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REGULARITY OF WEAK SOLUTIONS FOR A CLASS OF ELLIPTIC PDES IN ORLICZ-SOBOLEV SPACES

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Abstract. We consider the elliptic partial differential equation in the divergence form

$$-\mathrm{div}(\nabla G(\nabla u(x))) + F_u(x, u(x)) = 0,$$

where G is a convex, anisotropic function satisfying certain growth and ellipticity conditions. We prove that weak solutions in $W^{1,G}$ are in fact of class $W^{2,2}_{\mathrm{loc}} \cap W^{1,\infty}_{\mathrm{loc}}$.

1. Introduction

We consider a quasilinear elliptic equation in the divergence form:

$$-\mathrm{div}(\nabla G(\nabla u(x))) + F_u(x, u(x)) = 0 \tag{P}$$

where $u\colon \Omega \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is an open connected set. Functions $G \in C^2(\mathbb{R}^n, \mathbb{R})$ and $F \in C^1(\Omega \times \mathbb{R}, R)$ are assumed to satisfy certain growth conditions given below. The objective of this paper is to show that for such G and F, every weak solution u, that belongs to the Orlicz–Sobolev space $W^{1,G}_{\mathrm{loc}}(\Omega)$, is of a class $W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

This result is inspired by Marcellini's articles [14] and [13] in which he proves analogous regularity theorem for weak solutions of an elliptic equation. One of the differences between our result and these by Marcellini is that we assume

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anisotropic growth conditions (i.e. G may be growing differently in different directions). We also generalize a similar result by Siepe [16] where anisotropic growth conditions are considered but the equation does not have a part dependent on x and u. $W^{2,2}_{\rm loc} \cap W^{1,\infty}_{\rm loc}$ regularity of weak solutions was also obtained by Cupini, Marcellini, Mascolo in [9]. Also in earlier articles by the same authors [7], [8], [10] we can find theorems concerning regularity with $p-q$ growth condition. Some other regularity results concerning $p-q$ growth can be found in [3], [4]. Anisotropic polynomial growth (with functional being a sum of different powers) is considered in [5], [12].

The difference between our approach and cited above is that we focus on weak solutions belonging to the anisotropic Orlicz-Sobolev space $W^{1,G}_{\rm loc}$ instead of $W^{1,q}_{\rm loc}$. This space is natural when one works in an anisotropic setting. Regularity of weak solutions in the anisotropic Orlicz–Sobolev spaces was established earlier by, for example, Cianchi in [6] and Alberico in [1], where they proved boundedness of weak solutions.

Our assumptions on G are as follows:

- (G_1) G is convex and $G(-\xi) = G(\xi)$ for all $\xi \in \mathbb{R}^n$,
- (G_2) $\lim\limits_{\xi \to 0} G(\xi)/|\xi| = 0$,
- (G_3) $G(\xi) \geq c_0|\xi|^2$ for some $c_0 > 0$ and for all $\xi \in \mathbb{R}^n$,
- (G_4) $\langle DG(\xi), \lambda\rangle \leq p\, G(\xi)|\lambda|/|\xi|$ for some $p \geq 2$ and for all ξ, $\lambda \in \mathbb{R}^n$,
- (G_5) there exists $\alpha \geq 2$ and $c_\alpha > 0$ such that $c_\alpha|\xi|^\alpha \leq G(\xi)$ for all $|\xi| \geq 1$,
- (G_6) $G(\xi) \leq c \sum\limits_{s=1}^{n} |\xi_s|^{2^*(\alpha/2-1)+2}$ for all $|\xi| \geq 1$, where 2^* is a critical Sobolev exponent,
- (G_7) there exists $\nu > 0$ such that $\langle D^2G(\xi)\lambda, \lambda\rangle \geq 2\nu G(\xi)|\lambda|^2/|\xi|^2$ for all ξ, $\lambda \in \mathbb{R}^n$.

Note that function G satisfies the definition of n-dimensional Young function in the sense of Barletta, Cianchi [2]. Thanks to that we can naturally define Orlicz–Sobolev space

$$W^{1,G}(\Omega) = \left\{ u \in L^1(\Omega) : u \text{ is weakly differentiable and } \int_\Omega G(\nabla u)\, dx \leq \infty \right\}.$$

Here we will deal mainly with weak formulation of the equation and therefore we do not need any specific properties of Orlicz–Sobolev spaces. For our purpose we only need to know that $W^{1,G}(\Omega)$ is a linear space. For more information on this subject we refer the reader to [2] or [17] and references therein.

One can notice that (G_4) implies that G has polynomial growth, namely $G(\xi) \leq c|\xi|^p$ for $|\xi| \geq 1$ (see [2]). Assumption (G_6) gives another upper bound on the growth, so without loss of generality we could assume that $p = 2^*(\alpha/2-1)+2$. However, for technical reasons it is more convenient to keep both constants. Especially, this specific form of exponent is useful in (3.15).

Assumption (G_7) guarantees that D^2G is positive definite with lower bound for its eigenvalues dependent on $G(\xi)/|\xi|^2$. Thus inequality (G_7) implies the strong ellipticity of the operator $\operatorname{div}(\nabla G(\nabla u(x)))$. It is also equivalent to uniform convexity of G (see Lemma 2.3 of [16]). When considering higher regularity such ellipticity assumption seem to be necessary. Our inequality is a natural generalization of for example condition (8.11) from [11].

We consider $F \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$. The derivative F_u is assumed to be bounded or to have bounded derivatives, F_u can also be also a sum of such functions. For this reason, we shall write $F_u = \overline{F} + \widehat{F}$ where

(F_1) $\overline{F}$ is continuous and $|\overline{F}(x,u)| \leq Q$ for all $x \in \Omega$, $u \in \mathbb{R}$,
(F_2) $\widehat{F}$ is continuous and has bounded derivatives: $|\widehat{F}_{x_s}(x,u)| \leq Q$ and $|\widehat{F}_u(x,u)| \leq Q$ for all $x \in \Omega$, $u \in \mathbb{R}$.

Now we are in position to state our main theorem.

THEOREM 1.1. *Let $v \in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a weak solution of* (P). *Solution v is of class $W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and for every $R > 0$ and every ball $B_R \subset\subset \Omega$ there exist positive constants $\overline{C}$ and $\widetilde{C}$ such that*

$$\int_{B_{R/2}} |D^2 v|^2 \, dx \leq \overline{C} \int_{B_R} 1 + G(Dv) \, dx \tag{1.1}$$

and

$$\sup_{B_{R/2}} |Dv|^2 \leq \widetilde{C} \int_{B_R} 1 + G(Dv) \, dx. \tag{1.2}$$

In [14] Marcellini proved an analogous regularity result for equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, u(x), Du) = b(x, u(x), Du)$$

with a^i satisfying general (but isotropic) growth conditions. Our problem is a special case of this equation (with $a^i(x,u,\xi) = G_{\xi_i}(\xi)$). Assumptions ($G_1$)–($G_7$) give more restrictive growth conditions for G that given in [14] and [9]. In [14] weak solutions are assumed to be in $W^{1,G}$ but with isotropic function G determined by the upper bounds for a. In [13, 9] authors assume that weak solutions are of class $W^{1,q}$, where $G \prec |\cdot|^q$. Our assumptions admit anisotropic behaviour. Furthermore, the space $W^{1,G}$ in which we work is strictly connected with the equation. That is, the weak formulation can be naturally considered in this space.

We believe that our theorem, despite restrictive growth conditions and some technical complications, may be useful in many applications. In applications weak solutions can be obtained as critical points of certain functional defined on $W^{1,G}_{\mathrm{loc}}$. For example we get the following:

Remark 1.2. It follows form Theorem 1.1 that every local minimum of the functional

$$I(u) = \int_\Omega G(\nabla u) + F(x,u)\,dx$$

in the space $W^{1,G}_{\mathrm{loc}}(\Omega)$ is of class $W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and satisfies inequalities (1.1) and (1.2).

2. Difference quotients and properties of weak derivatives

The purpose of this section is to introduce the notation and to collect auxiliary facts that will be used. In the proof of Theorem 1.1 we will use many properties of difference quotients. The difference quotient of $u\colon \Omega \to \mathbb{R}$ in the direction of versor e_s is defined by

$$\Delta^s_h u(x) = \frac{u(x+he_s) - u(x)}{h},$$

where $h \neq 0$. Note that difference quotient $\Delta^s_h u$ is defined only on the set

$$\Omega_{|h|} = \{x \in \Omega : d(x, \partial\Omega) > |h|\}.$$

The upper index s in Δ^s_h will be usually omitted and be assumed to be fixed. The next lemma is a direct consequence of the definition of difference quotients and linearity of $W^{1,G}$ (cf. [14]).

Lemma 2.1. *Assume that $u, w \in W^{1,G}(\Omega)$ and $h \in \mathbb{R}$, $h \neq 0$. Then:*

(a) $\Delta_h u \in W^{1,G}(\Omega_{|h|})$.
(b) *If $\psi\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous, odd, convex function, nondecreasing on $[0,\infty)$ then $\psi(\Delta_h u) \in W^{1,G}(\Omega_{|h|})$.*
(c) *If ψ is as above and $\eta \in C^1_0(\Omega)$ then $\eta\psi(\Delta_h u) \in W^{1,G}(\Omega_{|h|})$.*
(d) $D_i(\Delta_h u) = \Delta_h(D_i u)$.
(e) *If $\operatorname{supp} u \subset \Omega_{|h|}$ or $\operatorname{supp} w \subset \Omega_{|h|}$ then*

$$\int_\Omega u\Delta_h w\,dx = -\int_\Omega w\Delta_{-h} u\,dx.$$

(f) $\Delta_h(uw)(x) = u(x+he_s)\Delta_h w(x) + w(x)\Delta_h u(x)$

The next lemma establishes relation between difference quotients and weak derivatives. It is a classical result that can be found, for example, in [14, Lemma 3.1].

Lemma 2.2. *Let $\Omega_0 \subset\subset \Omega$, $|h| < d(\Omega_0, \partial\Omega)$, $1 \le p < \infty$. For every $v \in W^{1,p}(\Omega)$ we have*

$$\int_{\Omega_0} |\Delta^s_h v|^p\,dx \le \int_\Omega |D_s v|^p\,dx \le \int_\Omega |Dv|^p\,dx$$

Following lemmas are inspired by [11, Lemma 8.2]. The first states that if there exists a weak derivative in $L^p_{\mathrm{loc}}(\Omega)$, then the difference quotients converge to it in L^p_{loc}. The second shows that boundedness of norms of the difference quotients implies existence of the weak derivative. It is our main tool in proving the existence of weak derivatives.

Lemma 2.3. *Assume that for some $v \in L^p_{loc}(\Omega)$ there exists $D_s v \in L^p_{\mathrm{loc}}(\Omega)$. Then $\Delta_h v \to D_s v$ in $L^p_{\mathrm{loc}}(\Omega)$.*

Lemma 2.4. *Let $v \in L^p(\Omega)$, $1 < p < \infty$. Assume that there exists $M > 0$ and a sequence $h_n \to 0$, such that*

$$\int_{\Omega_{|h_n|}} |\Delta^s_{h_n} v|^p \, dx \le M.$$

Then

$$\int_{\Omega} |D_s v|^p \, dx \le M$$

and $\Delta^s_{h_n} v \to D_s v$ in $L^p_{\mathrm{loc}}(\Omega)$.

Note that when Ω is bounded, then convergence in L^p_{loc} can be replaced with convergence in L^p. The case of bounded Ω will be considered in the proof of the Theorem 1.1. To finish this section we recall well known facts about the ess sup norm. The proof can be found in [15].

Lemma 2.5. *Assume that Ω is bounded.*

(a) *If $v \in L^\infty(\Omega)$, then $\|v\|_{L^\infty(\Omega)} = \lim_{p\to\infty} \|v\|_{L^p(\Omega)}$.*

(b) *If $v \in L^{p_k}(\Omega)$ for some sequence $p_k \to \infty$ and $\sup_k \|v\|_{L^{p_k}(\Omega)} < \infty$, then $v \in L^\infty(\Omega)$.*

The following is a simple consequence of the above Lemma and Lemma 2.2:

Lemma 2.6. *Let $\Omega_0 \subset\subset \Omega$, $|h| < d(\Omega_0, \partial\Omega)$. For every $v \in W^{1,\infty}(\Omega)$ we have*

$$\operatorname*{ess\,sup}_{\Omega_0} |\Delta^s_h v| \le \operatorname*{ess\,sup}_{\Omega} |D_s v|$$

3. Proof of Theorem 1.1

We have divided the proof into several steps. First, we provide some estimates on the integral on the left side of Euler equation:

$$\int_\Omega \sum_{i=1}^n (G_{\xi_i}(Dv)) D_i\varphi(x) + F_u(x,v)\varphi(x) \, dx = 0, \tag{3.1}$$

where $v \in W^{1,G}_{\mathrm{loc}}$ is a weak solution of (P) and $\varphi \in W^{1,G}(\Omega)$ is such that $\mathrm{supp}(\varphi) \subset\subset \Omega$. In the next steps we prove inequalities (1.1) and (1.2).

Before we start the proof, we shall introduce some auxiliary notation:

- letter c will be a positive constant which can vary from line to line, it may depend on the other constants given in the assumptions or appearing in calculations, but to simplify the proof we do not provide explicit formulas;
- B_r denotes the open ball in $\mathbb{R}^n$ of radius r, form now on all the balls will be concentric;
- $2h_0 \le \operatorname{dist}(B_R, \partial\Omega)$ and $|h| \le h_0$;
- for $0<\rho<R$, by $\eta \in C_0^2(\Omega)$ we will denote a function constantly equal to 1 on a ball B_ρ such that

$$\operatorname{supp}\eta \subset B_R, \quad |D\eta| \le \frac{1}{R-\rho} \quad \text{and} \quad |D^2\eta| \le \frac{1}{(R-\rho)^2};$$

- for $|h| \le h_0$ and $t \in [0,1]$ we define $\xi_h^t = tDv(x+he_s) + (1-t)Dv(x)$ and $\lambda_h^t = Dv(x+the_s)$;
- by $\psi \in C^1(\mathbb{R})$ we will understand an odd function which is convex on $[0,\infty)$ and for which $0 \le \psi'(t) \le c_{\psi'}$. Notice that for such ψ we have

$$|\psi(t)| \le \psi'(t)|t|. \tag{3.2}$$

Step 1. Auxiliary estimates. Since $v \in W^{1,G}$ is a weak solution, it satisfies (3.1) for any $\varphi \in W^{1,G}(\Omega)$ such that $\operatorname{supp}(\varphi) \subset\subset \Omega$. It follows from Lemma 2.1 (a), (b) and (c) the the function $\varphi = \Delta_{-h}(\eta^2\psi(\Delta_h v))$ is admissible.

For this particular φ, using Lemma 2.1 (d), (e) and (f) we can rewrite the first summand in equation (3.1) as

$$\begin{aligned}
\int_{B_R}\sum_{i=1}^n (G_{\xi_i}(Dv))D_i\varphi(x)\,dx &= \int_{B_R}\sum_{i=1}^n G_{\xi_i}(Dv)\Delta_{-h}\big(D_i(\eta^2\psi(\Delta_h v))\big)\,dx\\
&= -\int_{B_R}\sum_{i=1}^n \Delta_h(G_{\xi_i}(Dv))\big(D_i(\eta^2)\psi(\Delta_h v) + \eta^2\psi(\Delta_h v)D_i(\Delta_h v)\big)\,dx\\
&= -\int_{B_R}\sum_{i=1}^n \Delta_h G_{\xi_i}(Dv)D_i(\eta^2)\psi(\Delta_h v)\,dx\\
&\quad -\int_{B_R}\sum_{i=1}^n \Delta_h(G_{\xi_i}(Dv))\Delta_h(D_i v)\eta^2\psi'(\Delta_h v)\,dx.
\end{aligned}$$

Hence equation (3.1) can be rewritten in the form:

$$\begin{aligned}
&\int_{B_R}\sum_{i=1}^n \Delta_h(G_{\xi_i}(Dv))\Delta_h(D_i v)\eta^2\psi'(\Delta_h v)\,dx \\
&= -\int_{B_R}\sum_{i=1}^n \Delta_h G_{\xi_i}(Dv)D_i(\eta^2)\psi(\Delta_h v)\,dx + \int_{B_R} F_u(x,v)\Delta_{-h}(\eta^2\psi(\Delta_h v))\,dx.
\end{aligned} \tag{3.3}$$

We will denote these three integrals by J_1, J_2, J_3 (so $J_1 = -J_2 + J_3$). Our goal in this step is to provide bounds for every integral in (3.3). Observe that

$$\begin{aligned}\Delta_h(G_{\xi_i}(Dv)) &= \frac{1}{h}(G_{\xi_i}(Dv(x+he_s)) - G_{\xi_i}(Dv(x))) \\ &= \frac{1}{h}\int_0^1 \frac{d}{dt}\, G_{\xi_i}(tDv(x+he_s) + (1-t)Dv(x))\, dt \\ &= \int_0^1 \sum_{j=1}^n G_{\xi_i\xi_j}(\xi_h^t)\Delta_h(D_j v)\, dt.\end{aligned}$$

Applying this to J_1 we get

$$J_1 = \int_{B_R}\int_0^1 \Big(\sum_{i,j=1}^n G_{\xi_i\xi_j}(\xi_h^t)\Delta_h(D_i v)\Delta_h(D_j v)\Big)\eta^2\psi'(\Delta_h v)\, dt\, dx.$$

This gives

$$J_1 \geq 2\nu \int_{B_R}\int_0^1 \frac{G(\xi_h^t)}{|\xi_h^t|^2}|\Delta_h(Dv)|^2\eta^2\psi'(\Delta_h v)\, dt\, dx$$

by assumption(G_7)

In J_2 we will also transform the difference quotient $\Delta_h(G_{\xi_i}(Dv))$ but this time we will use λ_h^t instead of ξ_h^t.

$$\Delta_h(G_{\xi_i}(Dv)) = \frac{1}{h}\int_0^1 \frac{d}{dt}\, G_{\xi_i}(\lambda_h^t)\, dt = \int_0^1 \frac{\partial}{\partial x_s}\, G_{\xi_i}(\lambda_h^t)\, dt.$$

Applying this to J_2 and integrating by parts we get

$$\begin{aligned}J_2 &= \int_{B_R}\int_0^1 \sum_{i=1}^n G_{\xi_i}(\lambda_h^t) D_s\left(D_i(\eta^2)\psi(\Delta_h v)\right)\, dt\, dx \\ &= 2\int_{B_R}\int_0^1 \langle DG(\lambda_h^t), D_s D_i(\eta^2)\rangle\psi(\Delta_h v)\, dt\, dx \\ &\quad + \int_{B_R}\int_0^1 \langle DG(\lambda_h^t), D_i(\eta^2)\rangle\psi'(\Delta_h v)\Delta_h(D_s v)\, dt\, dx.\end{aligned}$$

We will denote those two integrals by $J_{2,1}$ and $J_{2,2}$. By definition of η we have $|D_i(\eta^2)| \leq 2/(R-\rho)$ and $|D_i D_j(\eta^2)| \leq 4/(R-\rho)^2$. From (G_4) and (3.2) we obtain

$$|J_{2,1}| \leq \int_{B_R}\int_0^1 p\frac{G(\lambda_h^t)}{|\lambda_h^t|}\frac{4}{(R-\rho)^2}\,\psi'(\Delta_h v)|\Delta_h v|\, dt\, dx.$$

Applying (G_4), properties of η and inequality $|2ab| \le \nu a^2 + b^2/\nu$ yields

$$
\begin{aligned}
|J_{2,2}| &\le \int_{B_R}\int_0^1 p\,\frac{G(\lambda_h^t)}{|\lambda_h^t|}\,2\eta|D_i\eta|\psi'(\Delta_h v)|\Delta_h(D_s v)|\,dt\,dx \\
&\le \int_{B_R}\int_0^1 2\bigg(\frac{G(\lambda_h^t)}{|\lambda_h^t|^2}\eta^2\psi'(\Delta_h v)|\Delta_h(D_s v)|^2\bigg)^{1/2} \\
&\qquad\times\bigg(\frac{p^2}{(R-\rho)^2}\,G(\lambda_h^t)\psi'(\Delta_h v)\bigg)^{1/2}\,dt\,dx \\
&\le \nu\int_{B_R}\int_0^1 \frac{G(\lambda_h^t)}{|\lambda_h^t|^2}\,\eta^2\psi'(\Delta_h v)|\Delta_h(D_s v)|^2\,dt\,dx \\
&\qquad+\frac{p^2}{\nu(R-\rho)^2}\int_{B_R}\int_0^1 G(\lambda_h^t)\psi'(\Delta_h v)\,dt\,dx.
\end{aligned}
$$

Now we consider J_3. From $F_u = \widehat{F} + \overline{F}$ and Lemma 2.1 (e) we have

$$
\begin{aligned}
|J_3| &= \bigg|\int_{B_R}\overline{F}(x,v)\Delta_{-h}\big(\eta^2\psi(\Delta_h v)\big)\,dx - \int_{B_R}\Delta_h\widehat{F}(x,v)\eta^2\psi(\Delta_h v)\,dx\bigg| \\
&\le \int_{B_R}|\overline{F}(x,v)||\Delta_{-h}(\eta^2\psi(\Delta_h v))|\,dx + \int_{B_R}\big|\Delta_h\widehat{F}(x,v)\big|\eta^2|\psi(\Delta_h v)|\,dx \\
&= J_{3,1}+J_{3,2}.
\end{aligned}
$$

From (F_1) and Lemma 2.2 we get

$$
J_{3,1} \le Q\int_{B_R}\big|\Delta_{-h}\big(\eta^2\psi(\Delta_h v)\big)\big|\,dx \le Q\int_{B_{R+|h|}}\bigg|\frac{d}{dx_s}(\eta^2\psi(\Delta_h v))\bigg|\,dx.
$$

Since $\operatorname{supp}\eta \subset B_R$, we can take B_R instead of $B_{R+|h|}$ in the last integral. Having this in mind and using 2.1 (d) we get

$$
J_{3,1} \le Q\int_{B_R} 2\eta|D_s\eta||\psi(\Delta_h v)| + \eta^2\psi'(\Delta_h v)|\Delta_h(D_s v)|\,dx.
$$

Applying inequality $ab \le \tau a^2 + b^2/\tau$ to the second integral yields

$$
\begin{aligned}
J_{3,1} \le Q\int_{B_R} &2\eta|D_s\eta||\psi(\Delta_h v)|\,dx \\
&+\tau\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h(Dv)|^2\,dx + \frac{1}{4\tau}\int_{B_R}\eta^2\psi'(\Delta_h v)\,dx.
\end{aligned}
$$

Now we deal with $J_{3,2}$. From (F_2) we get

$$
\begin{aligned}
J_{3,2} &= \int_{B_R}\bigg|\eta^2\psi(\Delta_h v)\frac{1}{h}\int_0^1\frac{d}{dt}\widehat{F}(x+the_s, v+th\Delta_h v)\,dt\bigg|\,dx \\
&\le \int_{B_R}\int_0^1\big|\eta^2\psi(\Delta_h v)\big|\big(\big|\widehat{F}_{x_s}\big| + \big|\widehat{F}_u\big||\Delta_h v|\big)\,dt\,dx \\
&\le Q\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h v|(1+|\Delta_h v|)\,dx.
\end{aligned}
$$

Combining the above inequalities gives

$$
\begin{aligned}
(3.4)\quad |J_3| \le{}& Q\int_{B_R} 2\eta|D_s\eta||\psi(\Delta_h v)|\,dx + \tau\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h(Dv)|^2\,dx\\
&+\frac{1}{4\tau}\int_{B_R}\eta^2\psi'(\Delta_h v)\,dx + Q\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h v|(1+|\Delta_h v|)\,dx.
\end{aligned}
$$

In equation (3.3) we have $J_1 = J_2 + J_3$. Taking all previous inequalities we easily get

$$
\begin{aligned}
(3.5)\quad & 2\nu\int_{B_R}\int_0^1 \frac{G(\xi_h^t)}{|\xi_h^t|^2}\,\eta^2\psi'(\Delta_h v)|\Delta_h(Dv)|^2\,dt\,dx\\
&\quad -\nu\int_{B_R}\int_0^1 \frac{G(\lambda_h^t)}{|\lambda_h^t|^2}\,\eta^2\psi'(\Delta_h v)|\Delta_h(D_s v)|^2\,dt\,dx\\
&\quad -\tau\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h(Dv)|^2\,dx\\
&\le Q\int_{B_R}2\eta|D_s\eta||\psi(\Delta_h v)|\,dx\\
&\quad + Q\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h v|(1+|\Delta_h v|)\,dx\\
&\quad + \frac{1}{4\tau}\int_{B_R}\eta^2\psi'(\Delta_h v)\,dx + \frac{p^2}{\nu(R-\rho)^2}\int_{B_R}\int_0^1 G(\lambda_h^t)\psi'(\Delta_h v)\,dt\,dx\\
&\quad + \frac{4p}{(R-\rho)^2}\int_{B_R}\int_0^1\frac{G(\lambda_h^t)}{|\lambda_h^t|}|\Delta_h v|\psi'(\Delta_h v)\,dt\,dx.
\end{aligned}
$$

Obviously, $|\Delta_h(D_s v)|^2 \le |\Delta_h(Dv)|^2$, thus the left hand side of (3.5) is bounded from below by

$$
\int_{B_R}\int_0^1\left(2\nu\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\frac{G(\lambda_h^t)}{|\lambda_h^t|^2} - \tau\right)\eta^2\psi'(\Delta_h v)|\Delta_h(Dv)|^2\,dt\,dx.
$$

Step 2. Uniform estimates. Now we want to show that the right side of (3.5) is bounded by quantities that do not depend on h. It is easy to obtain upper bounds for the last three summands. It follows immediately from Lemma 2.2, properties of ψ and η and recall that $|h| \le h_0$ that

$$
\begin{gathered}
Q\int_{B_R}2\eta|D_s\eta||\psi(\Delta_h v)|\,dx \le \frac{2c_{\psi'}Q}{R-\rho}\int_{B_{R+h_0}}|D_s v|\,dx,\\
Q\int_{B_R}\eta^2\psi'(\Delta_h v)|\Delta_h v|(1+|\Delta_h v|)\,dx \le c_{\psi'}Q\int_{B_{R+h_0}}|D_s v| + |D_s v|^2\,dx,\\
\frac{1}{4\tau}\int_{B_R}\eta^2\psi'(\Delta_h v)\,dx \le \frac{c_{\psi'}}{4\tau}|B_R|,\\
\frac{p^2}{\nu(R-\rho)^2}\int_{B_R}\int_0^1 G(\lambda_h^t)\psi'(\Delta_h v)\,dt\,dx \le \frac{p^2 c_{\psi'}}{\nu(R-\rho)^2}\int_{B_{R+h_0}}G(Dv)\,dx.
\end{gathered}
$$

The last inequality holds since from definition of λ_h^t we have

$$\int_{B_R}\int_0^1 G(\lambda_h^t)\,dt\,dx \le \int_{B_{R+h_0}} G(Dv)\,dx.$$

Obtaining bound for the last summand of (3.5) requires more work. Let us define a sequence of functions $G_k\colon \mathbb{R}^n \to \mathbb{R}$ by

$$G_k(x) = \begin{cases} G(x) & \text{for } |x| \le k, \\ c_0|x|^2 & \text{for } |x| > k. \end{cases}$$

This sequence is nondecreasing and $G_k \le G$ by (G_3). Let χ_A denote the characteristic function of the set A. With this notation, we have

$$\begin{aligned}\int_{B_R}\int_0^1 \frac{G_k(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx &= \int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\le 1\}}\frac{G(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx \\ &\quad + \sum_{i=1}^{k-1}\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\in[i,i+1]\}}\frac{G(\lambda_h^t)}{|\lambda_h^t|}|\Delta_h v|\,dt\,dx \\ &\quad + \int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\ge k\}}\frac{c_0|\lambda_h^t|^2}{|\lambda_h^t|}|\Delta_h v|dt\,dx.\end{aligned}$$

For $|\lambda_h^t| \le 1$ we have $G(\lambda_h^t)/|\lambda_h^t| \le M$, where $M = \sup\limits_{|\xi|=1} G(\xi)$. Thus, by Lemma 2.5, we get $|\Delta_h v| \le 1$ and consequently,

$$\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\le 1\}}\frac{G(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx \le M|B_{R+h_0}|.$$

Similarly, for $|\lambda_h^t| \in [i, i+1]$, by Lemma 2.5 we get

$$\begin{aligned}&\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\in[i,i+1]\}}\frac{G(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx \\ &\qquad \le \int_{B_{R+|h|}}\int_0^1 \chi_{\{|\lambda_h^t|\in[i,i+1]\}}\frac{G(\lambda_h^t)}{i}(i+1)\,dt\,dx \\ &\qquad \le 2\int_{B_{R+h_0}}\int_0^1 \chi_{\{|\lambda_h^t|\in[i,i+1]\}}G(\lambda_h^t)\,dt\,dx.\end{aligned}$$

Summing these inequalities over i we obtain

$$\sum_{i=1}^{k-1}\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\in[i,i+1]\}}\frac{G(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx \le 2\int_{B_{R+2h_0}} G(Dv)\,dx.$$

Applying Lemma 2.2 and inequality $ab \le (a^2+b^2)/2$ we conclude that

$$\begin{aligned}\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\ge k\}}|\lambda_h^t||\Delta_h v|\,dt\,dx &\le \frac{1}{2}\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\ge k\}}|\lambda_h^t|^2\,dt\,dx \\ &\quad + \frac{1}{2}\int_{B_R}\int_0^1 \chi_{\{|\lambda_h^t|\ge k\}}|\Delta_h v|^2\,dt\,dx \le \int_{B_{R+h_0}} |Dv|^2\,dx.\end{aligned}$$

Hence

$$\int_{B_R}\int_0^1 \frac{G_k(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx \le M|B_{R+h_0}| + 2\int_{B_{R+2h_0}} G(Dv)\,dx + \int_{B_{R+h_0}} |Dv|^2\,dx.$$

From the above inequality, Monotone Convergence Theorem and the definition of G_k we get

$$\begin{aligned}
&\int_{B_R}\int_0^1 \frac{G(\lambda_h^t)}{|\lambda_h^t|}|\Delta_h v|\psi'(\Delta_h v)dt\,dx\\
&\le c_{\psi'}\int_{B_R}\int_0^1 \frac{G(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx = c_{\psi'}\lim_{k\to\infty}\int_{B_R}\int_0^1 \frac{G_k(\lambda_h^t)}{|\lambda_h^t|}\,|\Delta_h v|\,dt\,dx\\
&\le c_{\psi'}\left(M|B_{R+h_0}| + 2\int_{B_{R+2h_0}} G(Dv)\,dx + \int_{B_{R+h_0}} |Dv|^2\,dx\right).
\end{aligned}$$

Finally, the right hand side of (3.5) can be bounded from above by quantities independent of h:

$$\begin{aligned}
(3.6)\qquad &\int_{B_R}\int_0^1 \eta^2\psi'(\Delta_h v)\left(2\nu\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\frac{G(\lambda_h^t)}{|\lambda_h^t|^2} - \tau\right)|\Delta_h(Dv)|^2\,dt\,dx\\
&\le \frac{c_{\psi'}Q}{R-\rho}\int_{B_{R+h_0}} |D_s v|\,dx + c_{\psi'}Q\int_{B_{R+h_0}} |D_s v| + |D_s v|^2\,dx\\
&\quad + \frac{c_{\psi'}}{4\tau}|B_R| + \frac{p^2 c_{\psi'}}{\nu(R-\rho)^2}\int_{B_{R+h_0}} G(Dv)\,dx\\
&\quad + c_{\psi'}\left(M|B_{R+h_0}| + 2\int_{B_{R+2h_0}} G(Dv)\,dx + \int_{B_{R+h_0}} |Dv|^2\,dx\right).
\end{aligned}$$

Step 3. Proof of inequality (1.1). We first prove that for sufficiently small τ from any sequence of $h \to 0$ we can extract a subsequence such that

$$2\nu\,\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\,\frac{G(\lambda_h^t)}{|\lambda_h^t|^2} - \tau > \nu c_0 - \tau > 0 \quad \text{almost everywhere.}$$

If $\xi_h^t \to Dv$ and $\lambda_h^t \to Dv$ in L^2 when $h \to 0$, then (after passing to a subsequence) ξ_h^t and λ_h^t are also convergentalmost everywhere. This gives

$$2\nu\,\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\,\frac{G(\lambda_h^t)}{|\lambda_h^t|^2} - \tau \to \nu\,\frac{G(Dv)}{|Dv|^2} - \tau \ge \nu c_0 - \tau > 0.$$

To finish this step we choose $\psi(t) = t$. Recall that $B_\rho \subset B_R$ and η is equal to 1 on B_ρ. Replacing B_R with B_ρ we obtain

$$\begin{aligned}
&\int_{B_R}\int_0^1 \eta^2\psi'(\Delta_h v)\left(2\nu\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\frac{G(\lambda_h^t)}{|\lambda_h^t|^2} - \tau\right)|\Delta_h(Dv)|^2\,dt\,dx\\
&\qquad\ge \int_{B_\rho}\int_0^1 \left(2\nu\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\frac{G(\lambda_h^t)}{|\lambda_h^2|^2} - \tau\right)|\Delta_h(Dv)|^2\,dt\,dx\\
&\qquad\qquad\ge (\nu c_0 - \tau)\int_{B_\rho} |\Delta_h(Dv)|^2\,dx.
\end{aligned}$$

Now we see that $\int_{B_\rho} |\Delta_h(Dv)|^2\,dx \le c$, where the constant c comes from (3.6), hence is independent of h. By Lemma 2.4 for $p=2$, there exist a second order weak derivative and $\int_{B_\rho} |D_s(Dv)|^2\,dx \le c$ for the same constant c.

Since h_0 was chosen arbitrary we can replace balls B_{R+h_0} and B_{R+2h_0} by B_R in upper bounds of (3.6). Additionally, by assumption (G_3) we have $|D_s v| \le 1 + G(Dv)/c_0$ and $|D_s v|^2 \le G(Dv)/c_0$. Hence we have the following upper bound for the right hand side of (3.6):

$$\begin{aligned}
&\frac{c_{\psi'}Q}{R-\rho}\int_{B_{R+h_0}} |D_s v|\,dx + c_{\psi'}Q\int_{B_{R+h_0}} |D_s v| \\
&\qquad + |D_s v|^2\,dx + \frac{c_{\psi'}}{4\tau}|B_R| \\
&\qquad + \frac{p^2 c_{\psi'}}{\nu(R-\rho)^2}\int_{B_{R+h_0}} G(Dv)\,dx \\
&\qquad + c_{\psi'}\bigg(M|B_{R+h_0}| + 2\int_{B_{R+2h_0}} G(Dv)\,dx + \int_{B_{R+h_0}} |Dv|^2\,dx\bigg) \\
&\le c\int_{B_R} 1 + G(Dv)\,dx.
\end{aligned}$$

Finally,

$$\int_{B_\rho} |D_s(Dv)|^2\,dx \le c\int_{B_R} 1 + G(Dv)\,dx,$$

Thus we have finished the proof of (1.1).

Step 4. Proof of inequality (1.2). We now turn to the proof of inequality (1.2). Applying bounds on η, $D\eta$ i ψ' to inequality (3.5) and using (3.2) we get

$$\begin{aligned}
&\int_{B_R}\int_0^1 \bigg(2\nu\,\frac{G(\xi_h^t)}{|\xi_h^t|^2} - \nu\,\frac{G(\lambda_h^t)}{|\lambda_h^t|^2} - \tau\bigg)\eta^2\psi'(\Delta_h v)|\Delta_h(Dv)|^2\,dt\,dx \\
&\quad\le c\int_{B_R}\int_0^1 \bigg(1 + 2|\Delta_h v| + |\Delta_h v|^2 + G(\lambda_h^t) + \frac{G(\lambda_h^t)}{|\lambda_h^t|}|\Delta_h v|\bigg)\psi'(\Delta_h v)\,dt\,dx \\
&\quad\le c\int_{B_R}\int_0^1 1 + \bigg(|\Delta_h v| + |\Delta_h v|^2 + G(\lambda_h^t) + \frac{G(\lambda_h^t)}{|\lambda_h^t|}|\Delta_h v|\bigg)\psi'(\Delta_h v)\,dt\,dx.
\end{aligned}$$

Note that $\Delta_h v$ converge to $D_s v$ and $\Delta_h(Dv)$ to $D_s(Dv)$ in L^2, by Lemma 2.3. In addition, observe that λ_h^t is a shift of Dv and ξ_h^t is a linear combination of Dv and a shift of Dv.

Hence λ_h^t, ξ_h^t, being shifts in argument, converge to Dv in L^1. In the same manner, any function g that depend on λ_h^t, ξ_h^t, converges in L^1 to an analogous function dependent on Dv. For example

$$\frac{G(\xi_h^t)}{|\xi_h^t|^2} \to \frac{G(Dv)}{|Dv|^2} \quad \text{in } L^1.$$

Thus in the above inequality we can pass to the limit with $h \to 0$ in L^1.

As in the proof of inequality (1.1), assumption (G_3) gives

$$\int_{B_R} \eta^2 \psi'(D_s v)\Big(\nu \frac{G(|Dv|)}{|Dv|^2} - \tau\Big)|D_s(Dv)|^2\,dx \leq c\int_{B_R} 1 + G(Dv)\psi'(D_s v)\,dx. \tag{3.7}$$

Again, by (G_3),

$$\nu\,\frac{G(Dv)}{|Dv|^2} - \tau \geq \frac{G(Dv)}{|Dv|^2}\Big(\nu - \frac{\tau}{c_0}\Big).$$

Therefore, for sufficiently small τ, we can rewrite (3.7) as

$$\int_{B_R} \eta^2 \psi'(D_s v)\,\frac{G(|Dv|)}{|Dv|^2}\,|D_s(Dv)|^2\,dx \leq c\int_{B_R} 1 + G(Dv)\psi'(D_s v)\,dx. \tag{3.8}$$

Now we shall prove that this inequality remains true if the assumptions that $\psi' < c_{\psi'}$ is dropped. Take $\widetilde{\psi}$ that satisfies all the previous assumptions on ψ but its derivative $\widetilde{\psi}'$ is unbounded. For such a $\widetilde{\psi}$ define a sequence $\widetilde{\psi}_k$ by:

$$\begin{aligned} \widetilde{\psi}_k(t) &= \widetilde{\psi}(t) \quad &&\text{for } |t| < k,\\ \widetilde{\psi}_k'(t) &= \widetilde{\psi}'(k) \quad &&\text{for } |t| \geq k. \end{aligned}$$

Every $\widetilde{\psi}_k$ has bounded derivative, thus inequality (3.8) holds for $\widetilde{\psi}_k$. By the Monotone Convergence Theorem, inequality (3.8) is satisfied also for $\widetilde{\psi}$.

Note that ψ' is even, thus we can rewrite inequality (3.8) as

$$\int_{B_R} \eta^2 \psi'(|D_s v|)\,\frac{G(Dv)}{|Dv|^2}|D_s(Dv)|^2\,dx \leq c\int_{B_R} 1 + G(Dv)\psi'(|D_s v|)\,dx. \tag{3.9}$$

Define $\Phi\colon (0,\infty) \to \mathbb{R}$ by

$$\Phi(t) = 1 + c_\Phi \int_0^t \sqrt{\psi'(\tau)}\cdot \tau^{\alpha/2-1}\,d\tau, \tag{3.10}$$

where $c_\Phi > 0$. Straightforward calculations and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ yields

$$|D(\eta\Phi(|D_s v|))|^2 \leq 2|D\eta|^2(\Phi(|D_s v|))^2 + 2\eta^2 c_\Phi^2 \psi'(|D_s v|)|D_s v|^{\alpha-2}|D(D_s v)|^2 \tag{3.11}$$

and

$$\begin{aligned}(\Phi(|D_s v|))^2 &\leq \Big(1 + c_\Phi\sqrt{\psi'(|D_s v|)}\cdot|D_s v|^{\alpha/2-1}\cdot|D_s v|\Big)^2\\ &\leq 2 + 2c_\Phi^2\psi'(|D_s v|)|D_s v|^\alpha \leq 2 + 2c\,\psi'(|D_s v|)G(Dv).\end{aligned} \tag{3.12}$$

From assumption (G_5) we have $|D_s v|^{\alpha-2} \leq cG(Dv)/|Dv|^2$. Applying (3.9), (3.11) and (3.12) we get

$$\begin{aligned}\int_{B_R} &|D(\eta\Phi(|D_s v|))|^2\,dx\\ &\leq 2\int_{B_R} |D\eta|^2(\Phi(|D_s v|))^2\,dx + 2\int_{B_R} \eta^2 c_\Phi^2 \psi'(|D_s v|)\,\frac{G(Dv)}{|Dv|^2}\,|D(D_s v)|^2\,dx\\ &\leq 2\int_{B_R} 2+2c\,\psi'(|D_s v|)G(Dv)\,dx + c\int_{B_R} 1+G(Dv)\psi'(|D_s v|)\,dx\\ &= c\int_{B_R} 1+G(Dv)\psi'(|D_s v|)\,dx.\end{aligned}$$

By Sobolev inequality and the definition of η,

$$\bigg(\int_{B_\rho} (\Phi(|D_s v|))^{2^*}\,dx\bigg)^{2/2*} \leq C\int_{B_R} |D(\eta\Phi(|D_s v|))|^2\,dx.$$

Combining this with the previous inequality we get

$$\begin{aligned}\bigg(\int_{B_\rho} (\Phi(|D_s v|))^{2^*}\,dx\bigg)^{2/2*} &\leq C\int_{B_R} |D(\eta\Phi(|D_s v|))|^2\,dx\\ &\leq c\int_{B_R} (1+G(Dv)\psi'(|D_s v|))\,dx.\end{aligned} \tag{3.13}$$

Choose $\gamma \geq 0$, $c_\Phi = \gamma + \alpha/2$ and

$$\psi(t) = \frac{1}{2\gamma+1}\,t^{2\gamma+1} \quad \text{for } t \geq 0.$$

Obviously, $\psi'(t) = t^{2\gamma}$. Now we have

$$\Phi(t) = 1 + \Big(\gamma + \frac{\alpha}{2}\Big)\int_0^t \tau^{\gamma+\alpha/2-1}\,d\tau = 1 + t^{\gamma+\alpha/2}$$

and thus

$$(\Phi(|D_s v|))^{2^*} \geq 1 + \bigg(\Big(\gamma+\frac{\alpha}{2}\Big)\int_0^{|D_s v|} \tau^{\gamma+\alpha/2-1}\,d\tau\bigg)^{2^*} = 1 + |D_s v|^{2^*(\gamma+\alpha/2)}.$$

With the above inequality and the chosen ψ we rewrite inequality (3.13) as

$$\bigg(\int_{B_\rho} 1 + |D_s v|^{2^*(\gamma+\alpha/2)}\,dx\bigg)^{2/2^*} \leq c\int_{B_R} 1 + G(Dv)|D_s v|^{2\gamma}\,dx.$$

Clearly,

$$\int_{B_\rho} 1 + |D_s v|^{2^*(\gamma+\alpha/2)}\,dx \leq c\bigg(\int_{B_R} 1 + G(Dv)|D_s v|^{2\gamma}\,dx\bigg)^{2^*/2}.$$

Adding the above inequality over $s = 1, \ldots, n$ and using inequality

$$\sum_{i=i}^n a_i^\beta \leq \bigg(\sum_{i=1}^n a_i\bigg)^\beta$$

we obtain

$$\tag{3.14} \begin{aligned} \int_{B_\rho} n + \sum_{s=1}^{n} |D_s v|^{2^*(\gamma+\alpha/2)}\, dx &\le c \sum_{s=1}^{n} \Big(\int_{B_R} 1 + G(Dv)|D_s v|^{2\gamma}\, dx \Big)^{2^*/2} \\ &\le c \Big(\int_{B_R} n + G(Dv) \sum_{s=1}^{n} |D_s v|^{2\gamma}\, dx \Big)^{2^*/2}. \end{aligned}$$

For all nonnegative and nondecreasing functions h_1 i h_2 following inequality

$$n \sum_{i=1}^{n} h_1(a_i) \cdot h_2(a_i) \ge \sum_{i=1}^{n} h_1(a_i) \cdot \sum_{i=1}^{n} h_2(a_i)$$

holds. We shall apply it to the left hand side of (3.14) together with the assumption (G_6).

Let A_1 denote the set $\{x : |Dv(x)| \ge 1\}$. We will give lower bounds for the left hand side of (3.14) previously splitting the integral into two integrals on sets $B_\rho \cap A_1$ and $B_\rho \setminus A_1$. For the first one we have

$$\tag{3.15} \begin{aligned} &\int_{B_\rho \cap A_1} n + \sum_{s=1}^{n} |D_s v|^{2^*(\gamma+\alpha/2)}\, dx \\ &\quad = \int_{B_\rho \cap A_1} n + \sum_{s=1}^{n} |D_s v|^{2^*(\alpha/2-1)+2} \cdot |D_s v|^{2^*(\gamma+1)-2}\, dx \\ &\quad \ge \int_{B_\rho \cap A_1} n + \frac{1}{n} \Big(\sum_{s=1}^{n} |D_s v|^{2^*(\alpha/2-1)+2} \Big) \Big(\sum_{s=1}^{n} |D_s v|^{2^*(\gamma+1)-2} \Big)\, dx \\ &\quad \ge c \int_{B_\rho \cap A_1} 1 + G(Dv) \Big(\sum_{s=0}^{n} |D_s v|^{2^*(\gamma+1)-2} \Big)\, dx. \end{aligned}$$

Recall that $M = \sup\limits_{|\xi|=1} G(\xi)$, $\gamma \ge 0$ and $2^* \ge 2$. It is easy to check that, for $|\xi| \le 1$,

$$\sum_{s=1}^{n} |\xi_s|^{2^*(\gamma+1)-2} \le |\xi|^2 \quad \text{and} \quad G(\xi) \sum_{s=1}^{n} |\xi_s|^{2^*(\gamma+1)-2} \le M.$$

Therefore

$$\begin{aligned} \int_{B_\rho \setminus A_1} n + \sum_{s=1}^{n} |D_s v|^{2^*(\gamma+\alpha/2)}\, dx &\ge \int_{B_\rho \setminus A_1} n\, dx \ge \frac{1}{2} \int_{B_\rho \setminus A_1} n + \frac{n}{M} M\, dx \\ &\ge \frac{1}{2} \int_{B_\rho \setminus A_1} n + \frac{n}{M}\, G(Dv) \Big(\sum_{s=0}^{n} |D_s v|^{2^*(\gamma+1)-2} \Big)\, dx. \end{aligned}$$

As a consequence we have the inequality

$$\int_{B_\rho} n + \sum_{s=1}^{n} |D_s v|^{2^*(\gamma+\alpha/2)}\, dx \ge c \int_{B_\rho} 1 + G(Dv) \Big(\sum_{s=0}^{n} |D_s v|^{2^*(\gamma+1)-2} \Big)\, dx.$$

Thus inequality (3.14) can be written in the form

$$\text{(3.16)}\quad \left(\int_{B_\rho} 1+G(Dv)\left(\sum_{s=0}^{n}|D_sv|^{2^*(\gamma+1)-2}\right)dx\right)^{2/2*} \leq c\int_{B_R} 1+G(Dv)\left(\sum_{s=1}^{n}|D_sv|^{2\gamma}\right)dx.$$

In order to show that $|Dv|\in L^\infty$ we introduce a sequence of integrals of increasing powers of $|D_sv|$. To do this we will use inequality (3.16).

Let us introduce the following notation: $\gamma_0=0$, $\gamma_{i+1}=2^*(\gamma_i+1)/2-1$ and $R_i=R/2+R/2^{i+1}$. It is easy to see that $\gamma_i=(2^*/2)^i-1$, $\gamma_i\to\infty$ and $R_i\to R/2$. If in (3.16) we replace γ with γ_i, R with R_i and ρ with R_{i+1} then we get

$$\text{(3.17)}\quad \left(\int_{B_{R_{i+1}}} 1+G(Dv)\left(\sum_{s=0}^{n}|D_sv|^{2\gamma_{i+1}}\right)dx\right)^{2/2^*} \leq c\int_{B_{R_i}} 1+G(Dv)\left(\sum_{s=1}^{n}|D_sv|^{2\gamma_i}\right)dx.$$

Observe that in the left hand side of (3.17) we have higher powers of $|D_sv|$ then the powers on the right hand side. Now let

$$E_i=\left(\int_{B_{R_i}} 1+G(Dv)\left(\sum_{s=1}^{n}|D_sv|^{2\gamma_i}\right)dx\right)^{1/(\gamma_i+1)}.$$

In particular,

$$E_0=\int_{B_R} 1+G(Dv)\,dx.$$

By the definition of γ_i we have $(\gamma_{i+1}+1)/(\gamma_i+1)=2^*/2$. It follows by (3.17) that

$$E_{i+1}\leq c^{1/(\gamma_i+1)}E_i\leq\left(\prod_{j=0}^{i}c^{1/(\gamma_j+1)}\right)E_0.$$

An easy computations shows that

$$\lim_{i\to\infty}\prod_{j=0}^{i}c^{1/(\gamma_j+1)}=\exp\left(\ln c\sum_{j=0}^{\infty}\left(\frac{2}{2^*}\right)^j\right)=c^{1/(1-2/2^*)}.$$

It follows that

$$\infty>c^{1/(1-2/2^*)}E_0=c^{1/(1-2/2^*)}\int_{B_R} 1+G(Dv)\,dx\geq\lim_{i\to\infty}E_{i+1}.$$

Recall that $B_{R/2} \subset B_{R_{i+1}}$. Using ($G_3$) and Lemma 2.5 we obtain

$$\begin{aligned}\lim_{i\to\infty} E_{i+1} &= \lim_{i\to\infty}\left(\int_{B_{R_{i+1}}} 1 + G(Dv)\left(\sum_{s=1}^{n} |D_s v|^{2\gamma_{i+1}}\right) dx\right)^{1/(\gamma_{i+1}+1)} \\ &\geq \lim_{i\to\infty}\left(c_0 \int_{B_{R/2}} \left(|Dv|^2 \sum_{s=1}^{n} |D_s v|^{2\gamma_{i+1}}\right) dx\right)^{1/(\gamma_{i+1}+1)} \\ &\geq \lim_{i\to\infty}\left(c_0 \int_{B_{R/2}} \left(\sum_{s=1}^{n} |D_s v|^{2\gamma_{i+1}+2}\right) dx\right)^{1/(\gamma_{i+1}+1)} = \operatorname*{ess\,sup}_{B_{R/2}} |Dv|^2.\end{aligned}$$

Finally,

$$\infty > c\int_{B_R} 1 + G(Dv)\, dx = cE_0 \geq \lim_{i\to\infty} E_{i+1} = \operatorname*{ess\,sup}_{B_{R/2}} |Dv|^2,$$

which proves that $|Dv| \in L^\infty$ and the proof of the inequality (1.2) is finished.

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