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MOMENTS OF HERMITE–GAUSSIAN FUNCTIONALS

Abstract. Moments of finite products of Hermite–Gaussian functionals are expressed by covariances of a Gaussian sequence.

Introduction. Mixed moments of Hermite–Gaussian functionals play an important role in stochastic analysis of Wiener chaos (for extensive treatment of the ideas corresponding to Wiener chaos, also those regarding moments, see [J], [PT]). In this paper, we present a new method of computing such moments. It allows us to formulate a necessary and sufficient condition (see Proposition 2.1 below) for vanishing of a moment of even order in the case of non-negative correlations of Gaussian random variables from Wiener chaos.

1. Hermite polynomials. Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, equipped with the standard inner product $(\cdot,\cdot)_d$ and the Euclidean norm $\|\cdot\|_d$. Let $(\Omega,\mathcal{F},P)$ be a fixed probability space. The Hermite polynomial H_n of degree $n\ge 1$ on $\mathbb{R}$ is defined by

$$H_n(x)=(-1)^n\exp(x^2/2)\frac{d^n}{dx^n}(\exp(-x^2/2)),\qquad x\in\mathbb{R},\ n\ge 1.$$

Additionally, we assume that $H_0\equiv 1$. The first Hermite polynomials are $H_1(x)=x$, $H_2(x)=x^2-1$. The polynomials H_n divided by $n!$ are the coefficients of the expansion in powers of t of the generating function $w(t,x)=\exp(tx-t^2/2)$, $x,t\in\mathbb{R}$. In fact, we have

$$w(t,x)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x),\qquad x,t\in\mathbb{R}. \tag{1.1}$$

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Note that for a standard Gaussian variable η we have

$$w(t,x) = \exp(tx)E\exp(it\eta) = E\exp(tx+it\eta) = E\sum_{n=0}^{\infty}\frac{t^n}{n!}(x+i\eta)^n.$$

Now, using the Lebesgue dominated convergence theorem and comparing the above expansion with (1.1), we get

$$H_n(x) = E(x+i\eta)^n, \qquad x\in\mathbb{R},\ n\geq 0. \tag{1.2}$$

Hence,

$$\sum_{n=0}^{\infty}\left|\frac{t^n}{n!}H_n(x)\right| \leq \sum_{n=0}^{\infty}\frac{|t|^n}{n!}E[(|x|+|\eta|)^n] \leq E\exp[|t|(|x|+|\eta|)] < \infty. \tag{1.3}$$

Therefore, the sum in (1.1) converges absolutely for all $t,x\in\mathbb{R}$.

Another well known relationship between Hermite polynomials and Gaussian random variables is the result below (see [N]).

LEMMA 1.1. *Let (X,Y) be a two-dimensional Gaussian vector such that $E(X)=E(Y)=0$, $E(X^2)=E(Y^2)=1$, $E(XY)=\rho$, where ρ is the correlation coefficient of X and Y. Then, for all $n,m\geq 0$,*

$$E[H_n(X)H_m(Y)] = \begin{cases} n!\rho^n & \text{if } n=m,\\ 0 & \text{if } n\neq m. \end{cases} \quad \blacksquare$$

Now, let $X=(X_1,\dots,X_d)$ be a Gaussian random vector such that $E(X_i)=0$ and $E(X_i^2)=1$ for $i=1,\dots,d$. The aim of this note is to compute the expectation

$$E[H_{n_1}(X_1)H_{n_2}(X_2)\cdots H_{n_d}(X_d)].$$

To formulate our result, we need some notations and definitions. For $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ and $k=(k_1,\dots,k_d)\in\mathbb{N}_0^d=(\mathbb{N}\cup\{0\})^d$, we write

$$|x|=\sum_{i=1}^{d}x_i,\qquad x^k=\prod_{i=1}^{d}x_i^{k_i},\qquad |k|=\sum_{i=1}^{d}k_i,\qquad k!=\prod_{i=1}^{d}k_i!.$$

For $k=(k_1,\dots,k_d)\in\mathbb{N}_0^d$ the integer $|k|$ will be called the *length* of the vector k. The set of all square matrices of dimension d with elements from $\mathbb{R}$ (resp. $\mathbb{N}_0$) is denoted by $\mathcal{M}_d(\mathbb{R})$ (resp. $\mathcal{M}_d(\mathbb{N}_0)$). If $R\in\mathcal{M}_d(\mathbb{R})$, the jth column and ith row are denoted by R_j and R^i respectively. From time to time, we shall use the shorthand notation $R=[R_j^i]$. As usual, we identify rows and columns of R with vectors from $\mathbb{R}^d$. If $R\in\mathcal{M}_d(\mathbb{R})$ and $K\in\mathcal{M}_d(\mathbb{N}_0)$, we denote

$$|K|=(|K^1|,\dots,|K^d|),\qquad |R|=(|R^1|,\dots,|R^d|),$$

$$K! = K^1! \cdots K^d! = \prod_{i,j=1}^{d} K_j^i!, \qquad R^K = {R^1}^{K^1} \cdots {R^d}^{K^d} = \prod_{i,j=1}^{d} {R_j^i}^{K_j^i}$$

with the convention $0^0 = 1$. For $K = [K_j^i] \in \mathcal{M}_d(\mathbb{N}_0)$, let $u(K)$ denote the upper diagonal matrix of K, i.e.

$$u(K) := [U_j^i], \quad \text{where} \quad U_j^i := \begin{cases} K_j^i & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$$

For $n \in \mathbb{N}_0^d$ let us introduce the following families of matrices:

$$\mathcal{M}_d^0(\mathbb{N}_0) = \{K \in \mathcal{M}_d(\mathbb{N}_0) : \operatorname{diag}(K) = 0,\ K \text{ is symmetric}\},$$
$$\mathcal{M}_{d,n}^0(\mathbb{N}_0) = \{K \in \mathcal{M}_d^0(\mathbb{N}_0) : |K| = n\},$$

where $\operatorname{diag}(K)$ denotes the main diagonal of the matrix K.

The Hermite polynomials on $\mathbb{R}^d$ are defined as tensor products of the Hermite polynomials on $\mathbb{R}$: for $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we put

$$H_n(x) = \prod_{i=1}^{d} H_{n_i}(x_i).$$

Similarly to the one-dimensional case, the polynomials H_n divided by $n!$ are the coefficients of expansion in powers of $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ of the generating function

$$w(t,x) = \exp(-\|t\|_d^2/2 + (t,x)_d), \qquad t, x \in \mathbb{R}^d.$$

That is,

$$w(t,x) = \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} H_n(x), \qquad t, x \in \mathbb{R}^d.$$

2. Main result. We can now formulate the main result of this note.

THEOREM 2.1. *Let $X = (X_1, \ldots, X_d)$, $d \geq 2$, be a Gaussian random vector such that $E(X_i) = 0$ and $E(X_i^2) = 1$ for $i = 1, \ldots, d$. Then, for the Hermite polynomial H_n on $\mathbb{R}^d$ of degree $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ we have*

$$\begin{aligned} EH_n(X) &= E[H_{n_1}(X_1)H_{n_2}(X_2) \cdots H_{n_d}(X_d)] \\ &= \begin{cases} \sum_{K \in \mathcal{M}_{d,n}^0} \frac{n!}{\sqrt{K!}} Q^{u(K)} & \text{if } \mathcal{M}_{d,n}^0 \neq \emptyset, \\ 0 & \text{if } \mathcal{M}_{d,n}^0 = \emptyset, \end{cases} \end{aligned} \tag{2.4}$$

where Q denotes the covariance matrix of X.

Proof. From the definition of H_n and from (1.3), we conclude that

$$\begin{aligned} H_n(x) &= H_{n_1}(x_1)H_{n_2}(x_2) \cdots H_{n_d}(x_d) \\ &= E[(x_1 + i\eta_1)^{n_1}(x_2 + i\eta_2)^{n_2} \cdots (x_d + i\eta_d)^{n_d}], \end{aligned}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $\eta_1, \ldots, \eta_d$ is a sequence of independent standard Gaussian variables, independent of X. From the above and from (1.2) we deduce that (with E_{η_i} denoting the expectation with respect to η_i, $i = 1, \ldots, d$)

$$\begin{aligned} E \sum_{n\in\mathbb{N}_0^d} \left|\frac{t^n}{n!} H_n(X)\right| &= E \prod_{i=1}^d \sum_{n_1=0}^\infty \left|\frac{t_i^{n_i}}{n_i!} H_{n_i}(X_i)\right| \\ &\le E[E_{\eta_1} e^{|t_1|(|X_1|+|\eta_1|)} E_{\eta_2} e^{|t_2|(|X_2|+|\eta_2|)} \ldots E_{\eta_d} e^{|t_d|(|X_d|+|\eta_d|)}] \\ &\le E_{\eta_1} e^{|t_1\eta_1|} E_{\eta_2} e^{|t_2\eta_2|} \ldots E_{\eta_d} e^{|t_d\eta_d|} E e^{|t_1X_1|+|t_2X_2|+\cdots+|t_dX_d|} < \infty. \end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem, we have

$$Ew(t, X) = E \sum_{n\in\mathbb{N}_0^d} \frac{t^n}{n!} H_n(X) = \sum_{n\in\mathbb{N}_0^d} \frac{t^n}{n!} E[H_n(X)]. \tag{2.5}$$

On the other hand,

$$\begin{aligned} Ew(t, X) &= E \exp((t, X)_d - \|t\|_d^2/2) = \exp((Qt, t)_d/2 - \|t\|_d^2/2) \\ &= \exp\left(\tfrac{1}{2}((Q - I)t, t)_d\right) = \exp\Big(\sum_{1\le i<j\le d} \rho_{ij} t_i t_j\Big), \end{aligned}$$

where I is the identity operator on $\mathbb{R}^d$. Consequently,

$$Ew(t, X) = \sum_{m=0}^\infty \frac{1}{m!} \Big(\sum_{1\le i<j\le d} \rho_{ij} t_i t_j\Big)^m. \tag{2.6}$$

Let us compute the components of the above sum. For simplicity, denote by S_m the set of all vectors

$$k = (k_{12}, \ldots, k_{1d}, k_{23}, \ldots, k_{2d}, \ldots, k_{d-1,d}) \in \mathbb{N}_0^{d(d-1)/2}$$

such that $|k| = m$. It follows that

$$\begin{aligned} \Big(\sum_{1\le i<j\le d} \rho_{ij} t_i t_j\Big)^m &= \sum_{k\in S_m} \frac{m!}{k!} \rho_{12}^{k_{12}} \cdots \rho_{1d}^{k_{1d}} \rho_{23}^{k_{23}} \cdots \rho_{2d}^{k_{2d}} \cdots \rho_{d-1,d}^{k_{d-1,d}} \\ &\quad \times (t_1t_2)^{k_{12}} \cdots (t_1t_d)^{k_{1d}} (t_2t_3)^{k_{23}} \cdots (t_2t_d)^{k_{2d}} \cdots (t_{d-1}t_d)^{k_{d-1,d}} \\ &= \sum_{\substack{K\in\mathcal{M}_d^0 \\ \|K\|=m}} \frac{m!}{\sqrt{K!}} Q^{u(K)} t^{|K|}, \end{aligned}$$

where $\|K\| = |K^1| + \cdots + |K^d|$. From the above and from (2.6) we have

$$Ew(t, X) = \sum_{m=0}^\infty \sum_{\substack{K\in\mathcal{M}_d^0 \\ \|K\|=m}} \frac{1}{\sqrt{K!}} Q^{u(K)} t^{|K|} = \sum_{n\in\mathbb{N}_0^d} \sum_{K\in\mathcal{M}_{d,n}^0} \frac{1}{\sqrt{K!}} Q^{u(K)} t^n. \tag{2.7}$$

Now, comparing (2.5) and (2.7) we get (2.4), and the theorem follows. ∎

We see at once that $\mathcal{M}^0_{d,n} = \emptyset$ if $|n|$ is an odd integer. When $|n|$ is even, we have the result below.

Proposition 2.1. *Let $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ be such that $|n|$ is an even integer and let $n_{i_0} = \max_{1 \le i \le d} n_i$. Then*

$$\mathcal{M}^0_{d,n} \neq \emptyset \iff n_{i_0} \le \sum_{\substack{i=1 \\ i \neq i_0}}^{d} n_i.$$

Proof. Without loss of generality, we may assume that

$$n_1 \ge \cdots \ge n_d.$$

($\Rightarrow$) Assume that $n_1 > n_2 + \cdots + n_d$ and let $K \in \mathcal{M}^0_{d,n}$. Then the first row of K is

$$K^1 = (0, k_{12}, k_{13}, \dots, k_{1d}) \quad \text{and} \quad n_1 = |K^1| = k_{12} + k_{13} + \cdots + k_{1d}.$$

Hence, there exists $2 \le i \le d$ such that $k_{1i} > n_i$. Therefore, $k_{i1} = k_{1i} > n_i$ and $|K^i| > n_i$. Consequently, $|K| \neq n$ and this contradicts the assumption that $K \in \mathcal{M}^0_{d,n}$.

($\Leftarrow$) Notice first that if $n_1 = n_2 + \cdots + n_d$ then the matrix

$$K = \begin{bmatrix} 0 & n_2 & n_3 & \dots & n_d \\ n_2 & 0 & 0 & \dots & 0 \\ n_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_d & 0 & 0 & \dots & 0 \end{bmatrix}_{d \times d}$$

belongs to $\mathcal{M}^0_{d,n}$, so $\mathcal{M}^0_{d,n} \neq \emptyset$. Now, let $n_1 < n_2 + \cdots + n_d$. Then $p := n_2 + \cdots + n_d - n_1 > 0$ and we see at once that p is even. For our further considerations, the following lemma will be necessary.

Lemma 2.1. *Let $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ be a non-increasing sequence such that $|n|$ is even and $|n| > 2n_1$. Then there exists a sequence $m = (m_1, \dots, m_k) \in \mathbb{N}_0^k$ with $2k + 1 \le d$ such that*

$$s = (0, n_2 - m_1, n_3 - m_1, n_4 - m_2, n_5 - m_2, \dots,$$
$$n_{2k} - m_k, n_{2k+1} - m_k, n_{2k+2}, \dots, n_d) \in \mathbb{N}_0^d$$

and $|s| = n_2 + \cdots + n_d - 2|m| = n_1$.

Proof of Lemma 2.1. Let $\{e_i\}_{i=1}^d$ be the standard basis in $\mathbb{R}^d$ and

$$r := \begin{cases} (d+1)/2 & \text{if } d \text{ is odd}, \\ d/2 & \text{if } d \text{ is even}. \end{cases}$$

Moreover, define a sequence $\{p_j\}_{j=1}^r$ by

$$p_1 = 0, \qquad p_j = \sum_{i=2}^{j} n_{2i-1}, \qquad j = 2, \dots, r,$$

and a sequence $\{S^{(i)}\}_{i=0}^{p_r}$ of vectors in $\mathbb{N}_0^d$ as follows:

$$S^{(0)} = (0, n_2, n_3, \dots, n_d), \qquad S^{(p_j+l)} = S^{(p_j+l-1)} - e_{2j} - e_{2j+1},$$

where $1 \le l \le n_{2j+1}$, $j = 1, \dots, r-1$. It can be seen that for $j > 1$,

$$S^{(p_j+l)} = S^{(0)} - n_3 e_2 - n_3 e_3 - \dots - n_{2j-1} e_{2j-2} - n_{2j-1} e_{2j-1} - l e_{2j} - l e_{2j+1}.$$

From the definition of $\{S^{(i)}\}_{i=0}^{p_r}$ we have

$$|S^{(p_j+l)}| = |S^{(p_j+l-1)}| - 2, \qquad 1 \le l \le n_{2j+1},\ j = 1, \dots, r-1,$$

i.e. the lengths $|S^{(p_j+l)}|$ decrease in arithmetic progression with common difference 2. By assumption,

$$|S^{(0)}| = n_2 + n_3 + \dots + n_d > n_1.$$

On the other hand, for $r = (d+1)/2$,

$$\begin{aligned}
|S^{(p_r)}| &= |S^{(0)}| - 2(n_3 + n_5 + \dots + n_d) \\
&= n_2 + n_3 + \dots + n_d - 2(n_3 + n_5 + \dots + n_d) \\
&= n_2 + (n_3 + n_4) + (n_5 + n_6) + \dots + (n_{d-2} + n_{d-1}) \\
&\quad + n_d - 2(n_3 + n_5 + \dots + n_d) \\
&\le n_1 + 2n_3 + 2n_5 + \dots + 2n_{d-2} + 2n_d - 2(n_3 + n_5 + \dots + n_d) = n_1.
\end{aligned}$$

Similarly for $r = d/2$,

$$\begin{aligned}
&|S^{(p_r)}| = |S^{(0)}| - 2(n_3 + n_5 + \dots + n_{d-1}) \\
&= n_2 + n_3 + \dots + n_d - 2(n_3 + n_5 + \dots + n_{d-1}) \\
&= n_2 + (n_3 + n_4) + (n_5 + n_6) + \dots + (n_{d-1} + n_d) - 2(n_3 + n_5 + \dots + n_{d-1}) \\
&\le n_1 + 2n_3 + 2n_5 + \dots + 2n_{d-1} - 2(n_3 + n_5 + \dots + n_{d-1}) = n_1.
\end{aligned}$$

We conclude that there exists $1 \le i_0 \le p_r$ such that $|S^{(i_0)}| = n_1$. Put $a = S^{(0)} - S^{(i_0)}$. Then we can define

$$2k := \#\{a_i : a_i \ne 0\}$$

and

$$\begin{aligned}
m := (m_1, \dots, m_k) &:= (n_2, n_4, \dots, n_{2k}) - (S_2^{(i_0)}, S_4^{(i_0)}, \dots, S_{2k}^{(i_0)}) \\
&= (n_3, n_5, \dots, n_{2k+1}) - (S_3^{(i_0)}, S_5^{(i_0)}, \dots, S_{2k+1}^{(i_0)}).
\end{aligned}$$

From the construction of m and the definition of the vector s, we obtain

$$|s| = |S^{(0)}| - 2|m| = |S^{(0)}| - (|S^{(0)}| - |S^{(i_0)}|) = |S^{(i_0)}| = n_1. \ \blacksquare$$

Using Lemma 2.1 we can construct a vector $m = (m_1, \dots, m_k) \in \mathbb{N}_0^k$, where $2k+1 \leq d$, such that $m_i \leq n_{2i+1}$ for $i = 1, \dots, k$ and $2|m| = p$, i.e. $n_1 = n_2 + \cdots + n_d - 2|m|$. Therefore, we can find a matrix K which belongs to $\mathcal{M}^0_{d,n}$. Namely, we set $K = A - B + C$, where

$$A = \begin{bmatrix} 0 & n_2 & n_3 & \dots & n_d \\ n_2 & 0 & 0 & \dots & 0 \\ n_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_d & 0 & 0 & \dots & 0 \end{bmatrix}_{d\times d}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}_{d\times d},$$

where $B_{11} = [0]_{1\times 1}$, $B_{12} = [m_1\ m_1\ m_2\ m_2\ \dots\ m_k\ m_k\ 0\ \dots\ 0]_{1\times(d-1)}$ and B_{22} is a null $(d-1)\times(d-1)$ matrix, and finally

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & & 0 & \dots & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 & \dots & & 0 & \dots & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & \dots & & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & \dots & & 0 & \dots & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 & \dots & & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \vdots \\ 0 & & \dots & & & 0 & 0 & m_k & 0 & \dots & 0 \\ 0 & & \dots & & & 0 & m_k & 0 & 0 & \dots & 0 \\ 0 & & \dots & & & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & & \dots & & & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{d\times d}$$

Therefore, $\mathcal{M}^0_{d,n} \neq \emptyset$ and the proof of Proposition 2.1 is complete. ∎

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