1\}$ with the required diagonal. Combining this with a measurable projection on the set $\{x \in X : f_1(x) < 1\}$ shows the reduction step by Lemma 2.2.

For any $n < N$, define a function $\sigma_n\colon X \to [0,1]$ by

$$\sigma_n(x) = n - S_{K_n-2}(x) = n - \sum_{i=1}^{K_n(x)-2} f_i(x).$$

Function σ_n is measurable as a consequence of Corollary 2.6. Note that

$$\sigma_n \leq f_{K_n-1} + f_{K_n}.$$

By the minimality of K_n we have

$$\sigma_n = n - S_{K_n-1} + f_{K_n-1} > f_{K_n-1}.$$

Hence, by (iii), we have

$$2\sigma_n > f_{K_n-1} + f_{K_n}.$$

Substituting values of $f_{K_n-1} = d_1$ and $f_{K_n} = d_2$ and σ_n as σ, Corollary 2.6 yields the measurability of a function $a_n\colon X \to [0,1]$ given by

$$a_n = \frac{\sigma_n(\sigma_n - f_{K_n})}{2\sigma_n - f_{K_n-1} - f_{K_n}}.$$

In particular, functions $\sigma_n - a_n$, $f_{K_n-1} - a_n$, $f_{K_n} - \sigma_n + a_n$ are measurable, and so are functions $(\sigma_n - a_n)^{1/2}$, $(f_{K_n-1} - a_n)^{1/2}$, $(f_{K_n} - \sigma_n + a_n)^{1/2}\colon X \to [0,1]$.

We define a sequence of vectors $(v_n)_{n<N}$ as follows. Let

$$v_1 = \sum_{i=1}^{K_1-2} (f_i)^{1/2} e_i + (a_1)^{1/2} e_{K_1-1} - (\sigma_1 - a_1)^{1/2} e_{K_1},$$

with the understanding that the sum is omitted when $K_1 = 2$. For $1 < n < N$, define

$$v_n = (f_{K_{n-1}-1} - a_{n-1})^{1/2} e_{K_{n-1}-1} + (f_{K_{n-1}} - \sigma_{n-1} + a_{n-1})^{1/2} e_{K_{n-1}}$$
$$+ \sum_{i=K_{n-1}+1}^{K_n-2} (f_i)^{1/2} e_i + (a_n)^{1/2} e_{K_n-1} - (\sigma_n - a_n)^{1/2} e_{K_n}.$$

The vectors v_n are well defined, since we have $K_{n-1} + 2 \le K_n$ for $n \ge 2$ due to assumption (ii). If $K_{n-1} + 2 = K_n$, then the last sum should be omitted. In the case when $N < \infty$, we also define the ultimate vector

$$v_N = (f_{K_{N-1}-1} - a_{N-1})^{1/2} e_{K_{N-1}-1} + (f_{K_{N-1}} - \sigma_{N-1} + a_{N-1})^{1/2} e_{K_{N-1}}$$
$$+ \sum_{i=K_{N-1}+1}^{\infty} (f_i)^{1/2} e_i.$$

A direct calculation using (2.2) shows that $(v_n)_{n=1}^N$ forms an orthonormal set of vectors in H. For details, we refer the reader to the proof of [16, Theorem 4.3]. Despite the fact that [16, Theorem 4.3] deals only with the case $\sum f_i = N = \infty$, a similar calculation yields the case $N < \infty$. Indeed, this follows by the fact that the support of the ultimate vector v_N overlaps only with the preceding vector v_{N-1}. Moreover,

$$\|v_N\|^2 = f_{K_{N-1}-1} - a_{N-1} + f_{K_{N-1}} - \sigma_{N-1} + a_{N-1} + \sum_{i=K_{N-1}+1}^{\infty} f_i$$
$$= \sum_{i=K_{N-1}-1}^{\infty} f_i - \left(N - 1 - \sum_{i=1}^{K_{N-1}-2} f_i \right) = \sum_{i=1}^{\infty} f_i - N + 1 = 1.$$

We can represent vectors $(v_n)_{n=1}^N$ as rows of an $N \times \infty$ matrix V:

$$\begin{bmatrix} \sqrt{f_1} \cdots & \sqrt{a_1} & -\sqrt{\sigma_1 - a_1} & & & \\ & \sqrt{f_{K_1-1} - a_1} & \sqrt{f_{K_1} - \sigma_1 + a_1} & \sqrt{f_{K_1+1}} \cdots & \sqrt{a_2} & -\sqrt{\sigma_2 - a_2} \\ & & & & \sqrt{f_{K_2-1} - a_2} & \sqrt{f_{K_2} - \sigma_2 + a_2} \cdots \\ & & & & & \ddots \end{bmatrix},$$

where the empty spaces are zeros. That is, row n of the matrix V represents coefficients of the vector v_n with respect to the basis $(e_i)_{i \in \mathbb{N}}$,

$$v_n = \sum_{k \in \mathbb{N}} V_{n,k} e_k.$$

Claim   *For any n and k, the function $V_{n,k} \colon X \to [0,1]$ is measurable.*

Observe that

$$
V_{n,k} = \begin{cases}
0 & k < K_{n-1} - 1, \\
\sqrt{f_{K_{n-1}-1} - a_{n-1}} & k = K_{n-1} - 1, \\
\sqrt{f_{K_{n-1}} - \sigma_{n-1} + a_{n-1}} & k = K_{n-1}, \\
\sqrt{f_k} & K_{n-1} < k < K_n - 1, \\
\sqrt{a_n} & k = K_n - 1, \\
\sqrt{\sigma_n - a_n} & k = K_n, \\
0 & k > K_n.
\end{cases}
$$

In the above, we set $K_0 = a_0 = \sigma_0 = 0$. If $N < \infty$, then we also set $K_N = \infty$; consequently, the last three cases are vacuous. The sets defined by the above cases, such as $\{K_{n-1} = k\}$, are measurable. Therefore, the measurability of K_i, f_i, a_i, σ_i together with Lemmas 2.2 and 2.3 yields the claim.

The sought measurable projection P is given by the formula

$$
Pv = \sum_{n=1}^{N} \langle v, v_n \rangle v_n, \qquad v \in H.
$$

A computation shows that diagonal of P is as desired, *i.e.*,

$$
\langle Pe_i, e_i \rangle = \|Pe_i\|^2 = \sum_{n \in \mathbb{N}} |\langle e_i, v_n \rangle|^2 = f_i.
$$

Hence, it remains to show that the mapping $x \mapsto P(x) \in \mathcal{B}(H)$ is weakly measurable. For any $k \in \mathbb{N}$,

$$
Pe_k = \sum_{n=1}^{N} \langle e_k, v_n \rangle v_n = \sum_{n=1}^{N} V_{n,k} v_n = \sum_{n=1}^{\min(2+k/2, N)} V_{n,k} v_n.
$$

The limitation in the last sum follows from the fact that $V_{n,k} = 0$ for $k < K_{n-1} - 1$ and the fact that $K_n \geq 2n - 1$ for all $n < N$. Since a product of a measurable vector-valued function by a measurable scalar-valued function is measurable, the above claim implies that the mapping $x \mapsto P(x)e_k$ is measurable as well. Since $k \in \mathbb{N}$ is arbitrary, P is a measurable projection. ∎

In order to weaken the assumptions of Theorem 2.8, we need to introduce the concept of a measurable permutation.

Definition 2.9 We say that $\pi \colon X \times \mathbb{N} \to \mathbb{N}$ is a *measurable permutation*, if for all $n \in \mathbb{N}$, $\pi(\,\cdot\,, n)$ is measurable and for all $x \in X$, $\pi(x, \,\cdot\,)$ is a permutation of $\mathbb{N}$. The inverse permutation $\pi^{-1} \colon X \times \mathbb{N} \to \mathbb{N}$ is given by

$$
\pi^{-1}(x, n) = \big(\pi(x, \,\cdot\,) \big)^{-1}(n), \qquad (x, n) \in X \times \mathbb{N}.
$$

Note that if π is measurable, then so is the inverse permutation. Indeed, for any $k \in \mathbb{N}$,

$$
\big\{ (x, n) : \pi^{-1}(x, n) = k \big\} = \big\{ (x, n) : \pi(x, k) = n \big\}
$$

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is a measurable set. For any $n \in \mathbb{N}$, we let $\pi(n) \colon X \to \mathbb{N}$ denote a function given by $\pi(n)(x) = \pi(x, n)$, $x \in X$.

We need the following two useful lemmas.

Lemma 2.10 *Let $\pi \colon X \times \mathbb{N} \to \mathbb{N}$ be a measurable permutation. Suppose that $P \colon X \to \mathcal{B}(H)$ is a measurable projection with diagonal $(f_i)_{i \in \mathbb{N}}$. Then there exists a measurable projection $\widetilde{P} \colon X \to \mathcal{B}(H)$ with diagonal $(f_{\pi(i)})_{i \in \mathbb{N}}$.*

Proof A measurable permutation π defines a permutation mapping $U \colon X \to \blacksquare$ $\mathcal{B}(H)$ given by

$$U(x)e_i = e_{\pi(x,i)} \qquad i \in \mathbb{N}, x \in X.$$

It is immediate that $x \mapsto U(x)$ is weakly measurable and $U(x)$ is a unitary operator. Define $\widetilde{P} \colon X \to \mathcal{B}(H)$ by

$$\widetilde{P}(x) = (U(x))^* \circ P(x) \circ U(x) \qquad \text{for } x \in X.$$

A straightforward argument shows that $\widetilde{P}$ is a measurable projection with diagonal $(f_{\pi(i)})_{i \in \mathbb{N}}$. Indeed, for any $(x, i) \in X \times \mathbb{N}$,

$$\left\langle \widetilde{P}(x)e_i, e_i \right\rangle = \left\langle P(x)U(x)e_i, U(x)e_i \right\rangle = \left\langle P(x)e_{\pi(x,i)}, e_{\pi(x,i)} \right\rangle = f_{\pi(i)}(x).$$

This proves the lemma. ∎

Lemma 2.11 *Let $f_i \colon X \to [0,1]$, $i \in [k] := \{1, \ldots, k\}$, be a finite sequence of measurable functions. Then there exists a measurable permutation $\pi \colon X \times [k] \to [k]$ such that functions $g_i(x) = f_{\pi(x,i)}(x)$, $i \in [k]$, are measurable and in (weakly) decreasing order.*

Proof Let $a_1, \ldots, \breve{a}_i, \ldots$ denote the sequence $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots$ with a_i omitted. Define functions $g_i \colon X \to [0,1]$ by

$$g_1 = \max(f_1, f_2, \ldots, f_k),$$
$$g_2 = \min_{i \le k} \max(f_1, \ldots, \breve{f}_i, \ldots, f_k),$$
$$g_3 = \min_{i < j \le k} \max(f_1, \ldots, \breve{f}_i, \ldots, \breve{f}_j, \ldots, f_k),$$

and so on. For a fixed $x \in X$, define

$$\pi(1) = \min\{i : f_i = g_1\},$$
$$\pi(2) = \min\{i : f_i = g_2 \text{ and } i \ne \pi(1)\},$$
$$\pi(3) = \min\{i : f_i = g_3 \text{ and } i \ne \pi(1), \pi(2)\},$$

and so on. Then one can show by induction that functions g_i and $\pi(\cdot, i)$ are measurable for every $i = 1, \ldots, k$. Finally, the identity $g_i(x) = f_{\pi(x,i)}(x)$ follows by the above definition. ∎

We now weaken the assumptions used in Theorem 2.8.

Theorem 2.12 *Suppose measurable functions $f_i \colon X \to [0,1]$, $i \in \mathbb{N}$, satisfy for all $x \in X$,*

(i) $\sum_i f_i(x) = \infty$ *or* $\sum_i f_i(x) \in \mathbb{N}$, *and*

(ii) $f_i(x) \in [0, \frac{1}{2}]$ *for all* $i > 1$.

Then there is a measurable projection $P \colon X \to \mathcal{B}(H)$ *with diagonal* $(f_i)_{i \in \mathbb{N}}$.

Proof Without loss of generality, we can assume that there exists $N \in \mathbb{N} \cup \{\infty\}$ such that (2.1) holds. Indeed, we can split a measurable space X into countable collection of sets on which $\sum_i f_i$ is constant and then apply Lemma 2.2.

The idea of the proof is to find a measurable permutation $\pi \colon X \times \mathbb{N} \to \mathbb{N}$ if $N = \infty$, or $\pi \colon X \times [N] \to [N]$ if $N < \infty$, such that the permuted sequence $g_i = f_{\pi(i)}$ satisfies

$$(2.3) \qquad g_{K_n^{(g_i)}-1} \geq g_{K_n^{(g_i)}} \qquad \text{for all } 1 \leq n < N.$$

That is, $(g_i)_{i \in \mathbb{N}}$ satisfies Theorem 2.8(iii). For simplicity, we assume that $N = \infty$; the case $N < \infty$ follows by obvious modifications.

To achieve this we merely follow the proof in [16, Lemma 4.2]. For every $x \in X$, we divide the sequence f_i into blocks corresponding to intervals defined by K_n given by

$$(2.4) \qquad I_n = I_n(x) = \{i \in \mathbb{N} : K_{n-1} < i \leq K_n\}, \qquad n \in \mathbb{N},$$

with the convention that $K_0 = 0$. On every such interval, we order $(f_i)_{i \in I_n}$ in a decreasing order using Lemma 2.11. More precisely, for every $n \in \mathbb{N}$, there are countably many choices for an interval I_n. For $a < b \in \mathbb{N}$, we define a measurable set

$$X_{(n,a,b)} = \{x \in X : I_n(x) = [a, b]\}.$$

We apply Lemma 2.11 to $(f_i)_{i \in [a,b]}$ restricted to $X_{(n,a,b)}$ to get a local measurable permutation of I_n. We combine these permutations into one global permutation $\pi \colon X \times \mathbb{N} \to \mathbb{N}$ that sorts every block of functions $(f_i)_{i \in I_n}$ in a decreasing order. By Lemma 2.2, π is a measurable permutation. Indeed, for a fixed $i \in \mathbb{N}$, the measurable space X splits into countably many subsets $\{x \in X : i \in I_n(x) = [a, b]\}$ indexed by triplets $(n, a, b) \in \mathbb{N}^3$. Clearly, $\pi(\,\cdot\,, i)$ is measurable on each such subset and hence on X.

Define $g_i = f_{\pi(i)}$ for $i \in \mathbb{N}$. By Lemma 2.3, functions g_i are measurable. We claim that the sequence (g_i) satisfies (2.3). To see this, we must consider functions K_n corresponding to (g_i). The values of K_n on the sequence (g_i) cannot differ much from analogous values on (f_i). More precisely, we have

$$(2.5) \qquad K_{n-1}^{(f_i)} + 2 \leq K_n^{(g_i)} \leq K_n^{(f_i)}, \qquad n \geq 2.$$

Indeed, by the minimality of $K_{n-1}^{(f_i)}$, we have for $n \geq 2$,

$$\sum_{i=1}^{K_{n-1}^{(f_i)}+1} g_i = \sum_{i=1}^{K_{n-1}^{(f_i)}} f_i + g_{K_{n-1}^{(f_i)}+1} < (n - 1/2) + 1/2 = n.$$

Thus, $K_n^{(g_i)} > K_{n-1}^{(f_i)} + 1$. On the other hand, we have for $n \geq 1$,

$$\sum_{i=1}^{K_n^{(f_i)}} g_i = \sum_{i=1}^{K_n^{(f_i)}} f_i \geq n.$$

This yields $K_n^{(g_i)} \leq K_n^{(f_i)}$ and completes the proof of (2.5). Since $(g_i)_{i \in I_n}$ is in decreasing order, (2.4) and (2.5) yield (2.3).

By Theorem 2.8 there exists a measurable projection $\widetilde{P}$ with diagonal $(g_i)_{i \in \mathbb{N}} = (f_{\pi(i)})_{i \in \mathbb{N}}$. Applying Lemma 2.10 for the inverse permutation π^{-1} yields a measurable projection P with diagonal $(g_{\pi^{-1}(i)})_{i \in \mathbb{N}} = (f_i)_{i \in \mathbb{N}}$. ∎

Our goal now is to prove Theorem 2.12 without assumption (ii). To this end, we need some auxiliary functions.

Definition 2.13 For a sequence of measurable functions $f_i \colon X \to [0,1]$, $i \in \mathbb{N}$, we define functions $\mathrm{Pos}(n) \colon X \to \mathbb{N}$ and $\mathrm{pos}(n) \colon X \to \mathbb{N}$ as follows. Let $\mathrm{Pos}(n)(x) = k$ if $f_k(x)$ is the n-th number in the sequence $(f_i(x))$ that is greater than $\frac{1}{2}$. Likewise, we let $\mathrm{pos}(n)(x) = k$ if $f_k(x)$ is the n-th number in the sequence $(f_i(x))$ that is at most $\frac{1}{2}$.

In order for $\mathrm{Pos}(n)$ and $\mathrm{pos}(n)$ to be defined on the whole X, we must assume that there are at least n indices $i \in \mathbb{N}$ such that $f_i(x) > \frac{1}{2}$ and $f_i(x) \leq \frac{1}{2}$, respectively.

Lemma 2.14 *Functions $\mathrm{Pos}(n)$ and $\mathrm{pos}(n)$ are measurable for all n.*

Proof Fix n and k. Sets $\{\mathrm{Pos}(n) = k\}$ and $\{\mathrm{pos}(n) = k\}$ are measurable, since

$$\{\mathrm{Pos}(n) = k\} = \bigcup_{1 \leq k_1 < \cdots < k_n = k} \bigcap_{i=1}^{k} \left(\{f_{k_i} > \tfrac{1}{2}\} \cap \bigcap_{j \neq k_i, j < k} \{f_j \leq \tfrac{1}{2}\} \right),$$

$$\{\mathrm{pos}(n) = k\} = \bigcup_{1 \leq k_1 < \cdots < k_n = k} \bigcap_{i=1}^{k} \left(\{f_{k_i} \leq \tfrac{1}{2}\} \cap \bigcap_{j \neq k_i, j < k} \{f_j > \tfrac{1}{2}\} \right). \quad \blacksquare$$

As a corollary of Lemmas 2.3 and 2.14, we obtain that $f_{\mathrm{Pos}(n)}$ and $f_{\mathrm{pos}(n)}$ are measurable functions. The following result is the next step toward proving Theorem 2.1.

Theorem 2.15 *Suppose that measurable functions $f_i \colon X \to [0,1]$, $i \in \mathbb{N}$, satisfy that for all $x \in X$,*

(i) *there are infinitely many $f_i(x) \leq \frac{1}{2}$;*
(ii) *there are infinitely many $f_i(x) > \frac{1}{2}$;*
(iii) *$\sum_i f_{\mathrm{pos}(i)}(x) = \infty$.*

Then there is a measurable projection $P \colon X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.

Proof To simplify the notation, we let $f^i = f_{\text{pos}(i)}$. By Lemma 2.14, $f^i\colon X \to [0, 1/2]$, $i \in \mathbb{N}$ is a sequence of measurable functions. Let $2\mathbb{N} - 1$ denote the set of odd numbers. We will use a partition of $\mathbb{N}$ into sets A^m, given by

$$A^m = \{2^{k-1}m : k \in \mathbb{N}\}, \qquad m \in 2\mathbb{N} - 1.$$

Next, we decompose H as an orthogonal sum of subspaces

$$H^m = \overline{\text{span}}\{e_i : i \in A^m\}, \qquad m \in 2\mathbb{N} - 1.$$

We also split the sequence $(f_i)_{i \in \mathbb{N}}$ into countably many subsequences $(g_i^m)_{i \in \mathbb{N}}$, where m is odd, by the following procedure.

Let K_n, $n \in \mathbb{N}$, be the function as in Definition 2.4 corresponding to the sequence $(f^i)_{i \in \mathbb{N}}$ rather than $(f_i)_{i \in \mathbb{N}}$. Let $I_n = I_n(x)$, $n \in \mathbb{N}$, be the interval of $\mathbb{N}$ defined by (2.4). Recall that intervals I_n, $n \in \mathbb{N}$, form a partition of $\mathbb{N}$, which depends on a choice of $x \in X$. For $i \in \mathbb{N}$, define a function $n_i\colon X \to \mathbb{N}$ by

$$n_i = \max\{n \in \mathbb{N} : K_{n-1} < i\}.$$

Equivalently, $n_i \in \mathbb{N}$ is the unique number such that $i \in I_{n_i}$. Each function n_i is measurable, since

$$\{n_i = n\} = \{K_{n-1} < i\} \cap \{K_n \geq i\}.$$

Define a mapping $\pi\colon X \times (2\mathbb{N} - 1) \times \mathbb{N} \to \mathbb{N}$ such that $\pi(x, m, i)$ is the i-th element of the (infinite) set

$$(2.6) \qquad \bigcup_{a \in A^m} I_{n_a} = \bigcup_{k \in \mathbb{N}} I_{n_{2^{k-1}m}}.$$

Claim *For every* $(m, i) \in (2\mathbb{N} - 1) \times \mathbb{N}$, *the mapping* $\pi(\,\cdot\,, m, i)$ *is measurable.*

Since each interval I_n is non-empty, the i-th element of the set (2.6) belongs to one of the intervals $I_{n_{m2^{k-1}}}$, where $1 \leq k \leq i$. Observe that there are only countably many choices for such intervals. Hence, we can split X into a countable collection of measurable subsets of the form

$$\big\{x \in X : I_{n_m}(x) = [a_1, b_1],\ I_{n_{2m}}(x) = [a_2, b_2], \ldots, I_{n_{2^{i-1}m}}(x) = [a_i, b_i]\big\},$$

for some choice of $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_i \leq b_i \in \mathbb{N}$. Consequently, the function $\pi(m, i) := \pi(\,\cdot\,, m, i)$ takes a constant value on such measurable subsets. By Lemma 2.2, $\pi(m, i)$ is measurable on X.

Consequently, $\pi\colon X \times (2\mathbb{N} - 1) \times \mathbb{N} \to \mathbb{N}$ is a measurable permutation after identifying $(2\mathbb{N} - 1) \times \mathbb{N}$ with $\mathbb{N}$ using the mapping $(i, m) \mapsto 2^{i-1}m$. For m odd, define sequence $(g_i^m)_{i \in \mathbb{N}}$ as

$$(2.7) \qquad g_i^m = \begin{cases} f_{\text{Pos}(\frac{m+1}{2})} & i = 1, \\ f^{\pi(m, i-1)} & i > 1. \end{cases}$$

In particular,

$$(g_i^1) = (f_{\mathrm{Pos}(1)}, f^1, \ldots, f^{K_2}, f^{K_3+1}, \ldots, f^{K_4}, f^{K_7+1}, \ldots, f^{K_8}, \ldots),$$
$$(g_i^3) = (f_{\mathrm{Pos}(2)}, f^{K_2+1}, \ldots, f^{K_3}, f^{K_5+1}, \ldots, f^{K_6}, f^{K_{11}+1}, \ldots),$$
$$(g_i^5) = (f_{\mathrm{Pos}(3)}, f^{K_4+1}, \ldots, f^{K_5}, f^{K_9+1}, \ldots, f^{K_{10}}, f^{K_{19}+1}, \ldots),$$

and so on. By Lemma 2.3 and the above claim, we deduce that all functions g_i^m are measurable. Since

$$\sum_{i=K_{n-1}+1}^{K_n} f^i \geq \frac{1}{2} \qquad \text{for } n \in \mathbb{N},$$

we have that for every odd m, $\sum_{i=1}^{\infty} g_i^m = \infty$ and $g_i^m \leq \frac{1}{2}$ for $i > 1$. Hence, by Theorem 2.12 there is a measurable projection Q^m on the space H^m with diagonal $(g_i^m)_{i \in \mathbb{N}}$. Therefore,

$$Q = \bigoplus_{m \in 2\mathbb{N}-1} Q^m \in \mathcal{B}\Big(\bigoplus_{m \in 2\mathbb{N}-1} H^m \Big) = \mathcal{B}(H)$$

is a projection with diagonal $(g_i^m)_{i \in \mathbb{N}}^{m \in 2\mathbb{N}-1}$.

We want, however, a projection with diagonal $(f_i)_{i \in \mathbb{N}}$. This is a consequence of the fact that the sequence $(g_i^m)_{i \in \mathbb{N}}^{m \in 2\mathbb{N}-1}$ is obtained from $(f_i)_{i \in \mathbb{N}}$ using a measurable permutation. More precisely, let $\widetilde{\pi} \colon X \times \mathbb{N} \to \mathbb{N}$ be a measurable permutation defined for $x \in X$ by

$$\widetilde{\pi}(x, n) = \begin{cases} \mathrm{Pos}(\frac{n+1}{2})(x), & n \in 2\mathbb{N}-1, \\ \mathrm{pos}(\pi(x, m, i-1))(x), & n = 2^{i-1}m, \ i > 1, \ m \in 2\mathbb{N}-1. \end{cases}$$

Indeed, one can check that for every $x \in X$, $\pi(x, \cdot)$ is a permutation of $\mathbb{N}$. The measurability follows from the claim, Lemma 2.3, and Lemma 2.14. Furthermore, by (2.7), we have $g_i^m = f_{\widetilde{\pi}(2^{i-1}m)}$. Therefore, Lemma 2.10 yields a measurable permutation with diagonal $(f_i)_{i \in \mathbb{N}}$. ∎

We also need the following slight variant of Theorem 2.15.

Theorem 2.16 *Let $k \in \mathbb{N}$. Suppose that measurable functions $f_i \colon X \to [0, 1]$, $i \in \mathbb{N}$, satisfy the following for all $x \in X$:*

  (i) *there are infinitely many $f_i(x) \leq \frac{1}{2}$;*
  (ii) *there are exactly k many $f_i(x) > \frac{1}{2}$;*
 (iii) *$\sum_i f_{\mathrm{pos}(i)}(x) = \infty$.*

Then there is a measurable projection $P \colon X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.

Proof The proof is a simple adaption of the proof of Theorem 2.15 to the finite case. The main difference is that we split the sequence $(f_i)_{i \in \mathbb{N}}$ into k finitely many subsequences $(g_i^m)_{i \in \mathbb{N}}$, where m is the remainder of division

by k. Hence,

$$(g_i^1) = (f_{\text{Pos}(1)}, f_{\text{pos}(1)}, f_{\text{pos}(k+1)}, f_{\text{pos}(2k+1)}, f_{\text{pos}(3k+1)}, \dots),$$
$$(g_i^2) = (f_{\text{Pos}(2)}, f_{\text{pos}(2)}, f_{\text{pos}(k+2)}, f_{\text{pos}(2k+2)}, f_{\text{pos}(3k+2)}, \dots),$$
$$\vdots$$
$$(g_i^k) = (f_{\text{Pos}(k)}, f_{\text{pos}(k)}, f_{\text{pos}(2k)}, f_{\text{pos}(3k)}, f_{\text{pos}(4k)}, \dots).$$

Similarly we decompose H as an orthogonal sum of k subspace $H^1, \dots, H^k$. Then Theorem 2.12 yields a measurable projection with diagonal $(g_i^m)_{i \in \mathbb{N}}$ for every $m = 1, \dots, k$. Then an application of Lemma 2.10 yields the required measurable projection. We leave the details to the reader. ∎

Combining Theorems 2.15 and 2.16 yields the following corollary.

Corollary 2.17 *Suppose that measurable functions $f_i \colon X \to [0,1]$, $i \in \mathbb{N}$, satisfy the following for all $x \in X$:*

(i) *there are infinitely many $f_i(x) \leq \frac{1}{2}$;*
(ii) *$\sum_i f_{\text{pos}(i)}(x) = \infty$.*

Then there is a measurable projection $P \colon X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.

Proof For fixed $k \in \mathbb{N} \cup \{0\}$, the set X_k of $x \in X$ such that the set $\{i \in \mathbb{N} : f_i(x) > \frac{1}{2}\}$ has k elements is measurable. Indeed,

$$X_k = \bigcup_{1 \leq i_1 < \cdots < i_k} \bigcap_{j=1}^{k} \left(\{f_{i_j} > \tfrac{1}{2}\} \cap \bigcap_{n \neq i_j} \{f_n \leq \tfrac{1}{2}\} \right).$$

In particular, the set $X_\infty = \{x \in X : \text{there are infinitely many } f_i(x) > \frac{1}{2}\}$ is measurable as $X_\infty = X \setminus \bigcup_k X_k$.

By Theorems 2.12, 2.15, and 2.16 there exists measurable projections P_0, P_k, $k \in \mathbb{N}$, and P_∞ defined on sets X_0, X_k, and X_∞, respectively. A projection $P = \bigcup_{k \in \mathbb{N} \cup \{0\}} P_k \cup P_\infty$ is defined on the entire X and is measurable by Lemma 2.2. ∎

We are now ready to complete the proof of Theorem 2.1.

Proof Let $f^i = f_{\text{pos}(i)}$ (see Definition 2.13). Let a and b be the functions as in (1.2). Let

$$S_\infty = \left\{ x \in X : \sum_i f^i(x) = \infty \right\} \quad \text{and} \quad S_{<\infty} = \left\{ x \in X : \sum_i f^i(x) < \infty \right\}.$$

The sets S_∞ and $S_{<\infty}$ are measurable. Applying Corollary 2.17 on the set

$$S_\infty = \{x \in X : a(x) = \infty\}$$

yields a measurable projection $P \colon S_\infty \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$. we must have $b(x) = \infty$ on the set $\{x \in X : a(x) < \infty\} = S_{<\infty}$. By the previous case, there is a measurable projection P' on $S_{<\infty}$ with diagonal $(1 - f_i)_{i \in \mathbb{N}}$. Hence, $P = \mathbf{I} - P'$ is a measurable projection on $S_{<\infty}$ with diagonal $(f_i)_{i \in \mathbb{N}}$. Applying Lemma 2.2 for $X = S_\infty \cup S_{<\infty}$ yields the desired measurable projection. ∎

3 Summable Case

The aim of this section is to prove the summable counterpart of Theorem 2.1.

Theorem 3.1 *Let $f_i\colon X \to [0,1]$, $i \in \mathbb{N}$, be measurable functions. Define functions $a, b\colon X \to [0, \infty]$ by (1.2). Assume for all $x \in X$, $a(x), b(x) < \infty$ and $a(x) - b(x) \in \mathbb{Z}$. Then there is a measurable projection $P\colon X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.*

We start from the easiest rank one case.

Lemma 3.2 *Assume that $\sum_i f_i(x) = 1$ for all $x \in X$. Then there is a measurable projection $P\colon X \to \mathcal{B}(H)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.*

Proof Fix an orthonormal basis (e_i) of H and define a vector-valued function $v_0 = \sum_i \sqrt{f_i} \cdot e_i$. A projection P onto a one-dimensional space given by $Pv = \langle v_0, v \rangle v_0$ is the desired projection. Indeed, $Pe_i = \langle v_0, e_i \rangle v_0 = \langle \sum_j \sqrt{f_j} e_j, e_i \rangle v_0 = \sqrt{f_i} \cdot v_0$, and the entries of P are

$$\langle Pe_j, e_i \rangle = \langle \sqrt{f_j} \cdot v_0, e_i \rangle = \sqrt{f_j} \langle v_0, e_i \rangle = \sqrt{f_j} \cdot \sqrt{f_i}. \qquad \blacksquare$$

We will need a measurable variant of the Schur–Horn theorem [28, 41]. Suppose that $(f_1, \ldots, f_n)$ and $(\lambda_1, \ldots, \lambda_n)$ are sequences in $\mathbb{R}^n$. Let $(\lambda_i^\downarrow)_{i=1}^n$ and $(f_i^\downarrow)_{i=1}^n$ be their decreasing rearrangements. Following [36] we introduce a majorization order: $(f_1, \ldots, f_n) \preceq (\lambda_1, \ldots, \lambda_n)$ if and only if

$$\sum_{i=1}^n f_i^\downarrow = \sum_{i=1}^n \lambda_i^\downarrow \quad \text{and} \quad \sum_{i=1}^k f_i^\downarrow \leq \sum_{i=1}^k \lambda_i^\downarrow \quad \text{for all } 1 \leq k \leq n.$$

The following result is a variant of a result shown by Benac, Massey, and Stojanoff [8, 9]. Although it was originally stated in the setting of measure spaces with almost everywhere majorization, it also holds for measurable spaces. This result also holds for real Hilbert spaces H_n, though the proof in [8] is shown only in the complex case. As we will see in Section 4, Theorem 3.3 answers Problem 1.2 for type I_n von Neumann algebras.

Theorem 3.3 ([8, Theorem 5.1]) *Let $A\colon X \to \mathcal{B}(H_n)$ be a measurable field of $n \times n$ self-adjoint matrices with associated measurable eigenvalues $\lambda_i\colon X \to \mathbb{R}$, $i = 1, \ldots, n$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $f_i\colon X \to \mathbb{R}$, $i = 1, \ldots, n$, be measurable functions. The following are equivalent.*

(i) *$(f_1(x), \ldots, f_n(x)) \preceq (\lambda_1(x), \ldots, \lambda_n(x))$ for all $x \in X$.*
(ii) *There is a measurable field of unitary matrices $U\colon X \to \mathcal{U}(H_n)$ such that the matrices $U^*(x)A(x)U(x)$ have the diagonal $(f_1(x), \ldots, f_n(x))$ for all $x \in X$.*

From Theorem 3.3, we can draw the following corollary.

Theorem 3.4 *Let $f_i\colon X \to [0,1]$, $i = 1, \ldots, n$, be measurable functions and $\sum f_i(x) \in \mathbb{N}$ for all x. Then there is a measurable projection $P\colon X \to \mathcal{B}(H_n)$ with diagonal $(f_i)_{i \in \mathbb{N}}$.*

Proof The sets $X_k = \{x \in X : \sum f_i(x) = k\}$ are measurable. On every X_k we have $(f_1(x), \ldots, f_n(x)) \preceq (\lambda_1, \ldots, \lambda_n)$, where

$$\lambda_1 = \cdots = \lambda_k = 1 \quad \text{and} \quad \lambda_{k+1} = \cdots = \lambda_n = 0.$$

For $x \in X$, let $A_k(x)$ be a diagonal $n \times n$ matrix with diagonal $\lambda_1(x), \ldots, \lambda_n(x)$. By Theorem 3.3 there is measurable unitary field of $n \times n$ matrices U such that the matrix $U^*(x) A_k(x) U(x)$ has diagonal $(f_i(x))_{i=1}^n$. The mapping $x \mapsto U^*(x) A_k(x) U(x)$ is the required measurable projection. ∎

First we prove Theorem 3.1 in the following special case.

Theorem 3.5 *Let $2 \le N \in \mathbb{N}$ or $N = \infty$. Theorem 3.1 holds under the additional assumptions that for all $x \in X$,*

(i) *$f_i(x) \in (0,1)$ for all $i \in \mathbb{N}$;*
(ii) *there are exactly N indices $i \in \mathbb{N}$ such that $f_i(x) \in (0, 1/2]$;*
(iii) *there are infinitely many $f_i(x) \in (1/2, 1)$.*

Proof Using Definition 2.13, we split the sequence $(f_i)_{i \in \mathbb{N}}$ into two sequences $(a_i)_{i=1}^N$ and $(b_i)_{i \in \mathbb{N}}$ defined as $a_i = f_{\mathrm{pos}(i)}$ and $b_i = f_{\mathrm{Pos}(i)}$. By Lemmas 2.3 and 2.14, the functions $a_i\colon X \to (0, \frac{1}{2}]$, $b_i\colon X \to (\frac{1}{2}, 1)$ are measurable. Observe that

$$a(x) = \sum_{i=1}^N a_i(x) \quad \text{and} \quad b(x) = \sum_{i=1}^\infty (1 - b_i(x)).$$

First, we will construct a projection P with diagonal that coincides with (f_i) except for the three terms corresponding to $(a_{i_1}, a_{i_2}, b_{i_3})$.

For $x \in X$, define

$$i_1(x) = \begin{cases} 1 & a_1(x) \ge a_2(x), \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad i_2(x) = 3 - i_1(x).$$

Hence, $i_1, i_2\colon X \to \{1, 2\}$ are measurable and $a_{i_1}(x) \ge a_{i_2}(x)$ for all x.

For $x \in X$, let $i_3(x)$ be the smallest i with $b_i(x) \ge 1 - a_{i_1}(x)$. The number $i_3(x)$ is well defined since the sum $b(x)$ is finite, and hence $\lim_{i \to \infty} b_i(x) = 1$. Function $i_3\colon X \to \mathbb{N}$ is measurable, since

$$\{i_3 = k\} = \bigcap_{i=1}^{k-1} \{b_i < 1 - a_{i_1}\} \cap \{b_k \ge 1 - a_{i_1}\} \qquad k \in \mathbb{N}.$$

Let $i_4(x)$ be the smallest $k \in \mathbb{N}$, $k \ge 3$, such that

$$b_{i_3}(x) + \sum_{i=k}^N a_i(x) \le 1.$$

Note that if $N < \infty$, it might happen that $i_4(x) = N + 1$, but this does not affect our construction. Let $i_5(x)$ be the smallest $k \in \mathbb{N}$ such that

$$1 - a_{i_2}(x) + \sum_{i=k, i \neq i_3}^{\infty} (1 - b_i(x)) \leq 1.$$

The existence of $i_5(x)$ follows from the convergence of the series defining $b(x)$.

Claim   *The functions $i_4, i_5 \colon X \to \mathbb{N}$ are measurable.*

By Lemma 2.3, the function b_{i_3} is measurable. For any $n \geq 3$, we have

$$\{i_4 = n\} = \bigcap_{k=3}^{n-1} \left\{ b_{i_3} + \sum_{i=k}^{N} a_i > 1 \right\} \cap \left\{ b_{i_3} + \sum_{i=n}^{N} a_i \leq 1 \right\}.$$

Hence, i_4 is measurable. Likewise, for any $n \geq 1$, we have

$$\{i_5 = n\} = \bigcap_{k=1}^{n-1} \left\{ 1 - a_{i_2} + \sum_{i=k, i \neq i_3}^{\infty} (1 - b_i) > 1 \right\} \cap \left\{ 1 - a_{i_2} + \sum_{i=n, \ i \neq i_3}^{\infty} (1 - b_i) \leq 1 \right\}.$$

Hence, i_5 is also measurable.

With the above definitions, we have $i_4 \geq 3$,

$$(3.1) \qquad\qquad b_{i_3} + \sum_{i=i_4}^{N} a_i \leq 1$$

and

$$(3.2) \qquad\qquad 1 - a_{i_2} + \sum_{i=i_5, \ i \neq i_3}^{\infty} (1 - b_i) \leq 1.$$

Define functions $\widetilde{b}_{i_3}(x)$ and $\widetilde{a}_{i_2}(x)$ such that

$$(3.3) \qquad\qquad \widetilde{b}_{i_3}(x) + \sum_{i=i_4}^{N} a_i(x) = 1$$

and

$$(3.4) \qquad\qquad 1 - \widetilde{a}_{i_2}(x) + \sum_{i=i_5, \ i \neq i_3}^{\infty} (1 - b_i(x)) = 1.$$

By the definition of functions i_4 and i_5, we have $0 \leq \widetilde{b}_{i_3} \leq 1$ and $0 \leq \widetilde{a}_{i_2} \leq 1$, respectively. Measurability of functions $\widetilde{b}_{i_3}, \widetilde{a}_{i_2} \colon X \to [0, 1]$ then follows from the above claim. Finally, $\widetilde{a}_{i_1}(x)$ is defined by the equality

$$(3.5) \qquad\qquad \widetilde{a}_{i_1} + \widetilde{a}_{i_2} + \widetilde{b}_{i_3} \ = \ a_{i_1} + a_{i_2} + b_{i_3}.$$

We claim that $0 \leq \widetilde{a}_{i_1} \leq 1$. Indeed, to see that $\widetilde{a}_{i_1} \leq 1$, we write

$$\widetilde{a}_{i_1} = a_{i_1} + (a_{i_2} - \widetilde{a}_{i_2}) + (b_{i_3} - \widetilde{b}_{i_3}),$$

where the first two terms are $\leq 1/2$ and the last one is ≤ 0. Likewise, $a_{i_1} + b_{i_3} \geq 1$ and $a_{i_2} \geq \widetilde{a}_{i_2}$ imply that $\widetilde{a}_{i_1} \geq 0$. Hence, $\widetilde{a}_{i_1} \colon X \to [0,1]$ is measurable as a sum/difference of measurable functions. By (3.1)–(3.4),

$$(3.6) \qquad \widetilde{a}_{i_2} \leq a_{i_2} \leq a_{i_1} \leq b_{i_3} \leq \widetilde{b}_{i_3}.$$

Hence, by (3.5) and (3.6), we have

$$(3.7) \qquad (b_{i_3}, a_{i_1}, a_{i_2}) \preceq (\widetilde{b}_{i_3}, \widetilde{a}_{i_1}, \widetilde{a}_{i_2}).$$

Indeed, $\widetilde{a}_{i_2} \leq a_{i_2}$ and (3.5) implies that $\widetilde{a}_{i_1} + \widetilde{b}_{i_3} \geq a_{i_1} + b_{i_3}$, with the latter being the sum of two largest terms of $(b_{i_3}, a_{i_1}, a_{i_2})$. We also claim that

$$(3.8) \qquad \widetilde{a}_{i_1} + \sum_{i=3}^{i_4-1} a_i + \sum_{i=1,\ i\neq i_3}^{i_5-1} b_i \in \mathbb{N}.$$

Indeed, by (3.3) and (3.4),

$$\mathbb{Z} \ni \sum_{i=1}^{N} a_i - \sum_{i=1}^{\infty}(1-b_i) = \sum_{i=3}^{N} a_i - \sum_{i=1,\ i\neq i_3} (1-b_i) + a_{i_1} + a_{i_2} - (1 - b_{i_3})$$

$$= \sum_{i=3}^{N} a_i - \sum_{i=1,\ i\neq i_3} (1-b_i) + \widetilde{a}_{i_1} + \widetilde{a}_{i_2} + \widetilde{b}_{i_3} - 1$$

$$= \sum_{i=3}^{i_4-1} a_i - \sum_{i=1,\ i\neq i_3}^{i_5-1} (1-b_i) + \widetilde{a}_{i_1} + \left(\widetilde{b}_{i_3} + \sum_{i=i_4}^{N} a_i\right)$$

$$\quad - \left(1 - \widetilde{a}_{i_2} + \sum_{i=i_5,\ i\neq i_3}^{\infty} (1-b_i)\right)$$

$$= \sum_{i=3}^{i_4-1} a_i - \sum_{i=1,\ i\neq i_3}^{i_5-1} (1-b_i) + \widetilde{a}_{i_1}.$$

We divide the functions a_i, b_i, $\widetilde{a}_{i_1}$, $\widetilde{a}_{i_2}$, $\widetilde{b}_{i_3}$ into three groups:

(I) $\{a_i : 3 \leq i < i_4\} \ \cup \ \{b_i : 1 \leq i < i_5,\ i \neq i_3\} \ \cup \ \{\widetilde{a}_{i_1}\}$,

(II) $\{a_i :\ i \geq i_4\} \ \cup \ \{\widetilde{b}_{i_3}\}$,

(III) $\{b_i :\ i \geq i_5,\ i \neq i_3\} \ \cup \ \{\widetilde{a}_{i_2}\}$.

The sum of the numbers in group I is a natural number by (3.8). The sum of the numbers in each of groups II and III is one by (3.3) and (3.4). Using the notation from the proof of Lemma 2.11, we order the functions in these groups in the following sequences:

$$(p_i^I) = (\widetilde{a}_{i_1}, a_3, \ldots, a_{i_4-1}, b_1, b_2, \ldots, \widecheck{b}_{i_3}, \ldots, b_{i_5-1}),$$

$$(p_i^{II}) = (\widetilde{b}_{i_3}, a_{i_4}, a_{i_4+1}, \ldots),$$

$$(p_i^{III}) = (\widetilde{a}_{i_2}, b_{i_5}, b_{i_5+1}, \ldots, \widecheck{b}_{i_3}, \ldots).$$

Note that the sequence (p_i^I) is finite, but it has a variable length. However, we can decompose the space X into measurable sets X_n such that the sequence

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(p_i^I) has length n on each set X_n. Likewise, if $N < \infty$, then the sequence (p_i^{II}) has variable length as well. However, for the sake of simplicity, we assume that $N = \infty$; hence, (p_i^{II}) is infinite. The case $N < \infty$ is a simple modification, and it does not cause any difficulties.

The sequences (p_i^I), (p_i^{II}), and (p_i^{III}) consist of measurable functions by the following fact.

Fact Let $s \colon X \to \mathbb{N}$, $f_i \colon X \to \mathbb{R}$ be measurable. Then a function at any fixed position in the sequence $f_1, \ldots, \check{f}_s, \ldots$ is measurable. The same is true for the sequence $f_s, f_{s+1}, \ldots$

The fact is quickly shown by restricting to the set $\{s = k\}$ for a given $k \in \mathbb{N}$. Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of H. For fixed $n \in \mathbb{N}$ define subspaces

$$H_n^I = \mathrm{span}\{e_1, \ldots, e_n\},$$
$$H_n^{II} = \overline{\mathrm{span}}\{e_{n+1}, e_{n+3}, \ldots\},$$
$$H_n^{III} = \overline{\mathrm{span}}\{e_{n+2}, e_{n+4}, \ldots\}.$$

Applying Theorem 3.4 to the sequence (p_i^I) of length n, we obtain a measurable projection $P_n^I \colon X_n \to \mathcal{B}(H_n^I)$ with diagonal $(p_i^I)_{i \in [n]}$. Applying Lemma 3.2 we obtain measurable projections P_n^{II} and P_n^{III} with diagonals $(p_i^{II})_{i \in \mathbb{N}}$ and $(p_i^{III})_{i \in \mathbb{N}}$, respectively. Consequently,

$$P_n \colon X_n \to \mathcal{B}(H), \qquad P_n = P_n^I \oplus P_n^{II} \oplus P_n^{III}$$

is a measurable projection with diagonal

(3.9) $$(p_1^I, \ldots, p_n^I, p_1^{II}, p_1^{III}, p_2^{II}, p_2^{III}, p_3^{II}, p_3^{III}, \ldots).$$

Define $P = \bigcup_{n \in \mathbb{N}} P_n \colon X \to \mathcal{B}(H)$. The sequence (3.9) can be written as

$$\widetilde{f}_{\pi(1)}, f_{\pi(2)}, \ldots, f_{\pi(n)}, \widetilde{f}_{\pi(n+1)}, \widetilde{f}_{\pi(n+2)}, f_{\pi(n+3)}, f_{\pi(n+4)}, \ldots$$

for some measurable permutation $\pi \colon X \times \mathbb{N} \to \mathbb{N}$, where

$$\widetilde{f}_{\pi(1)} = \widetilde{a}_{i_1}, \qquad \widetilde{f}_{\pi(n+1)} = \widetilde{b}_{i_3}, \qquad \widetilde{f}_{\pi(n+2)} = \widetilde{a}_{i_2}.$$

For each n, define subspaces

$$H_n^{IV} = \mathrm{span}\{e_1, e_{n+1}, e_{n+2}\},$$
$$H_n^V = H \ominus H_n^{IV} = \overline{\mathrm{span}}\{e_2, \ldots, e_n, e_{n+3}, e_{n+4}, \ldots\}.$$

Define a measurable field of 3×3 self-adjoint matrices $A_n \colon X_n \to \mathcal{B}(H_n^{IV})$ by

$$A_n = \begin{bmatrix} \langle Pe_1, e_1 \rangle & \langle Pe_{n+1}, e_1 \rangle & \langle Pe_{n+2}, e_1 \rangle \\ \langle Pe_1, e_{n+1} \rangle & \langle Pe_{n+1}, e_{n+1} \rangle & \langle Pe_{n+2}, e_{n+1} \rangle \\ \langle Pe_1, e_{n+2} \rangle & \langle Pe_{n+1}, e_{n+2} \rangle & \langle Pe_{n+2}, e_{n+2} \rangle \end{bmatrix}$$
$$= \begin{bmatrix} \widetilde{f}_{\pi(1)} & * & * \\ * & \widetilde{f}_{\pi(n+1)} & * \\ * & * & \widetilde{f}_{\pi(n+2)} \end{bmatrix}.$$

By (3.7), we have

$$\big(f_{\pi(n+2)}(x), f_{\pi(1)}(x), f_{\pi(n+1)}(x)\big) \preceq \big(\widetilde{f}_{\pi(n+2)}(x), \widetilde{f}_{\pi(1)}(x), \widetilde{f}_{\pi(n+1)}(x)\big) \quad x \in X_n. \blacksquare$$

Applying Theorem 3.3, there exists a measurable field of unitaries $U_n \colon X_n \to \mathcal{B}(H_n^{IV})$ such that $U_n^*(x)A(x)U_n(x)$ has diagonal $(f_{\pi(1)}(x), f_{\pi(n+1)}(x), f_{\pi(n+2)}(x))$.

Define a measurable projection

$$Q_n \colon X_n \longrightarrow \mathcal{B}(H) \qquad Q_n = (U_n \oplus \mathbf{I})^* P_n (U_n \oplus \mathbf{I}),$$

where $\mathbf{I}$ is the identity on H_n^V. Define $Q = \bigcup_{n \in \mathbb{N}} Q_n \colon X \to \mathcal{B}(H)$. By our construction, Q has diagonal $(f_{\pi(1)}, f_{\pi(2)}, \dots)$. Since π is a measurable permutation, Lemma 2.10 yields the desired measurable projection P. $\blacksquare$

The following result removes the last two superfluous hypotheses in Theorem 3.5.

Theorem 3.6 *Theorem 3.1 holds under the additional hypothesis that $f_i(x) \in (0,1)$ for all $i \in \mathbb{N}$ and $x \in X$.*

Proof We split X into two measurable subsets

$$X' = \big\{ x \in X : \text{ there are infinitely many } f_i(x) \in (1/2, 1) \big\}$$

and its complement $X \setminus X'$. Furthermore, X' is split into measurable sets X_N, $N \in \mathbb{N} \cup \{0, \infty\}$ such that there are exactly N terms $f_i(x) \in (0, 1/2]$ for $x \in X_N$. We can then apply Theorem 3.5 on each set X_N where $N \geq 2$, and use Lemma 2.2. Hence, it suffices to show that Theorem 3.5 also holds for $N = 0, 1$. However, this is a consequence of Theorem 2.12 applied to the sequence $(1 - f_i)_{i \in \mathbb{N}}$, which has all terms in $(0, 1/2)$ with the exception of at most one term in $[1/2, 1)$. Hence, there exists a measurable projection $P \colon X_N \to \mathcal{B}(H)$, $N = 0, 1$, with diagonal $(1 - f_i)_{i \in \mathbb{N}}$. Consequently, $\mathbf{I} - P$ is a measurable projection with diagonal $(f_i)_{i \in I}$.

Since the series defining $a(x)$ is finite, for every $x \in X$, there are only finitely many $f_i(x) = 1/2$. Thus, for every $x \in X \setminus X'$, there are infinitely many $f_i(x) \in (0, 1/2)$. Applying the above construction for the sequence of functions $(1 - f_i)_{i \in \mathbb{N}}$ on $X \setminus X'$ yields a measurable projection $P \colon X \setminus X' \to \mathcal{B}(H)$ with diagonal $(1 - f_i)_{i \in \mathbb{N}}$. Consequently, $\mathbf{I} - P$ is a measurable projection with diagonal $(f_i)_{i \in I}$. $\blacksquare$

Finally, we do away with the remaining superfluous assumption in Theorem 3.6. We adopt the following definition.

Definition 3.7 For a sequence of measurable functions $f_i \colon X \to [0, 1]$, $i \in \mathbb{N}$, we define functions $\mathrm{Pro}(n) \colon X \to \mathbb{N}$ and $\mathrm{pro}(n) \colon X \to \mathbb{N}$ as follows. Let $\mathrm{Pro}(n)(x) = k$ if $f_k(x)$ is the n-th number in the sequence $(f_i(x))$ that belongs to $(0, 1)$. Likewise, we let $\mathrm{pro}(n)(x) = k$ if $f_k(x)$ is the n-th number in the sequence $(f_i(x))$ that is either 0 or 1.

We have the following analogue of Lemma 2.14, which is shown in the same manner.

Lemma 3.8   *Functions* $\mathrm{Pro}(n)$ *and* $\mathrm{pro}(n)$ *are measurable for all* n.

Proof of Theorem 3.1   We split X into measurable subsets corresponding to the following two cases.

Case 1. There are only finitely many $f_i \in (0, 1)$.

For any finite sequence $k_1, \ldots, k_n$ of natural numbers, the set

$$X_{k_1,\ldots,k_n} = \bigcap_{j=1,\ldots,n} \{f_{k_j} \in (0,1)\} \cap \bigcap_{i \neq k_j} (\{f_i = 0\} \cup \{f_i = 1\})$$

is a measurable subset of X. Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of H. We split the space H into two orthogonal subspaces

$$H_{k_1,\ldots,k_n} = \mathrm{span}\{e_{k_i} : i = 1, \ldots, n\} \quad \text{and} \quad (H_{k_1,\ldots,k_n})^\perp = \overline{\mathrm{span}}\{e_j : j \neq k_i\}.$$

By Theorem 3.4, there is a measurable projection $P_{k_1,\ldots,k_n} : X_{k_1,\ldots,k_n} \to \mathcal{B}(H_{k_1,\ldots,k_n})$ with diagonal $(f_{k_i})_{i \in [n]}$. On the space $(H_{k_1,\ldots,k_n})^\perp$, there is an obvious diagonal projection $Q_{k_1,\ldots,k_n}$ with zeros and ones on the diagonal. The projection $P_{k_1,\ldots,k_n} \oplus Q_{k_1,\ldots,k_n}$ is a sought projection on $X_{k_1,\ldots,k_n}$ acting on the whole space H. Since there are countably many sets $X_{k_1,\ldots,k_n}$, and they are disjoint for different $k_1, \ldots, k_n$, we are done on a measurable set

$$\bigcup_{n=0}^{\infty} \bigcup_{k_1,\ldots,k_n \in \mathbb{N}} X_{k_1,\ldots,k_n}.$$

The case $n = 0$ corresponds to all zeros and ones in (f_i).

Case 2. There are infinitely many $f_i \in (0, 1)$.

We split X into measurable subsets

$$X_N = \{x \in X : \text{there are exactly } N \text{ values } f_i(x) = 0 \text{ or } 1\}, \quad N \in \mathbb{N} \cup \{0, \infty\}.$$

That way the functions $\mathrm{pro}(n)$ are defined on X_n for all $1 \leq n \leq N$. For simplicity, we assume that $N = \infty$. The case $N < \infty$ is a simple modification and does not cause any difficulty.

Define a measurable projection $\pi \colon X_\infty \times \mathbb{N} \to \mathbb{N}$ by

$$\pi(n) = \begin{cases} \mathrm{Pro}(n/2) & \text{if } n \text{ is even,} \\ \mathrm{pro}((n+1)/2) & \text{if } n \text{ is odd.} \end{cases}$$

Define orthogonal subspaces

$$H_0 = \overline{\mathrm{span}}\{e_{2n} : n \in \mathbb{N}\} \quad \text{and} \quad H_1 = \overline{\mathrm{span}}\{e_{2n-1} : n \in \mathbb{N}\}.$$

By Theorem 3.6, there exists a measurable projection $P_0 \colon X_\infty \to \mathcal{B}(H_0)$ with diagonal $(f_{\pi(2n)})_{n \in \mathbb{N}}$. Let $P_1 \colon X_\infty \to \mathcal{B}(H_1)$ be the obvious diagonal projection with zeros and ones on the diagonal $(f_{\pi(2n-1)})_{n \in \mathbb{N}}$. Then $P_0 \oplus P_1$ is a measurable projection with diagonal $(f_{\pi(n)})_{n \in \mathbb{N}}$. Applying Lemma 2.10 yields the desired measurable projection. ∎

4 Applications

In this section, we present applications of Theorem 1.4 to shift-invariant spaces and von Neumann algebras.

4.1 Shift-invariant Spaces

Shift-invariant (SI) spaces are closed subspaces of $L^2(\mathbb{R}^d)$ that are invariant under all shifts, *i.e.*, integer translations. That is, a closed subspace $V \subset L^2(\mathbb{R}^d)$ is SI if $T_k(V) = V$ for all $k \in \mathbb{Z}^d$, where $T_k f(x) = f(x - k)$ is the translation operator. The theory of shift-invariant spaces plays an important role in many areas, most notably in the theory of wavelets, spline systems, Gabor systems, and approximation theory [12–14, 39, 40]. The study of analogous spaces for $L^2(\mathbb{T}, H)$ with values in a separable Hilbert space H in terms of the range function, often called doubly-invariant spaces, is quite classical and goes back to Helson [26].

In the context of SI spaces, a *range function* is any mapping

$$(4.1) \qquad J\colon \mathbb{T}^d \longrightarrow \{Y \subset \ell^2(\mathbb{Z}^d) : Y \text{ is a closed subspace}\},$$

where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is identified with its fundamental domain $[-1/2, 1/2)^d$. We say that J is *measurable* if the associated orthogonal projections $P_J(\xi)$ of $\ell^2(\mathbb{Z}^d)$ onto $J(\xi)$ are weakly operator measurable in the sense of Definition 1.3. We follow the convention that identifies range functions if they are equal almost everywhere. A fundamental result due to Helson [26, Theorem 8, p. 59] gives a one-to-one correspondence between SI spaces V and measurable range functions J; see also [14, Proposition 1.5]. This is achieved using a fiberization operator $\mathcal{T}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{T}^d, \ell^2(\mathbb{Z}^d))$ given by

$$\mathcal{T}f(\xi) = (\widehat{f}(\xi + k))_{k \in \mathbb{Z}^d} \qquad \text{for } f \in L^2(\mathbb{R}^d),\ \xi \in \mathbb{T}^d,$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$ is the Fourier transform of $f \in L^1(\mathbb{R}^d)$, and extended unitarily to $L^2(\mathbb{R}^d)$ by the Plancherel theorem. Then the one-to-one correspondence between SI spaces $V \subset L^2(\mathbb{R}^n)$ and measurable range functions J is encapsulated by the formula

$$V = \big\{f \in L^2(\mathbb{R}^d) : \mathcal{T}f(\xi) \in J(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^d \big\}.$$

Spectral function of SI spaces were introduced by Rzeszotnik and the first author in [19, 20]; see also [23, 25]. While there are several equivalent ways of introducing the spectral function of a SI space, the most relevant definition uses a range function.

Definition 4.1 The *spectral function* of an SI space V is a measurable mapping $\sigma_V \colon \mathbb{R}^d \to [0, 1]$ given by

$$\sigma_V(\xi + k) = \|P_J(\xi)e_k\|^2 = \langle P_J(\xi)e_k, e_k \rangle \qquad \text{for } \xi \in \mathbb{T}^d,\ k \in \mathbb{Z}^d,$$

where $(e_k)_{k \in \mathbb{Z}^d}$ denotes the standard basis of $\ell^2(\mathbb{Z}^d)$ and $\mathbb{T}^d = [-1/2, 1/2)^d$. In other words, $(\sigma_V(\xi + k))_{k \in \mathbb{Z}^d}$ is a diagonal of a projection $P_J(\xi)$.

Note that $\sigma_V(\xi)$ is well defined for a.e. $\xi \in \mathbb{R}^d$, since $\{k + \mathbb{T}^d : k \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. Theorem 1.4 yields the following characterization of spectral functions.

Theorem 4.2 *Let $\sigma \colon \mathbb{R}^d \to [0,1]$ be a measurable function. For $\xi \in \mathbb{R}^d$, define*

$$a(\xi) = \sum_{\substack{k \in \mathbb{Z}^d \\ \sigma(\xi+k)<1/2}} \sigma(\xi + k) \quad and \quad b(\xi) = \sum_{\substack{k \in \mathbb{Z}^d \\ \sigma(\xi+k)\geq 1/2}} \big(1 - \sigma(\xi + k)\big).$$

The following are equivalent.

(i) *There exists a SI space $V \subset L^2(\mathbb{R}^d)$ such that its spectral function σ_V coincides almost everywhere with σ:*

$$\sigma_V(\xi) = \sigma(\xi) \quad for\ a.e.\ \xi \in \mathbb{R}^d.$$

(ii) *For a.e. $\xi \in \mathbb{R}^d$, we either have*
   - *$a(\xi), b(\xi) < \infty$ and $a(\xi) - b(\xi) \in \mathbb{Z}$, or*
   - *$a(\xi) = \infty$ or $b(\xi) = \infty$.*

Proof The implication (i) $\implies$ (ii) follows from the necessity part of Kadison Theorem 1.1. Indeed, by Definition 4.1, the sequence $(\sigma_V(\xi + k))_{k \in \mathbb{Z}^d}$ is a diagonal of a projection $P_J(\xi)$ acting on $\ell^2(\mathbb{Z}^d)$ for a.e. $\xi \in \mathbb{T}^d$. Hence, (ii) holds pointwise almost everywhere.

The converse implication (ii) $\implies$ (i) is the crux of this result. Kadison's Theorem 1.1 merely implies the existence of projections $P(\xi)$ with desired diagonal $(\sigma(\xi + k))_{k \in \mathbb{Z}^d}$ for a.e. ξ. However, it does not guarantee the measurability of $\xi \mapsto P(\xi)$. This is where we need to apply Theorem 1.4 instead. Let X be the set of all points $\xi \in \mathbb{R}^d$ such that (ii) holds. This is a set of full measure; *i.e.*, $\mathbb{R}^d \setminus X$ is a null set. Theorem 1.4 yields the existence of a measurable projection $P \colon X \to \mathcal{B}(\ell^2(\mathbb{Z}^d))$ such that $P(\xi)$ has diagonal $(\sigma(\xi + k))_{k \in \mathbb{Z}^d}$ for a.e. ξ. This projection corresponds to a measurable range function J as in (4.1), which by Helson's theorem corresponds to a SI space $V \subset L^2(\mathbb{R}^d)$. $\blacksquare$

The notion of a spectral function can be extended to the setting of translation invariant (TI) subspaces of $L^2(G)$ of a second countable locally countable abelian group G, see [18]. In this case, $\Gamma \subset G$ is a closed co-compact subgroup and the spectral function represents diagonal entries of a measurable projection defined on $X = \widehat{G}/\Gamma^*$ with values in $\mathcal{B}(\ell^2(\Gamma^*))$, where Γ^* is the annihilator of Γ in the dual group $\widehat{G}$. Since Theorem 1.4 does not put any restriction on a measurable space X, an extension of Theorem 4.2 to this setting does not cause any extra difficulties.

4.2 Carpenter's Theorem for Type I$_\infty$ von Neumann Algebras

In this subsection we will answer Problem 1.2 for von Neumann algebras of type I$_\infty$ when T is a projection.

The structure theorem for type I von Neumann algebras [11, Section I.3.12] gives the following characterization. A type I_n, where $n \in \mathbb{N} \cup \{\infty\}$, von Neumann algebra $\mathfrak{M}$ is isomorphic with the tensor product $\mathfrak{M} \cong \mathcal{B}(H_n) \otimes L^\infty(X, \mu)$, where H_n is an n-dimensional Hilbert space and (X, μ) is a locally finite measure space. Hence, $\mathfrak{M}$ is isomorphic to $L^\infty(X, \mu, \mathcal{B}(H_n))$ acting on a Hilbert space $L^2(X, \mu, H_n)$.

For type I_∞ von Neumann algebras, let H be the direct integral of a measurable field of infinite-dimensional separable Hilbert spaces H_∞ on (X, μ) given by

$$(4.2) \qquad H = \int_X^\oplus H_\infty d\mu(x).$$

Let $\mathfrak{M}$ be the subalgebra of $\mathcal{B}(H)$ consisting of all decomposable operators

$$(4.3) \qquad T = \int_X^\oplus T(x) d\mu(x),$$

where $x \mapsto T(x) \in \mathcal{B}(H_\infty)$ is an essentially bounded measurable field of operators; see [22, Section 2.3]. The structure theorem for type I_∞ von Neumann algebra implies that every such algebra is of this form. Consider a MASA $\mathcal{A} \subset \mathcal{M}$ consisting of all decomposable operators T such that for μ-a.e. $x \in X$, $T(x)$ is a diagonal operator with respect to some fixed orthonormal basis $(e_i)_{i=1}^\infty$ of H_∞. Then Theorem 1.4 implies the following theorem.

Theorem 4.3 *Let $\mathfrak{M}$ be the von Neumann algebra of type I_∞ consisting of decomposable operators (4.3) acting on the direct integral Hilbert space (4.2). Let $\mathcal{A} \subset \mathfrak{M}$ be a MASA consisting of decomposable operators whose fibers $T(x)$ are diagonal operators with respect to some fixed orthonormal basis of H_∞.*

Suppose that P is a projection in $\mathfrak{M}$. Define the dimension functions $p, q \colon X \to \mathbb{N} \cup \{0, \infty\}$ by

$$p(x) = \mathrm{rank}(P(x)), \qquad q(x) = \mathrm{rank}(\mathbf{I} - P(x)) \qquad \text{for } x \in X.$$

Then the set $D_\mathcal{A}(P)$ of conditional expectations of the unitary orbit of P, which is given by (1.1), consists of operators T such that $T(x)$ is a diagonal operator on $\mathcal{H}_\infty$ with entries given by a sequence of measurable functions $f_i \colon X \to [0, 1]$, $i \in \mathbb{N}$, satisfying the following conditions for μ-a.e. $x \in X$:

(i) $\sum_{i \in \mathbb{N}} f_i(x) = p(x)$, $\sum_{i \in \mathbb{N}}(1 - f_i(x)) = q(x)$, *and*
(ii) *functions $a, b \colon X \to [0, \infty]$ given by (1.2) satisfy either*
- $a(x) = \infty$ *or* $b(x) = \infty$ *or*
- $a(x), b(x) < \infty$ *and* $a(x) - b(x) \in \mathbb{Z}$.

Proof Let $U \in \mathfrak{M}$ be a unitary operator. Since U is a decomposable operator its fibers $U(x)$ are unitary operators for a.e. x. Thus, U^*PU is a decomposable operator with fibers $U(x)^*P(x)U(x)$. Let $(f_i(x))_{i=1}^\infty$ be the diagonal of $U(x)^*P(x)U(x)$. Then, by the trace argument and by the necessity part in Kadison's Theorem 1.1, the diagonal sequence satisfies (i) and (ii), respectively.

The converse implication is a consequence of Theorem 1.4. Let $X' \subset X$ be the subset of full measure for which either (i) or (ii) hold for all $x \in X'$. By Theorem 1.4, there exists a measurable projection $Q \colon X' \to \mathcal{B}(H_\infty)$ with diagonal $(f_i)_{i \in \mathbb{N}}$. We extend Q to X in any way. Then

$$Q = \int_X^\oplus Q(x) d\mu(x)$$

is a projection in $\mathfrak{M}$ with diagonal $(f_i)_{i \in \mathbb{N}}$ modulo null sets. It remains to show that there exists a unitary $U \in \mathfrak{M}$ such that $Q = U^* P U$.

Measurable projections P and Q correspond to measurable range functions

$$J_P, J_Q \colon X \to \{Y \subset H_\infty : Y \text{ is a closed subspace}\}.$$

Let $P^\perp = \mathbf{I} - P$ and $Q^\perp = \mathbf{I} - Q$ be the projections on orthogonal subspaces, which correspond to measurable range functions

$$J_{P^\perp}(x) = (J_P(x))^\perp \quad \text{and} \quad J_{Q^\perp}(x) = (J_Q(x))^\perp.$$

By (i), we have

$$\dim J_P(x) = \dim J_Q(x) = p(x) \quad \text{and} \quad \dim J_{P^\perp}(x) = \dim J_{Q^\perp}(x) = q(x)$$

for μ-a.e. x. By Helson's theorem [27, Theorem 2, Section 1.3], there exists a sequence of measurable functions $G_i \colon X \to H_\infty$, $i \in \mathbb{N}$, such that for μ-a.e. $x \in X$, $\{G_i(x)\}_{i=1}^{p(x)}$ forms an orthonormal basis of $J_P(x)$. Let $\widetilde{G}_i$, $i \in \mathbb{N}$, be the corresponding orthonormal basis sequence for the range function J_Q. Likewise, there exist measurable functions $F_i \colon X \to H$ and $\widetilde{F}_i \colon X \to H_\infty$, $i \in \mathbb{N}$, such that $\{F_i(x)\}_{i=1}^{q(x)}$ and $\{\widetilde{F}_i(x)\}_{i=1}^{q(x)}$ are an orthonormal basis of $J_{P^\perp}(x)$ and $J_{Q^\perp}(x)$, resp. Let $U(x)$ be the unitary operator on H_∞ that maps the orthonormal basis $\{\widetilde{G}_i(x)\}_{i=1}^{p(x)} \cup \{\widetilde{F}_i(x)\}_{i=1}^{q(x)}$ onto $\{G_i(x)\}_{i=1}^{p(x)} \cup \{F_i(x)\}_{i=1}^{q(x)}$. Then our construction yields $Q(x) = U(x)^* P(x) U(x)$. Consequently,

$$U = \int_X^\oplus U(x) d\mu(x)$$

is the required unitary satisfying $Q = U^* P U$, which completes the proof. ∎

We conjecture that an analogue of Theorem 4.3 holds for self-adjoint operators $T \in \mathfrak{M}$ such that $T(x)$ has a finite spectrum for μ-a.e. $x \in X$. The necessary conditions are provided by the corresponding result for I_∞ factors, that is $\mathcal{B}(H_\infty)$, which was shown by Jasper and the first author [17, Theorem 1.3]. However, the sufficiency requires a construction of a measurable field of unitary operators and, a priori, it is not clear if this is possible. The lack of any obstruction for operators with two point spectrum, which are essentially projections, suggests an affirmative answer to this problem as well.

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