

## Reconfiguring Minimum Dominating Sets in Trees

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### Abstract

We provide tight bounds on the diameter of  $\gamma$ -graphs, which are reconfiguration graphs of the minimum dominating sets of a graph  $G$ . In particular, we prove that for any tree  $T$  of order  $n \geq 3$ , the diameter of its  $\gamma$ -graph is at most  $n/2$  in the single vertex replacement adjacency model, whereas in the slide adjacency model, it is at most  $2(n-1)/3$ . Our proof is constructive, leading to a simple linear-time algorithm for determining the optimal sequence of “moves” between two minimum dominating sets of a tree.

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## 1 Introduction

For a vertex  $v$  of a (simple) graph  $G = (V_G, E_G)$ , its *neighborhood*, denoted by  $N_G(v)$ , is the set of all vertices adjacent to  $v$ . The cardinality of  $N_G(v)$ , denoted by  $d_G(v)$ , is termed the *degree of  $v$* . A vertex of degree one is termed a *leaf*, and the only neighbor of a leaf is called its *support vertex* (or simply, its *support*). If a support vertex has at least two leaves as neighbors, we call it a *strong support*, otherwise it is a *weak support*. A set of vertices  $D \subseteq V_G$  of  $G$  is *dominating* if every vertex in the set  $V_G - D$  has a neighbor in  $D$ . The cardinality of a minimum dominating set in  $G$  is termed the *domination number of  $G$*  and denoted by  $\gamma(G)$ , and any minimum dominating set of  $G$  is referred to as a  $\gamma$ -set.

Over the years, researchers have published thousands of papers on domination in graphs, exploring the topic in a variety of contexts. In particular, quite recently, two closely related concepts of reconfiguration graphs of the minimum dominating sets were introduced. In both of these variants, for a given graph  $G$ , the vertex set of the reconfiguration graph is the collection of all  $\gamma$ -sets of  $G$ ; however, the difference lies in the adjacency concept. Namely, in the *single vertex replacement adjacency model*, introduced in 2008 by Subramanian and Sridharan [16], two  $\gamma$ -sets  $X$  and  $Y$  of  $G$  are adjacent if there are vertices  $x \in X$  and  $y \in Y$  such that  $X - \{x\} = Y - \{y\}$ , whereas in the *slide adjacency model*, introduced by Fricke et al. [5] in 2011, it is required that, in addition,  $xy \in E_G$ . The single vertex replacement adjacency model was further studied in [10, 14, 15], and the slide adjacency model was further studied in [2, 3, 4]. Finally, reconfiguration graphs for dominating sets that are not necessarily minimum or for other models of domination have also been considered, see for example [1, 7, 8, 9, 11, 12, 17].

Herein, we focus on reconfiguration graphs of trees. For simplicity of presentation, we shall assume that in the two aforementioned models, both the reconfiguration graphs are termed the  $\gamma$ -graphs and denoted by  $\Gamma_G$  because the model under consideration is always either clear from the context, or not relevant. In 2011, Fricke et al. [5] posed the following question (among others, just as interesting, some of them having been already solved completely, see [3, 4, 13]): *In the slide adjacency model, is  $\text{diam}(\Gamma_T) = O(n)$  for any tree  $T$  of order  $n$ ?* The partial answer for so-called caterpillars with one leg and for trees of diameter at most five was given by Bień [2], and only in 2018, Edwards et al. [4] answered the question in an affirmative way for all trees.

**Theorem 1** [4] *For any tree  $T$  of order  $n$ ,  $\text{diam}(\Gamma_T) \leq 2\gamma(T) \leq n$  in the single vertex replacement adjacency model, whereas in the slide adjacency model,  $\text{diam}(\Gamma_T) \leq 2(2\gamma(T) - |S_T|) \leq 2(n - 2)$ , where  $S_T$  is the set of support vertices in  $T$ .*

However, the upper bounds established in Theorem 1 are not tight; in the single vertex replacement adjacency model, the best lower bound is  $n/2$  [4] (being attained by the corona graph of a tree [6]), whereas in the slide adjacency



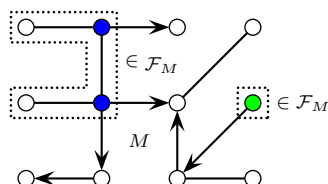


Figure 1: The mixed tree  $M$  has two arc-separators, marked with blue and green, respectively.

model it is  $2(n-1)/3$  [5] (being attained by the path of order  $n = 3k+1$ ,  $k \geq 1$ ). Therefore, in this paper, we undertake their study and close down these gaps. Namely, our result is the following theorem:

**Theorem 2** For any tree  $T$  of order  $n \geq 3$ , we have  $\text{diam}(\Gamma_T) \leq \gamma(T) - |S_T''| \leq n/2$  in the single vertex replacement adjacency model, whereas in the slide adjacency model,  $\text{diam}(\Gamma_T) \leq \min\{2(\gamma(T) - |S_T''|) - |S_T'|, 2(n-1)/3\}$ , where  $S_T'$  (resp.  $S_T''$ ) is the set of weak (resp. strong) support vertices in  $T$ .

**Notation.** For a vertex  $v$  of a graph  $G = (V_G, E_G)$ , the *closed neighborhood* of  $v$ , denoted by  $N_G[v]$ , is the set  $N_G(v) \cup \{v\}$ , and for a subset  $X \subseteq V_G$  of vertices, the *neighborhood* of  $X$ , denoted by  $N_G(X)$ , is defined to be  $\bigcup_{v \in X} N_G(v)$ , and the *closed neighborhood* of  $X$ , denoted by  $N_G[X]$ , is the set  $N_G(X) \cup X$ . Next, for a vertex  $v \in X$ , the *private neighborhood of  $v$  with respect to  $X$*  is the set  $\text{pn}_G(v, X) = N_G[v] - N_G[X - \{v\}]$ , that is, the set of vertices that are in the closed neighborhood of  $v$ , but are not in the closed neighborhood of any other vertex in  $X$ . A vertex in  $\text{pn}_G(v, X)$  is referred as to a *private neighbor* of  $v$  (with respect to  $X$ ), and private neighbor of  $v$  is *external* if it is distinct from  $v$  itself. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of  $G$  are denoted by  $L_G$ ,  $S_G'$ ,  $S_G''$ , and  $S_G$ , respectively.

For a mixed tree  $M = (V_M, E_M, A_M)$ , the sets of tails and heads of arcs in  $A_M$  are denoted by  $V_M^\circ$  and  $V_M^\blacktriangleright$ , respectively (notice that  $v \in V_M$  may be an element of both  $V_M^\circ$  and  $V_M^\blacktriangleright$ ). Next, let  $\mathcal{F}_M$  be the family of all maximal connected arc-free subgraphs of  $M$ , and let  $R = (V_R, E_R) \in \mathcal{F}_M$  be a subgraph of  $M$  such that  $V_R \cap V_M^\blacktriangleright = \emptyset$ . Then the set  $S_R = V_R \cap V_M^\circ$  is called an *arc-separator in  $M$* , whereas the graph  $R$  itself — the *certificate graph of  $S_R$* ; see Fig. 1 for an illustration. Observe that for any two distinct arc-separators  $S_1$  and  $S_2$  in  $M$ , we have  $S_1 \cap S_2 = \emptyset$ , and moreover, there is neither edge  $uv \in E_M$  nor arc  $(u, v) \in A_M$ , nor arc  $(v, u) \in A_M$  such that  $u \in S_1$  and  $v \in S_2$ .

**Observation 1** Every mixed tree possesses an arc-separator.

A rooted tree is a pair  $(T, r)$ , for simplicity denoted by  $T_r$ , where  $T = (V_T, E_T)$  is a tree and  $r \in V_T$  is a distinguished vertex termed the *root*. A vertex  $x \in V_T$  is labelled an *ancestor* of a vertex  $y$  in  $T_r$  if  $x$  belongs to the

unique path joining  $y$  and  $r$ , and if, in addition,  $xy \in E_T$ , then  $x$  is a *parent* of  $y$ . Next, symmetrically, the terms *descendant* of  $x$  and *child* of  $x$ , respectively, are used to describe such a vertex  $y$ . Note that  $x$  is both an ancestor and a descendant of itself. Finally, we use  $T_r(x)$  to describe the subtree of  $T_r$  induced by the descendants of  $x$  and rooted at  $x$ .

## 2 The proof of Theorem 2

The statement is trivially valid for the case of  $\gamma(T) = 1$ . Thus, assume now that  $T = (V_T, E_T)$  is a tree of order  $n \geq 4$ , with  $\gamma(T) \geq 2$ . We start with a simple general lemma.

**Lemma 1** *Let  $X$  and  $Y$  be two distinct minimal dominating sets of a graph  $G$ . If  $X - \{x\} = Y - \{y\}$  for some  $x \in X$  and  $y \in Y$ , then:*

- a)  $1 \leq \text{dist}_G(x, y) \leq 2$  holds;
- b) *If the girth of  $G$  is at least five, that is,  $G$  is acyclic or the shortest cycle in  $G$  is of the length at least five, then  $|\text{pn}(x, X) - \{x\}| \leq 1$  as well as  $|\text{pn}(y, Y) - \{y\}| \leq 1$ .*

**Proof:** (a) Because  $X$  and  $Y$  are minimal dominating sets of  $G$  and  $X - \{x\} = Y - \{y\}$ , we have that  $\text{pn}_G(x, X) = \text{pn}_G(y, Y) \neq \emptyset$ . Consequently,  $N_G(x) \cap N_G(y) \neq \emptyset$ , and hence  $1 \leq \text{dist}_G(x, y) \leq 2$ . (b) Next, if  $|\text{pn}(x, X) - \{x\}| \geq 2$  or  $|\text{pn}(y, Y) - \{y\}| \geq 2$ , then  $G$  would have a cycle of length three or four, which is a contradiction.  $\square$

The idea of our proof of Theorem 2 is to treat a  $\gamma$ -set of the tree  $T$  as a set of  $k$  *tokens*, where  $k = \gamma(T)$ , that can be relocated within  $T$ , in discrete time steps, maintaining domination of the tree. Specifically, assume  $V_T = \{1, 2, \dots, n\}$  and let  $D$  be the  $\gamma$ -set of  $T$  with the following property. When  $D$  is represented as the ordered  $k$ -tuple  $(v_1^D, \dots, v_k^D)$  of vertices in  $V_T$ , with  $v_{i-1}^D < v_i^D$ ,  $i \in [k] - \{1\}$ <sup>1</sup>, then the sequence  $v_1^D \dots v_k^D$  is lexicographically the smallest one over the alphabet  $V_T$ , taken over all  $\gamma$ -sets of  $T$ . Next, let the  $k$  tokens, where  $k = \gamma(T)$ , be once labeled with identifying numbers  $1, \dots, k$ , which we shall refer to as  $Id_i$ ,  $i \in [k]$ . Finally, let us initially locate these  $k$  tokens in such a way that the (unique) vertex occupied by the token  $Id_i$  is  $v_i^D$ ,  $i \in [k]$ . Because the  $\gamma$ -graph of a tree  $T$  is connected [5], in both adjacency models, any sequence of consecutive (feasible) vertex replacements/slides (*moves*), starting from the set  $D$  and finishing at another  $\gamma$ -set of  $T$ , may be thought of as relocating our  $k$ -tokens, keeping their identifiers unchanged. In other words, we may uniquely associate any  $\gamma$ -set  $X$  of  $T$  with the ordered  $k$ -tuple  $(v_1^X, \dots, v_k^X)$ , where  $v_i^X$  is the vertex occupied by token  $Id_i$ . Following this convention, we observe that for any two (ordered)  $\gamma$ -sets  $X$  and  $Y$  of  $T$ , vertices  $X$  and  $Y$  are adjacent in the graph  $\Gamma_T$  if and only if for all but one  $i \in [k]$ ,  $v_i^X = v_i^Y$  holds. Next, for  $i \in [k]$ ,

<sup>1</sup>Herein, we use the convention that  $[k]$  stands for the index set  $\{1, 2, 3, \dots, k\}$ ,  $k \geq 1$ .

let  $V_T^i$  be the set of all vertices that can ever be occupied by token  $Id_i$ , that is,  $V_T^i = \{v_i^X : X \text{ is a } \gamma\text{-set of } T\}$  (we emphasize that the set  $D$  defining the token labeling remains fixed).

**Lemma 2** *For any  $i \in [k]$ , the relevant vertex sets  $V_T^i$  are the same in both adjacency models. In particular, the induced subgraph  $T[V_T^i]$  is connected for any  $i \in [k]$  (in both adjacency models).*

**Proof:** Due to the fact that every  $\gamma$ -graph in the slide adjacency model is a spanning subgraph of the relevant  $\gamma$ -graph in the single vertex replacement adjacency model [2], all we need is to argue that in the latter model, if  $X$  and  $Y$  are two adjacent  $\gamma$ -sets in the  $\gamma$ -graph of  $T$ , then a single move of a token in  $T$  from a vertex in  $X$  to a vertex in  $Y$  can be simulated by at most two subsequent moves of that token in the former model.

Let  $X - \{x\} = Y - \{y\}$  for some  $x \in X, y \in Y$ . Assume without loss of generality that  $\text{dist}_T(x, y) = 2$  (see Lemma 2). First, observe that the unique vertex  $z \in N_T(x) \cap N_T(y)$  neither belongs to  $X$  nor to  $Y$  (otherwise, the set  $X - \{x\} (= Y - \{y\})$  would be a smaller dominating set of  $T$ , which is a contradiction). Next, the minimality of  $X$  and  $Y$  combined with Lemma 1 implies that  $\text{pn}(x, X) = \{z\} = \text{pn}(y, Y)$ , and hence the set  $Z = (X - \{x\}) \cup \{z\}$  is a  $\gamma$ -set of  $T$ , being adjacent to both  $X$  and  $Y$  in the  $\gamma$ -graph of  $T$ . Therefore, because  $\text{dist}_T(x, z) = \text{dist}_T(z, y) = 1$ , a single move of a token in  $T$  from  $x$  to  $y$  can be simulated by two subsequent moves of that token (from  $x$  to  $z$  and then from  $z$  to  $y$ ) in the slide adjacency model, as required.  $\square$

In the following sequence of lemmas we describe other properties of the sets  $V_T^i$ . These will be useful for the proof of Theorem 2.

**Lemma 3**  $V_T^i \cap V_T^j = \emptyset$  for any distinct  $i, j \in [k]$  (in both adjacency models).

**Proof:** By Lemma 2, we may restrict ourselves only to the slide adjacency model. Suppose on the contrary that there exist distinct  $i, j \in [k]$  such that  $V_T^i \cap V_T^j \neq \emptyset$ . Let  $\Pi$  be any (finite) walk in  $\Gamma_T$  starting at the  $\gamma$ -set  $D$  and traversing the edges of  $\Gamma_T$  until all vertices in  $\cup_{t=1}^k V_T^t$  have been visited/occupied by tokens (tokens are moving with respect to the  $\gamma$ -sets visited along the walk); clearly, such a walk  $\Pi$  exists as  $\Gamma_T$  is connected [5]. Because  $V_T^i \cap V_T^j \neq \emptyset$ , there exist two  $\gamma$ -sets of  $T$  being adjacent along  $\Pi$ , say  $Y$  and  $Z$ , such that one of the tokens, say  $Id_a$ , is moved from a vertex of  $T$ , say  $y$ , and placed for the first time at another vertex of  $T$ , say  $z$ , that has already been visited by another token, say  $Id_b$ , with  $b \neq a$ . Let  $X$  be the  $\gamma$ -set of  $T$  with  $Id_b$  occupying vertex  $z$  for the first time along the walk  $\Pi$ . Consider now the rooted subtree  $T' = T_z(y)$  of  $T_z$ , and, symmetrically, the rooted subtree  $T'' = T_y(z)$  of  $T_y$ , see Fig. 2 for an illustration. From the choice of  $y$  and  $z$ , acyclicity of  $T$  and  $\text{dist}_T(y, z) = 1$ , it follows that:

- $Z \cap V_{T'}$  dominates all vertices in  $V_{T'} - \{y\}$  and  $|Z \cap V_{T'}| = |Y \cap V_{T'}| - 1$ ;

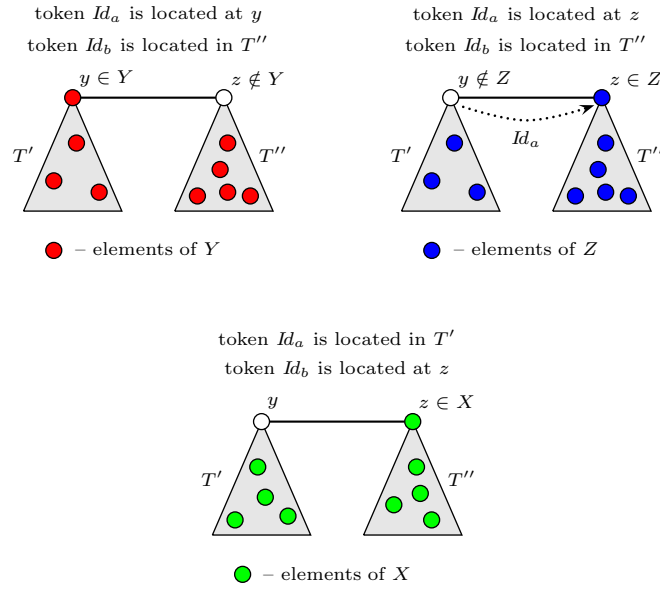


Figure 2: The set  $S = (Z \cap V_{T'}) \cup (X \cap V_{T''})$  is a dominating set of  $T$ , and  $|S| = \gamma(T) - 1$ ; notice that  $y$  may belong to  $X$ .

- $X \cap V_{T''}$  dominates all vertices in  $V_{T''} \cup \{y\}$ , and  $|X \cap V_{T''}| = |Y \cap V_{T''}|$ .

Consequently, because  $V_T = V_{T'} \cup V_{T''}$  and  $V_{T'} \cap V_{T''} = \emptyset$ , the set  $S = (Z \cap V_{T'}) \cup (X \cap V_{T''})$  is a dominating set of  $T$  with  $|S| = \gamma(T) - 1$ , which is a contradiction.  $\square$

**Lemma 4** For any  $i \in [k]$ , the distance between any two vertices in  $V_T^i$  is at most two in  $T$  (in both adjacency models).

**Proof:** By Lemma 2, we may again restrict ourselves only to the slide adjacency model. Suppose to the contrary that for some  $i \in [k]$ , there are two vertices  $y, z \in V_T^i$  such that  $\text{dist}_T(y, z) = 3$  (notice that in our supposition, we may, without loss of generality, restrict ourselves to vertices at the distance three because  $T[V_T^i]$  is connected by Lemma 2). Let  $\pi = v_0 v_1 v_2 v_3$  be the shortest path between  $v_0 = y$  and  $v_3 = z$  in  $T$ . Let  $Y$  and  $Z$  be two  $\gamma$ -sets of  $T$  such that token  $Id_l$  is located at vertex  $v_0 (= y)$  and at vertex  $v_3 (= z)$ , respectively. Consider the rooted subtree  $T' = T_{v_2}(v_1)$  of  $T_{v_2}$  and the rooted subtree  $T'' = T_{v_1}(v_2)$  of  $T_{v_1}$ , see Fig. 3 for an illustration. Now, because  $T$  is a tree,  $T[V_T^i]$  is connected (by Lemma 2), and  $V_T^i \cap V_T^j = \emptyset$  for any distinct  $i, j \in [k]$  (by Lemma 3), we observe that vertices  $v_1, v_2 \notin Y$  and  $v_1, v_2 \notin Z$ . Consequently:

- $Z \cap V_{T'}$  dominates all vertices in  $V_{T'}$  and  $|Z \cap V_{T'}| = |Y \cap V_{T'}| - 1$ ;

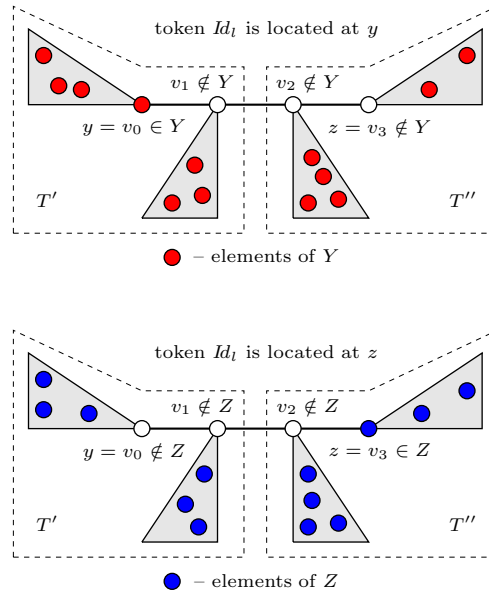


Figure 3: The set  $S = (Z \cap V_{T'}) \cup (Y \cap V_{T''})$  is a dominating set of  $T$ , and  $|S| = \gamma(T) - 1$ .

- $Y \cap V_{T''}$  dominates all vertices in  $V_{T''}$ .

Consequently, because  $V_T = V_{T'} \cup V_{T''}$  and  $V_{T'} \cap V_{T''} = \emptyset$ , the set  $S = (Z \cap V_{T'}) \cup (Y \cap V_{T''})$  is a dominating set of  $T$  with  $|S| = \gamma(T) - 1$ , which is a contradiction.  $\square$

**Lemma 5** *If  $s \in S'_T$ , then there exists  $i_s \in [k]$  such that  $V_T^{i_s} \subseteq \{s, l_s\}$ , where  $l_s$  is the unique leaf adjacent to  $s$  in  $T$ , and so  $\text{diam}(T[V_T^{i_s}]) \leq 1$  (in both adjacency models).*

**Proof:** By Lemma 2, we may focus only on the slide adjacency model. Let  $X$  be a  $\gamma$ -set of  $T$  such that  $s \in X$  (clearly, such a  $\gamma$ -set exists) and let  $Id_{i_s}$  be the token located at vertex  $s$ . It follows from the minimality of  $X$  that no other token occupies the leaf  $l_s$ . Therefore, in order to move  $Id_{i_s}$  from  $s$  to a vertex distinct from the leaf  $l_s$  in  $T$  while maintaining domination of  $l_s$ , there must have already been located another token at  $s$ , together with  $Id_{i_s}$ , which contradicts Lemma 3.  $\square$

**Lemma 6** *If  $s \in S''_T$ , then there exists  $i_s \in [k]$  such that  $V_T^{i_s} = \{s\}$ , and so  $\text{diam}(T[V_T^{i_s}]) = 0$  (in both adjacency models).*

**Proof:** It follows by arguments analogous to those in the proof of Lemma 5.  $\square$

We say that two (ordered)  $\gamma$ -sets  $X = (v_1^X, \dots, v_k^X)$  and  $Y = (v_1^Y, \dots, v_k^Y)$  of the given tree  $T$  are *inconsistent* at the coordinate  $i \in [k]$  if  $v_i^X \neq v_i^Y$ ; such a coordinate  $i$  itself, the vertices  $v_i^X$  and  $v_i^Y$  as well as the token  $Id_i$  are then also referred to as *inconsistent*, whereas the set  $X - (X \cap Y)$  of all inconsistent vertices in  $Y$  (with respect to  $Y$ ) is denoted by  $\text{In}(X, Y)$ , respectively.

Let  $X$  and  $Z$  be two (different) inconsistent  $\gamma$ -sets of the tree  $T$  (and so  $\text{In}(X, Z) \neq \emptyset$ ), and let  $M = (V_M, E_M, A_M)$  be the mixed tree, with the vertex set  $V_M = V_T$ , the edge set  $E_M$  and the arc set  $A_M$ , respectively, resulting from  $T$  by assigning the orientation to the edges (towards  $v_i^Z$ ) on the shortest path between  $v_i^X$  and  $v_i^Z$ , for each  $v_i^X \in \text{In}(X, Z)$ . Let  $R = (V_R, E_R)$  be the certificate graph of some arc-separator in  $M$  (such a graph  $R$  exists by Observation 1, and it is a subgraph of both  $T$  and  $M$ ). We have a sequence of observations.

- (A) In the mixed tree  $M$ , all maximal directed paths are vertex-disjoint and of length of at most two (by combining Lemma 2, Lemma 3, and Lemma 4).
- (B) Therefore,  $\text{In}(X, Z) = X - (X \cap Z) \subseteq V_M^\circ$  and  $Z - (X \cap Z) \subseteq V_M^\blacktriangleright - V_M^\circ$ , and thus  $(Z - (X \cap Z)) \cap V_R = \emptyset$  (by the definition of a certificate graph); in other words, there is no inconsistent vertex in  $Z$  that belongs to  $V_R$ .
- (C) Finally, it follows from the definition of an arc-separator that if  $l$  is a leaf of  $R$ , then  $l$  is a leaf of  $T$  or  $l = v_i^X$  ( $\neq v_i^Z$ ) for some inconsistent coordinate  $i \in [k]$ . Notice that in the former case,  $l = v_j^X = v_j^Z$  for some  $j \in [k]$  may also hold.

Next, let  $U_d$  denote the set of all inconsistent vertices  $v_i^X \in \text{In}(X, Z)$  such that  $\text{dist}_T(v_i^X, v_i^Z) = d$ ; notice  $d \in \{1, 2\}$  by Lemma 4. Observe that (see Fig. 4 for an illustration):

- (D) Because  $Z$  is a  $\gamma$ -set of  $T$  and  $\text{dist}_T(v_i^X, v_i^Z) \geq 1$  for every  $v_i^Z \in Z - (Z \cap X)$ , the set  $(Z \cap V_R) - U_2$  dominates all vertices in  $V_R$ , and so does the set  $(X \cap V_R) - U_2$  (because  $Z \cap V_R = X \cap V_R$  by the definition of an arc-separator). In other words, for the purpose of domination of  $R$ , vertices in the set  $\{v_i^Z : v_i^X \in U_2\} \subseteq Z - (Z \cap X)$  are useless.
- (E) By similar arguments, the set  $(Z \cap V_R) - \text{In}(X, Z)$  dominates all vertices in  $V_R - \text{In}(X, Z)$ , and so does the set  $(X \cap V_R) - \text{In}(X, Z)$ . In other words, no vertex in  $U_1 (= \text{In}(X, Z) \cap (V_R - U_2))$  has an external private neighbor in  $V_R$ , that is, any such vertex may be required only to dominate itself in  $R$ .
- (F) Finally,  $N_T(x_i) \cap (V_T - V_R) \subseteq N_T(z_i) \cap (V_T - V_R)$ .

Consequently, tokens at inconsistent vertices in  $\text{In}(X, Z) \cap V_R$  can be slid along the relevant arcs of  $M$  (recall that all maximal directed paths in  $M$  are vertex-disjoint), in a sequence, in total number  $|U_1| + 2|U_2|$  of slides, to make all of them consistent, and the resulting set  $Y$  is a  $\gamma$ -set of  $T$  (by the properties



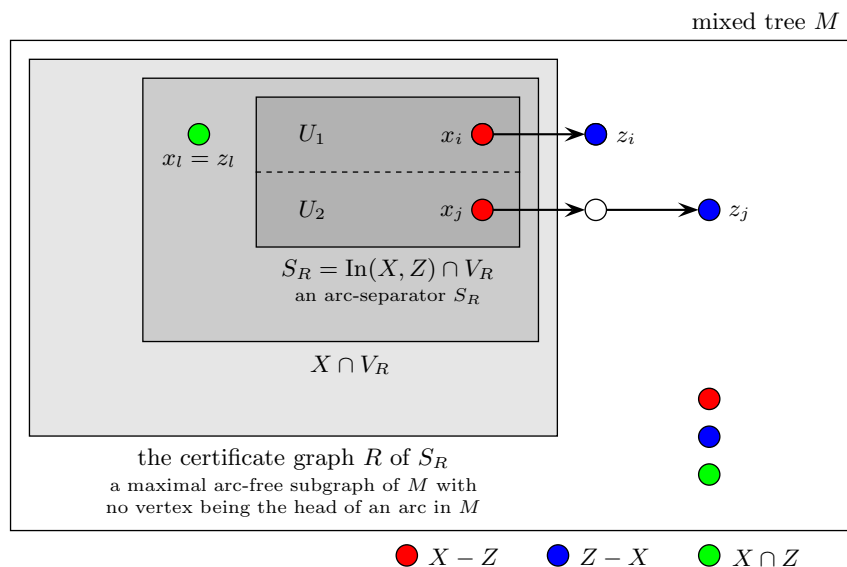


Figure 4: The set  $(X \cap V_R) - U_2$  dominates all vertices in  $V_R$ , while the set  $(X \cap V_R) - \text{In}(X, Z)$  dominates all vertices in  $V_R - \text{In}(X, Z)$ .

discussed above), with  $|\text{In}(Y, Z)| < |\text{In}(X, Z)|$ . Applying this approach repeatedly will eventually move all tokens from their initial positions  $v_1^X, \dots, v_k^X$  to the desired positions  $v_1^Z, \dots, v_k^Z$ , and — supported by Lemmas 4-6 — we may conclude that in the single vertex replacement adjacency model, the number of jumps is at most  $\gamma(T) - |S_T''| \leq n/2$ , and so  $\text{diam}(\Gamma_T) \leq \gamma(T) - |S_T''| \leq n/2$  in this model, whereas in the slide adjacency model, the number of slides is at most  $2(\gamma(T) - |S_T|) + |S_T'|$ , and hence  $\text{diam}(\Gamma_T) \leq 2(\gamma(T) - |S_T|) + |S_T'|$  in that model, as required.

Regarding the slide adjacency model and bounding the diameter of  $\Gamma_T$  in terms of the number of vertices, taking into account Lemmas 5 and 6, first observe that there are at least  $|S_T| \geq 2$  tokens that require at most  $|S_T|$  slides in total to make them consistent (recall that  $T$  is a tree of order at least four and  $\gamma(T) \geq 2$ ). Next, if the number of slides to make a token  $Id_i$  consistent is equal to 2, then  $|V_T^i| \geq 3$ , and hence the number of such “expensive” tokens is at most  $(|V_T| - 2|S_T|)/3 \leq (n - 4)/3$  (by Lemma 3). Therefore, a simple calculus shows that the maximum (total) number of slides is at most  $2 + 2(n - 4)/3 = 2(n - 1)/3$ , which finishes the proof of Theorem 2.

**Remark.** Let us note that the statements of Lemmas 3–4 cannot be carried over to the class of arbitrary graphs. As an example, consider the cycle  $G = C_{3k+1}$  in which  $V_G^i = V_G$  for any token  $Id_i$  (defined with respect to the  $\gamma$ -set  $D$ ).



### 3 Algorithmic result

Observe that in the proof of Theorem 2, the relevant graph  $R$  can be extended and defined to be the union of the certificate graphs of an arbitrary number of (distinct) arc-separators in the mixed tree  $M$ . This is a core property that gives rise to a simple linear-time algorithm for determining the optimal sequence of jumps between two minimum dominating sets of a tree. The algorithm consists of three phases: pre-processing, assigning levels and final phase.

**Pre-processing Phase.** We identify pairs of vertices  $(x_i, z_i) \in X \times Z$ , each of which corresponds to the placement of the (unique by Lemma 3) token  $Id_i$ .

In that phase (see Fig. 5(a,b) for an illustration), we perform a DFS-based approach starting from a leaf  $l \in L_T$ , and for each vertex  $v \in V_T$ , we recursively determine the number  $n_v^X$  (resp.  $n_v^Z$ ) of inconsistent vertices in the rooted subtree  $T_l(v)$  that belong to a  $\gamma$ -set  $X$  (resp. to a  $\gamma$ -set  $Z$ ). Notice that  $|n_v^X - n_v^Z| \leq 1$  (because otherwise, vertex  $v$  must have been visited by two distinct tokens by Lemma 4 — a contradiction with Lemma 3). Next, using these data, starting from the same leaf  $l$ , the second pass of DFS is sufficient to identify the aforementioned pairs of vertices. More specifically, for the currently handled vertex  $v$  (in a post-order manner while performing DFS), assuming that the  $i - 1$  pairs  $(x, z) \in X \times Z$  has already been identified in all subtrees of  $T_l(v)$  rooted at the children of  $v$  (if any), the following rules can be applied (they are exhaustive and distinct by Lemma 3 and Lemma 4).

- If  $v \in X \cap Z$ , then  $x_i := v$  and  $z_i := v$ . (Notice that  $n_v^X = n_v^Z$  in this case.)
- If  $v \in X - Z$  and  $n_v^X = n_v^Z$ , then  $x_i := v$ , whereas  $z_i$  is assigned the unique non-associated yet vertex in  $T_l(v)$  that belongs to  $Z$ .
- If  $v \in Z - X$  and  $n_v^X = n_v^Z$ , then  $z_i := v$ , whereas  $x_i$  is assigned the unique non-associated yet vertex in  $T_l(v)$  that belongs to  $X$ .
- Otherwise, continue: no vertices are associated, but if  $v \in X$ , then  $v$  is marked as “non-associated  $x$ ”, and if  $v \in Z$  then it is marked as “non-associated  $z$ ”.

**Assigning Levels Phase.** We assigns levels to vertices/tokens in  $X$ . These levels will constitute the ordering that the tokens will move with respect to.

Let  $M = (V_M, E_M, A_M)$  be the mixed tree defined in the proof of Theorem 2 (Section 2), resulting from  $T$  by assigning the orientation to the edges (towards  $z_i$ ) on the shortest path between  $x_i$  and  $z_i$ , for each  $x_i \in \text{In}(X, Z)$ ; see Fig. 6(a)

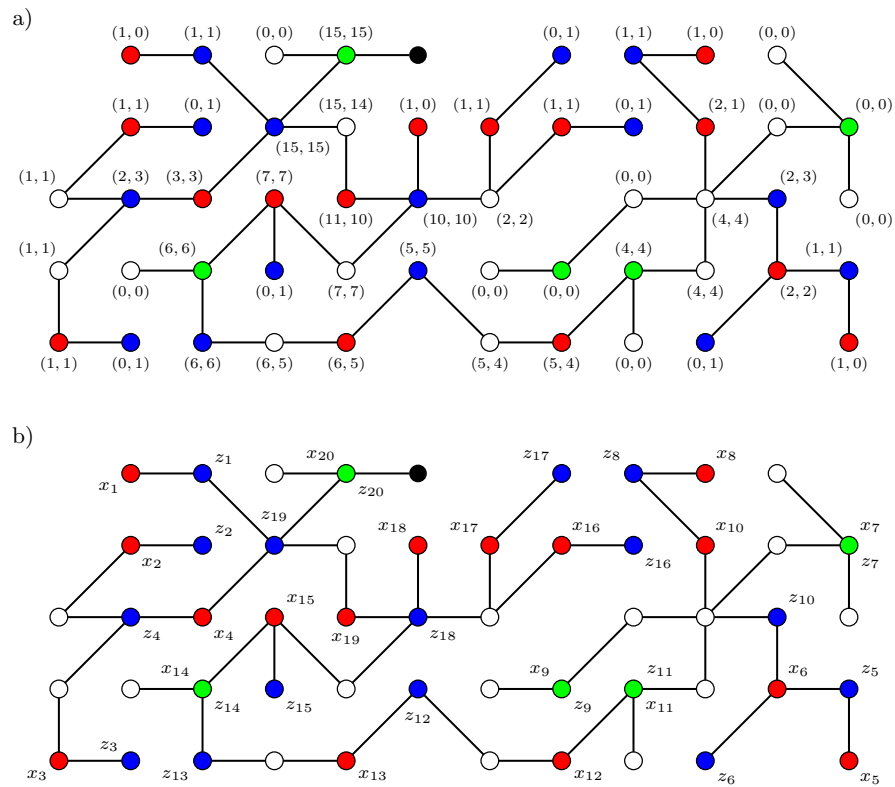


Figure 5: Pre-processing Phase. A tree  $T$  with  $\gamma(T) = 20$  and the  $\gamma$ -sets  $X$  and  $Z$  of  $T$ :  $X - Z$  is marked red,  $Z - X$  is marked blue, and  $X \cap Z$  is marked green. (a) Determining the numbers  $n_v^X$  and  $n_v^Z$  (depicted as pairs  $(n_v^X, n_v^Z)$ ), starting at the black leaf). (b) Identifying the pairs  $(x_i, z_i) \in X \times Z$ ; herein, children of a vertex are visited in a counterclockwise manner, with respect to the given plane embedding of  $T$ .

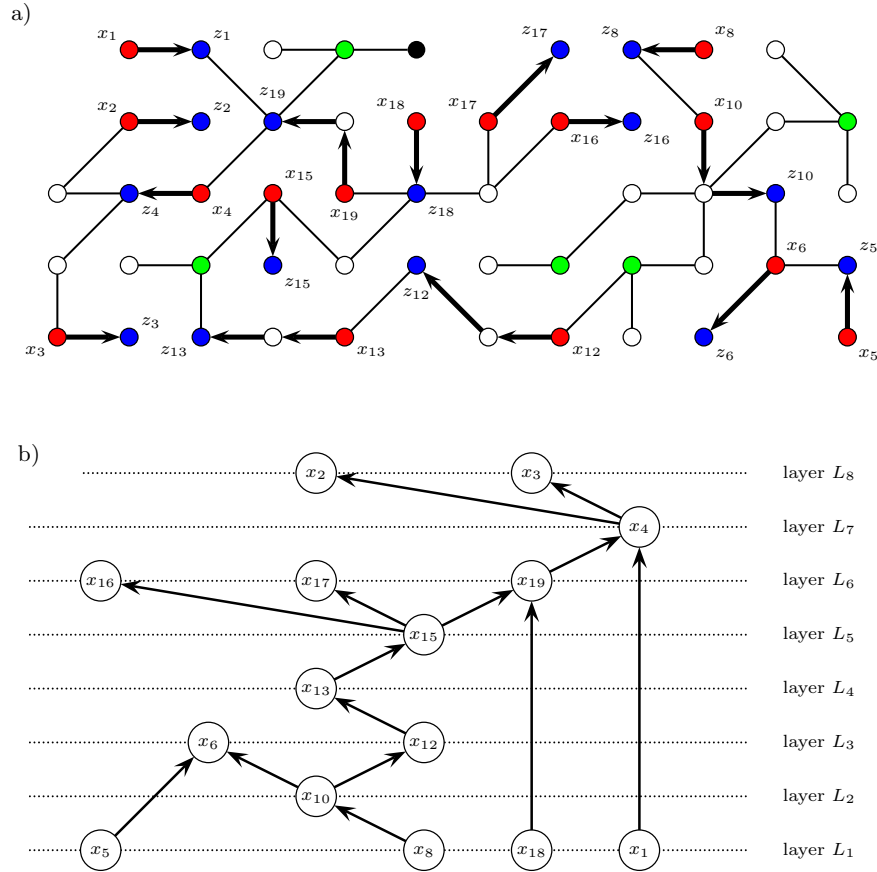


Figure 6: Assigning Levels Phase. (a) The mixed tree  $M$ . (b) The Hasse diagram  $H = (\text{In}(X, Z), A)$  of  $\langle \text{In}(X, Z), \prec \rangle$ .

for an illustration. Recall that all directed paths in  $M$  are vertex-disjoint and of a length of at most two. Define the partially ordered set  $\langle \text{In}(X, Z), \prec \rangle$ , where for two distinct  $x_i, x_j \in \text{In}(X, Z)$ ,  $\prec x_j$  if and only if there is no arc-free path between  $x_i$  and  $x_j$  in  $M$  and all the arcs on the (unique) path between  $x_i$  and  $x_j$  are oriented towards  $x_j$ . Next, consider the transitive reduction  $H = (\text{In}(X, Z), A)$  of  $\langle \text{In}(X, Z), \prec \rangle$  in the form of the Hasse diagram, with the layers  $L_1, \dots, L_t$ , where  $t \leq \gamma(T)$  (notice that because  $T$  is a tree, such a transitive reduction exists); see Fig. 6(b). These layers define now the labeling of inconsistent vertices in  $X$ : if  $x_i \in L_k$ , then  $x_i$  is assigned the level  $k$ . Observe that  $H$  is not necessarily connected, but it is a directed forest, that is, its underlying undirected graph is a forest (because  $T$  is a tree). Moreover, it can be computed, together with the layers  $L_1, \dots, L_t$ , in linear time by applying the third pass of a DFS-based approach on the tree  $T$ .

**Final Phase.** We move tokens from  $x_i$  to  $z_i$  with respect to the increasing order of the assigned levels to inconsistent vertices.

Before we proceed with the correctness proof of our 3-phase algorithm, let us point out that it was not our intention to optimize the number of DFS-phases in our algorithm. Therefore, we believe that with respect to this criterion, some improvement is possible, and we eventually conclude our paper with the following theorem.

**Theorem 3** *Given two  $\gamma$ -sets  $X$  and  $Z$  of a tree  $T$ , an optimal sequence of jumps through which  $X$  can be transformed into  $Z$  can be computed in linear time (in both adjacency models).*

**Proof:** For a level  $l \in \{1, \dots, t\}$ , let  $Y_{l+1}$  denote the set resulting from moving all tokens in  $L_l$  to the relevant vertices in  $Z$ . It follows from the definition/construction that  $Y_{t+1} = Z$ , and for each  $l \in \{1, \dots, t-1\}$ ,  $L_{l+1} \subseteq Y_{l+1}$  and  $\text{In}(Y_{l+1}, Z) = \text{In}(X, Z) - \bigcup_{i=1}^l L_i$ .

Due to the fact that  $L_1$  is the set of minimal elements in  $\langle \text{In}(X, Z), \prec \rangle$ ,  $L_1$  is the sum of a number of arc-separators in the mixed tree  $M_1 = M$  (exploited in Phase 2 and defined in the proof of Theorem 2). Consequently, it follows from the proof of Theorem 2 (i.e., the arguments from the paragraph just after Lemma 6) that the set  $Y_2$ , resulting from moving tokens located at inconsistent vertices in  $L_1$  towards the relevant vertices in  $Y_2$  (and so in  $Z$ ), in any order, is a  $\gamma$ -set of  $T$ .

But the same argument can be inductively (successively) applied to all the  $\gamma$ -sets  $Y_l$  and  $Z$ ,  $l \in \{1, 2, \dots, t\}$ , and the partially ordered set  $\langle \text{In}(Y_l, Z), \prec \rangle$ , defined now with respect to  $Y_l$  and  $Z$ . Namely, observe that  $L_{l+1}$  is the set of minimal elements in  $\langle \text{In}(Y_{l+1}, Z), \prec \rangle$ , which implies that  $L_{l+1}$  is the sum of a number of arc-separators in the relevant mixed tree  $M_l$  (defined now with respect to  $Y_l$  and  $Z$ ). Consequently, it follows from the proof of Theorem 2 that the set  $Y_{l+1}$  is a  $\gamma$ -set of  $T$  for each  $l \in \{1, \dots, t\}$ . Therefore, moving tokens with respect to the increasing order of the assigned levels to inconsistent vertices constitutes a feasible optimal reconfiguration of the  $\gamma$ -set  $X$  into the  $\gamma$ -set  $Z$ .

Finally, with respect to the complexity issue, all we need is to observe that all three phases can clearly be accomplished in linear time.  $\square$

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