On homotopies of morphisms and admissible mappings

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To the memory of A. Granas.

Abstract. The notion of homotopy in the category of morphisms introduced by Górniewicz and Granas is proved to be equivalence relation which was not clear for years. Some simple properties are proved and a coincidence point index is described.

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1. Introduction

In [9] Górniewicz and Granas defined a convenient category which allows to treat fixed point problems for a broad class of multivalued maps as coincidence problems for single-valued maps. Their notion of morphisms is still the object of interest in various aspects (see, e.g., [1,12,16–18,20,21]). The aim of this paper is to define a homotopy notion and to check its basic properties in their category. In particular, we prove the transitivity property of the homotopy what seems new (comp. the remark before Lemma 3.4. in [1, p. 602]). Note that in some papers, a different notion of homotopy is considered [8,13]. We prove some basic properties of the homotopy and also introduce a homotopy invariant—coincidence point index. This yields some known theorems on coincidence and fixed points (comp., e.g., [5,7,9,14,19,24]).

The index was in fact defined in [5] for a broader class of multivalued maps but then it appeared to be a set of indices. Note that our notion of homotopy appears to be not the usual one in the case of single-valued maps. It seems to be closer to the homology relation. Nevertheless, it is satisfactory from the point of view of index theory.

2. The Górniewicz–Granas category of morphisms

By a space, we understand a Hausdorff topological space and by a map—a continuous transformation.

**Definition 2.1.** A map $p : X \to Y$ is a **Vietoris map** provided

1. p is onto and closed (an image of a closed set is a closed set),
2. for each $y \in Y$ $p^{-1}(y)$ is a compact acyclic set (i.e., all its reduced Čech homology groups with rational coefficients are zero).

We will denote Vietoris maps by a double arrow $p : X \Longrightarrow Y$. We make use of the following basic properties of them. Observe that Vietoris maps are *proper* maps (i.e., the preimage of a compact set is compact).

Theorem 2.2. (see [11, Ch. VI §19 Thm 3.2]) *If $p : X \Longrightarrow Y$ then the induced linear map $p_* : H_*(X) \to H_*(Y)$ is an isomorphism. The composite of Vietoris maps is also a Vietoris map. The pullback of a Vietoris map is a Vietoris map.*

We use the Čech homology theory with compact carriers here and thereafter.

Given two spaces X and Y, we consider the set of all diagrams of the form $X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y$. Two such diagrams are said to be **equivalent** if there are maps $f : Z \to Z'$ and $g : Z' \to Z$ such that the following two diagrams commute
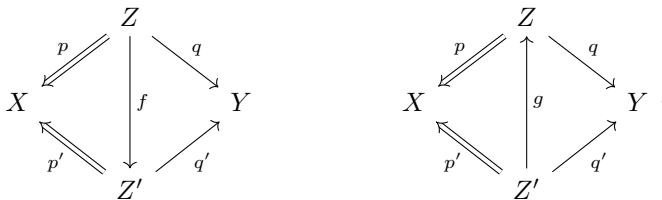
Clearly, this is an equivalence relation (see [9]). Let us note that in [8,13] a different (more restrictive) relation is considered. The maps f, g are required to be homeomorphisms there.

Definition 2.3. ([9]) An equivalence class of a diagram (p, q) with respect to the above relation is called a **morphism** and is denoted by

$$\varphi = \{X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y\} : X \longrightarrow Y.$$

Given two morphisms $\varphi = \{X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y\} : X \longrightarrow Y$ and

$\psi = \{Y \xleftarrow{\ p'\ } Z' \xrightarrow{\ q'\ } W\} : Y \longrightarrow W$, we write a commutative diagram:

$$Z \boxtimes Z' \xrightarrow{\bar{q}} Z' \xrightarrow{q'} W$$
$$\bar{p} \downarrow \qquad\qquad \downarrow p'$$
$$Z \xrightarrow{q} Y$$
$$p \downarrow$$
$$X$$

in which $Z \boxtimes Z' = \{(z, z') \in Z \times Z'; \; q(z) = p'(z')\}, \bar{p}(z, z') = z$ and $\bar{q}(z, z') = z'$. Then, we define the **composite** of φ and ψ to be a morphism $\quad \psi \circ \varphi = \{X \xleftarrow{p \circ \bar{p}} Z \boxtimes Z' \xrightarrow{q' \circ \bar{q}} W\} : X \longrightarrow W \quad$ ($\bar{p}$ is Vietoris being the pullback of p'). It is easy to check that the above definition is independent of the choice of representatives. The law of composition allows to regard the collection of morphisms on spaces as a category $\mathcal{M}$. The category of usual single-valued maps regarded as morphisms of the form $\{X \xleftarrow{id} X \longrightarrow Y\} \quad$ is a subcategory of $\mathcal{M}$. Górniewicz and Granas [9] established that the Čech homology functor extends to the category $\mathcal{M}$.

One easily see that morphisms can be treated as multivalued maps. We define an image of a point as the set $\varphi(x) = q\left(p^{-1}(x)\right)$ and it is obviously independent of the choice of representatives. We say that a morphism **determines** a multivalued map φ. Such a map φ is upper-semicontinuous (see, e.g., [19] for a detailed but easy proof). A broad class of multivalued maps can be viewed in this way. A multivalued map $\varphi : X \multimap Y$ is **acyclic** provided it is u.s.c. and the image of each point $\varphi(x)$ is an acyclic compact set.

Proposition 2.4. *A multivalued map $\varphi : X \multimap Y$ is determined by a morphism if and only if it is a composition of acyclic maps.*

Proof. Every acyclic map is determined by a morphism of the form

$$\varphi = \{X \xleftarrow{\pi_1} \Gamma_\varphi \xrightarrow{\pi_2} Y\} \;,$$

where Γ_φ is the graph of φ, $\Gamma_\varphi = \{(x, y) \in X \times Y; \; y \in \varphi(x)\}$ and π_1, π_2 are projections. Therefore, a composition of acyclic maps is determined by a composition of the above morphisms.

Now, given any morphism $\quad \varphi = \{X \xleftarrow{p} Z \xrightarrow{q} Y\} \quad$, it is sufficient to notice that a map $\psi : X \to Z, \psi(x) = p^{-1}(x)$ is acyclic and φ is a composition $\varphi = q \circ \psi$. $\qquad\square$

Remark 2.5. In [7], a broader class of **admissible** maps was considered. Those were maps admitting selections which were determined by morphisms.

Example. A multivalued map may be determined by different morphisms. Consider an acyclic map $\widetilde{\varphi} : S^n \to S^n, \; \widetilde{\varphi}(x) = \{y \in S^n; \; \|y - x\| \le \frac{3}{2}\}$, where $S^n = \{x \in \mathbb{R}^n; \; \|x\| = 1\}$.

It is determined by a morphism $\quad \varphi = \{S^n \xleftarrow{\pi_1} \Gamma_{\widetilde{\varphi}} \xrightarrow{\pi_2} S^n\}$. Therefore, the square $\varphi^2 = \varphi \circ \varphi$ determines a map $\widetilde{\psi} : S^n \to S^n, \widetilde{\psi}(x) = S^n$. But

$\widetilde{\psi}$ is also determined by any morphism of the form ψ^2,

$$\psi = \{S^n \xleftarrow{\ \pi_1\ } \Gamma_\varphi \xrightarrow{\ f \circ \pi_2\ } S^n\}\ ,$$

where $f : S^n \to S^n$ is a map of degree k.

It is easily seen that $\varphi_* : H_*(S^n) \to H_*(S^n)$ is an identity isomorphism and so is φ_*^2. But ψ_* is not an identity. Therefore, φ^2 and ψ^2 are different morphisms.

3. Homotopy of morphisms

There are various ways of defining a homotopy relation in the category $\mathcal{M}$. We have chosen the one which seems to be the most straightforward and has been used by some authors (see, e.g., [1,4,7]), though it was not obvious whether it was an equivalence relation.

Definition 3.1. Two diagrams

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y \ \text{ and } \ X \xleftarrow{\ p'\ } Z' \xrightarrow{\ q'\ } Y$$

are **homotopic** if there is a diagram $X \times [0,1] \xleftarrow{\ \bar{p}\ } \bar{Z} \xrightarrow{\ \bar{q}\ } Y$ and maps f_0, f_1 such that the following diagram commutes:

Here $i_0(x) = (x,0)$, $i_1(x) = (x,1)$.

Proposition 3.2. *The homotopy relation is an equivalence relation in the set of diagrams from X to Y.*

Proof. The only non-obvious property is the transitivity. Assume we have two commutative diagrams:
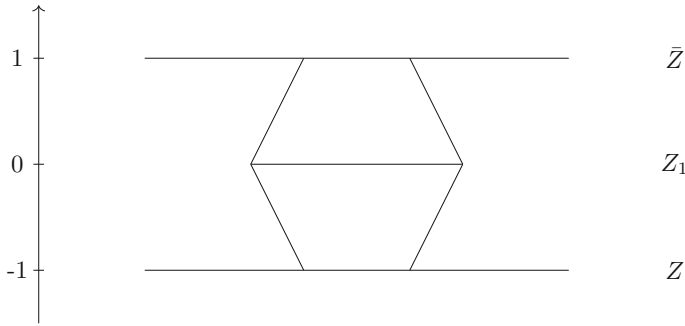

where $i_k(x) = (x,k)$.

FIGURE 1. The space K

We have the mapping cylinders:

$$Z_{f_1} = Z_1 \times [-1, 0] \cup Z/_\sim \ , \ (x, -1) \sim y \ \ iff \ \ y = f_1(x), x \in Z_1, y \in Z;$$

$$Z_{f_2} = Z_1 \times [0, 1] \cup \bar{Z}/_\sim \ , \ \ (x, 1) \sim y \ \ iff \ \ y = f_2(x), x \in Z_1, y \in \bar{Z}.$$

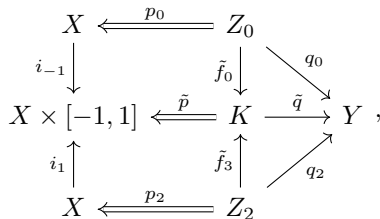Now, we define a space $K = Z_{f_1} \cup_{Z_1} Z_{f_2}$ which can be seen schematically in Fig. 1.

Define a map $\tilde{p} : K \Longrightarrow X \times [-1, 1]$ by a formula

$$\tilde{p}([x]) = \begin{cases} (p_1(x), t) & for \ [x] = [(x, t)], \ -1 \leq t \leq 1 \\ (\pi_1(p(x)), -1) & for \ x \in Z \\ (\pi_1(\bar{p}(x)), 1) & for \ x \in \bar{Z} \end{cases},$$

and a map $\tilde{q} : K \longrightarrow Y$ by

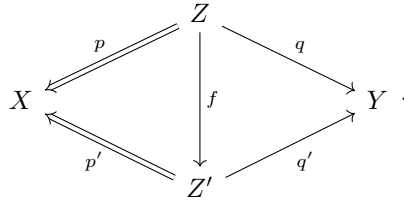$$\tilde{q}([x]) = \begin{cases} q_1(x) \ for \ [x] = [(x, t)], \ x \in Z_1 \\ q(x) \ \ for \ x \in Z \\ \bar{q}(x) \ \ for \ x \in \bar{Z} \end{cases}.$$

It follows immediately from the definition of K that $\tilde{p}, \tilde{q}$ are correctly defined continuous maps. Moreover, $\tilde{p}$ is a Vietoris map since the preimages of points are exactly the same as for p_1 whenever $-1 < t < 1$, and for $t = 1, t = -1$ the maps $\pi_1 \circ p$ and $\pi_1 \circ \bar{p}$, which are Vietoris maps, respectively.
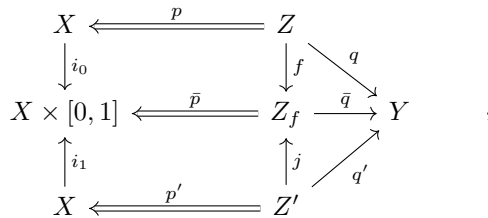
$$
\begin{array}{ccc}
X & \xLeftarrow{\ p_0\ } & Z_0 \\
{\scriptstyle i_{-1}}\downarrow & {\scriptstyle \tilde{f}_0}\downarrow & \searrow^{\scriptstyle q_0} \\
X \times [-1, 1] & \xLeftarrow{\ \tilde{p}\ } K \xrightarrow{\ \tilde{q}\ } & Y \ , \\
{\scriptstyle i_1}\uparrow & {\scriptstyle \tilde{f}_3}\uparrow & \nearrow_{\scriptstyle q_2} \\
X & \xLeftarrow{\ p_2\ } & Z_2
\end{array}
$$

where $\tilde{f}_0(x) = j \circ f_0(x), \tilde{f}_3(x) = \bar{j} \circ f_3(x)$, with inclusions $j : Z \hookrightarrow K$, $\bar{j} : \bar{Z} \hookrightarrow K$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

Proposition 3.3. *Assume that there exists a map $f : Z \to Z'$ such that the following diagram commutes*

$$X \xleftarrow{\;\;p\;\;} Z \xrightarrow{\;q\;} Y, \qquad X \xleftarrow{\;\;p'\;\;} Z' \xrightarrow{\;q'\;} Y$$

Then, the diagrams $X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} Y, \qquad X \xleftarrow{\;p'\;} Z' \xrightarrow{\;q'\;} Y \qquad$ are homotopic.

Proof. It is sufficient to write a commutative diagram

where the maps are defined by formulas:

$$\bar{p}([x]) = \begin{cases} (p(x), t) \; for \; [x] = [(x,t)], \, x \in Z, t \in [0,1] \\ p'(x) \qquad for \; x \in Z' \end{cases},$$
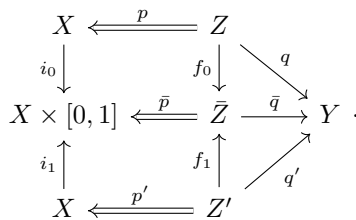
$$\bar{q}([x]) = \begin{cases} q(x) \; for \; [x] = [(x,t)], \, x \in Z, t \in [0,1] \\ q'(x) \; for \; x \in Z' \end{cases},$$

$\bar{f}(x) = [f(x)], j(x) = x.$ It is easily seen that $\bar{p}, \bar{q}$ are well defined. $\qquad\square$

Definition 3.4. Two morphisms

$$\varphi = \{X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} Y\} \;\; \text{and} \;\; \psi = \{X \xleftarrow{\;p'\;} Z' \xrightarrow{\;q'\;} Y\}$$

are **homotopic** if there is a morphism $\quad \chi = \{X \times [0,1] \xleftarrow{\;\bar{p}\;} \bar{Z} \xrightarrow{\;\bar{q}\;} Y\}$ and maps f_0, f_1 such that the following diagram commutes:

Here $i_0(x) = (x,0)$, $i_1(x) = (x,1)$.

The following proposition is an immediate consequence of Propositions 3.2 and 3.3:

Proposition 3.5. *The definition 3.4 does not depend on the choice of representatives.*

The following property is an immediate consequence of the definition of homotopy:

Proposition 3.6. *Homotopic morphisms induce the same homology homomorphism.*

Given three spaces X, Y, Z and maps $\alpha : X \to Z, \beta : Y \to Z$ we denote by $X \underset{\alpha,\beta}{\boxtimes} Y$ their fiber-product $X \underset{\alpha,\beta}{\boxtimes} Y = \{(x, y) \in X \times Y; \ \alpha(x) = \beta(y)\}$.
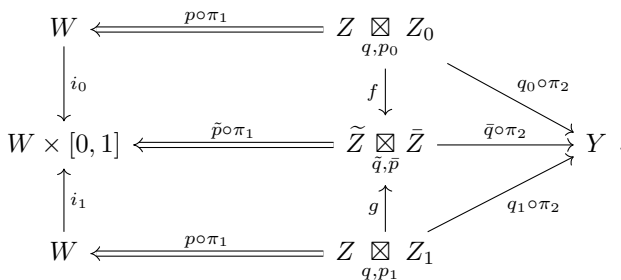
Proposition 3.7. *Compositions of homotopic morphisms are also homotopic.*

Proof. The proof is a diagram game. First, assume that

$$\varphi_0 = \{X \xleftarrow{\ p_0\ } Z_0 \xrightarrow{\ q_0\ } Y\}, \quad \text{and} \quad \varphi_1 = \{X \xleftarrow{\ p_1\ } Z_1 \xrightarrow{\ q_1\ } Y\}$$

are homotopic and let $\quad \psi = \{W \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } X\}$.

Define a morphism $\quad \widetilde{\psi} = \{W \times [0,1] \xleftarrow{\ \tilde{p}\ } \widetilde{Z} \xrightarrow{\ \tilde{q}\ } X \times [0,1]\}, \quad$ where $\widetilde{Z} = Z \times [0,1], \tilde{p}(x,t) = (p(x),t)$ and $\tilde{q}(x,t) = (q(x),t)$. We can write the following commutative diagram

Here $f(x,y) = ((x,0), f_0(y)), \ g(x,y) = ((x,1), f_1(y))$, where f_i are the maps from the diagram defining a homotopy between φ_0 and φ_1 (see Def. 3.4), and $\pi_1(x,y) = x, \pi_2(x,y) = y$. One easily checks that the above diagram satisfies the conditions of 3.1 and thus the morphisms $\varphi_0 \circ \psi$ and $\varphi_1 \circ \psi$ are homotopic.

Now let φ_0, φ_1 be as above and $\quad \psi = \{Y \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } W\}$. Consider the following diagram:
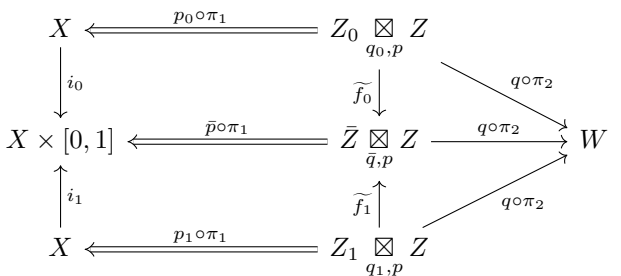
where π_i are projections and $\widetilde{f_i}(x,y) = (f_i(x), y)$. The above diagram defines a homotopy joining morphisms $\psi \circ \varphi_0$ and $\psi \circ \varphi_1$. Now we use the transitivity property of homotopy to end the proof. □

4. Acyclic maps and homotopy of morphisms

As the Example 2 shows, different morphisms may determine the same multivalued map. In view of 3.6 these morphisms may even be not homotopic in general. The reason for it was the fact that the determined map in 2 was not acyclic.

Proposition 4.1. *If $\varphi : X \multimap Y$ is an acyclic map then every two morphisms which determine a selection of φ, must be homotopic.*

Proof. Consider the graph $\Gamma_\varphi = \{(x, y) \in X \times Y;\ y \in \varphi(x)\}$ and maps $p_\varphi(x, y) = x, q_\varphi(x, y) = y$. Then $p_\varphi : \Gamma_\varphi \to X$ is a Vietoris map. Assume that $\quad \psi = \{X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y\} \quad$ determines a selection of φ (i.e. for each $x \in X$ we have an inclusion $\psi(x) \subset \varphi(x)$). Then we have a commutative diagram

where $f(x) = (p(x), q(x))$. The assertion follows from 3.3. $\qquad\square$

Corollary 4.2. *If two morphisms $\varphi_0, \varphi_1 : X \to Y$ have a common acyclic support (i.e., an acyclic map such that φ_0, φ_1 determine its selections) then, the induced homology homomorphisms are the same.*

Proposition 4.3. *There exist non-homotopic single-valued maps which are homotopic as morphisms.*

Proof. It is enough to consider a space X which is acyclic but not contractible. Then, the identity map $id : X \to X$ is not homotopic to any constant map $y_0(x) = y_0$. But they are homotopic as morphisms since they are selections of the acyclic map $\varphi : X \multimap X, \varphi(x) = X$.

5. A local coincidence index for morphisms

In [5], an index theory for a broad class of multi-valued maps including admissible maps as well some maps with non-connected images of points has been constructed (comp. [7], Ch. IV). It has been proved that such maps induce arbitrarily fine chain approximations on nerves of coverings in the sense of [22] (see [5, Thm 4.3]). Regarding a representative of a morphism $\varphi = \{X \xleftarrow{\ p\ } Y \xrightarrow{\ q\ } X\} \quad$ as a sequence of two acyclic maps p^{-1} and q there are chain maps induced by it. If X is a compact polyhedron and U is a polyhedral open subset of X such that $x \notin q(p^{-1}(x))$ for $x \in \partial U$, an index of (p, q) can be defined to be a 'local Lefschetz number' (comp.

[5,22]). It has been proved to be independent of the choice of approxima-
tion and representatives (since they are all homotopic). Therefore, the **index**
$i(X, \varphi, U)$ of a morphism φ is defined (see [5, Def. 5.6]) to be a rational
number (more exactly, it is an element of the coefficient field). The defi-
nition can be extended in quite a routine way to the case of a morphism
$\varphi = \{X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X\}$, where X is a metric ANR, $\overline{\varphi(X)}$ compact
and $U \subset X$ an open set with $x \notin \varphi(x)$ for $x \in \partial U$ (equivalently $p(x) \neq q(x)$
for $x \in p^{-1}(\partial U)$). The detailed proofs can be found in [5].

Let (K, τ) be a compact polyhedron with a fixed triangulation τ. Its
n-th barycentric subdivision is denoted by τ^n. A subset $U \subset K$ is *polyhedral*
provided there is an integer l such that τ^l induces a triangulation of the
closure $\overline{U}$ of U in K.

Let l be a natural number and F a field. We denote by $C_*(K, \tau^l)$ the
oriented chain complex $C_*(K, \tau^l; F)$. The carrier of $c \in C_*(K, \tau)$ (*carr c*)
is the smallest polyhedral subset $X \subset K$ such that $c \in C_*(X, \tau)$. By $b :
C_*(K, \tau) \to C_*(K, \tau^l)$ we denote the subdivision chain mapping which maps
each chain into its l-th barycentric subdivision.

By $\chi : C_*(K, \tau^l) \to C_*(K, \tau)$, we denote a chain mapping which is
induced by a simplicial approximation of the identity map $id : (K, \tau^l) \to
(K, \tau)$. It is known that b and χ are chain homotopy inverses to each other.
In the context of approximations the notion of a closed star is useful. Let
$B \subset K$ be a subset. The *k-th closed star* of B in (K, τ) is defined inductively
(simplices are closed here):

$$St^1(B, \tau) = St(B, \tau) := \bigcup \{\sigma \in \tau \ ; \ \sigma \cap B \neq \emptyset\},$$
$$St^k(B, \tau) := St(St^{k-1}(B, \tau), \tau).$$

Let $\Phi : (K, \tau) \multimap (L, \mu)$ be an u.s.c. multivalued mapping.

Definition 5.1. Let k and l be two natural numbers.

A chain mapping $\varphi : C_*(K, \tau^l) \to C_*(L, \mu^k)$ is called an (n, k)-
approximation of Φ provided the following conditions holds: for each sim-
plex $\sigma \in \tau^l$, there exists a point $y(\sigma) \in K$ such that

$$\sigma \subset St^n(y(\sigma), \tau^l), \ carr \ \varphi\sigma \subset St^n(\Phi(y(\sigma)), \mu^k).$$

Definition 5.2. A graded set

$$A(\Phi) = \{A(\Phi)_j\}_{j \in \mathbb{N}},$$

where $A(\Phi)_j \subset hom(C_*(K, \tau^j), C_*(L, \mu^j))$ is called an **approximation system**
(*A*-system) for Φ provided there is an integer $n = n(A)$ such that:

(i) If $\varphi \in A(\Phi)_j$ then $\varphi = \varphi_1 \circ b$, where φ_1 is an (n, j)-approximation of Φ.
(ii) For every $j \in \mathbb{N}$, there exists $j_1 \in \mathbb{N}$ such that for $m \geq l \geq j_1$ and for
all $\varphi = \varphi_1 \circ b \in A(\Phi)_l$ and $\psi = \psi_1 \circ b \in A(\Phi)_m$ the diagram

$$C_*(K, \tau^{l_1}) \xrightarrow{\varphi_1} C_*(L, \mu^l)$$

$$\chi \Big\uparrow \qquad\qquad\qquad \chi \Big\uparrow$$

$$C_*(K, \tau^{m_1}) \xrightarrow{\psi_1} C_*(L, \mu^m)$$

with $m_1 \geq l_1$ is homotopy commutative with a chain homotopy D satisfying the following smallness condition:

- For any simplex $\sigma \in \tau^{m_1}$, there exists a point $z(\sigma) \in K$ such that

$$\sigma \subset St^n(z(\sigma), \tau^j), \; carr \; D\sigma \subset St^n(\Phi(z(\sigma)), \mu^j).$$

Definition 5.3. Let $\Phi_1, \Phi_2 : K \multimap L$ be u.s.c. maps and let $H : K \times [0,1] \multimap L$ be an u.s.c. homotopy joining Φ_1 and Φ_2. Let $A(\Phi_1)$ and $A(\Phi_2)$ be A-systems for Φ_1 and Φ_2, respectively. They are H-**homotopic** provided there is an integer $m \in \mathbb{N}$ such that the following condition holds:

(I) For every $j \in \mathbb{N}$, there is $j_1 \in \mathbb{N}$ such that for any $l \geq j_1$ there are $\varphi = \varphi_1 \circ b \in A(\Phi_1)_l$ and $\psi = \psi_1 \circ b \in A(\Phi_2)_l$ such that $\varphi_1, \psi_1 : C_*(K, \tau^{l_1}) \to C_*(L, \mu^l)$ are chain homotopic with an H-small homotopy D, i.e.:

   (i) For $\sigma \in \tau^{l_1}$ there is a point $d(\sigma) \in K$ such that

$$\sigma \subset St^m(d(\sigma), \tau^j), \; carr \; D\sigma \subset St^m(H(d(\sigma) \times [0,1]), \mu^j).$$

One can define an index for an A-system as follows: Let K be a compact polyhedron with a given triangulation τ and let U be a polyhedral open subset of K (i.e., U is open and $\bar{U}$ is a subpolyhedron of K). Let $\Phi : \bar{U} \multimap K$ be such that $x \notin \Phi(x)$ for $x \in \partial U$, and let $A = A(\Phi)$ be an A-system for Φ. Denote by $p_U : C_*(K, \tau^k) \to C_*(\bar{U}, \tau^k)$ the natural linear projection and let $\varphi \in A(\Phi)_k$. We define a 'local Lefschetz number'

$$\lambda(p_U \circ \varphi) := \sum_{i=0}^{\dim K} (-1)^i \mathrm{tr} \, (p_U \circ \varphi)_i.$$

It has been proved in [22] that there exists sufficiently large k_0 such that the above defined number does not depend on the choice of $\varphi \in A(\Phi)_k, k \geq k_0$.

Thus, we can define the index properly.

Definition 5.4. $i_A(K, \Phi, U) := \lambda(p_U \circ \varphi)$ for $\varphi \in A(\Phi)_k, k \geq k_0$.

Proposition 5.5. *Properties of index:*

(1) *(Additivity) Let U_1, U_2 be open, disjoint and polyhedral subsets of U and $\Phi : \bar{U} \to K$ an u.s.c. mapping such that* $\mathrm{Fix}(\Phi) \subset U_1 \cup U_2$. *If $A(\Phi)$ is an A-system for Φ then*

$$i_A(K, \Phi, U) = i_A(K, \Phi, U_1) + i_A(K, \Phi, U_2).$$

(2) *(Excision) Let $U_1 \subset U$ be two open and polyhedral subsets of K and $\mathrm{Fix}(\Phi) \subset U_1$. Then,*

$$i_A(K, \Phi, U) = i_A(K, \Phi, U_1).$$

(3) *(Homotopy invariance) Let $H : \bar{U} \times [0,1] \to K$ be an u.s.c. homotopy such that $x \notin H(x,t)$ for $x \in \partial U$ and $t \in [0,1]$. Let A_0, A_1 be H-homotopic A-systems for $H_0 = H(\cdot, 0)$ and $H_1 = H(\cdot, 1)$,, respectively. Then,*

$$i_{A_0}(K, H_0, U) = i_{A_1}(K, H_1, U).$$

(4) *(Commutativity) Let $W \subset K$ be open and let $\Phi_1 : K \to L, \Phi_2 : L \to K$ be u.s.c. maps.*
Assume that $x \notin \Phi_2 \circ \Phi_1(x)$ for $x \in \partial W$ and $y \notin \Phi_1 \circ \Phi_2(y)$ for $y \in \partial(\Phi_2^{-1}(W))$. Assume further that if $y \in \mathrm{Fix}(\Phi_1 \circ \Phi_2) \setminus \Phi_2^{-1}(W)$ then $\Phi_2(y) \cap \mathrm{Fix}(\Phi_2 \circ \Phi_1)\big|_{\bar{W}} = \emptyset$. Then for any A-systems $A_1 = A(\Phi_1)$, $A_2 = A(\Phi_2)$

$$i_{A_1 \circ A_2}(L, \Phi_1 \circ \Phi_2, \Phi_2^{-1}(W)) = i_{A_2 \circ A_1}(K, \Phi_2 \circ \Phi_1, W).$$

(5) *(Mod-p property) Let the coefficient field $F = \mathbb{Z}_p, p$ prime. Let $W \subset K$ be open and $\Phi : K \to K$ an u.s.c. map such that $x \notin \Phi^p(x)$ for $x \in \partial W$. Assume that if $y \in \mathrm{Fix}(\Phi^p) \setminus W$ then $\Phi^k(y) \cap \mathrm{Fix}(\Phi^p)\big|_{\bar{W}} = \emptyset$ for $k < p$. Then*

$$i_A(K, \Phi, W) = i_{A^p}(K, \Phi^p, W).$$

Remark 5.6. Because of Prop. 5.5(2), one can also define the index $i_A(K, \Phi, V)$, when V is open and not polyhedral, if $\Phi : L \to K, V \subset L \subset K$, and L is a subpolyhedron of K. Then one puts

$$i_A(K, \Phi, V) := i_A(K, \Phi, U),$$

where $U \subset V$ is polyhedral and $\mathrm{Fix}(\Phi)\big|_{\bar{V}} \subset U$.

The existence of an A-system for maps determined by morphisms was proved in fact in [5], where more general maps were considered. Recall that we can associate an abstract simplicial complex with a locally finite covering α of a given space X, called its' nerve $N(\alpha)$. Simplices of $N(\alpha)$ are the nonempty intersections of the sets $V_i \in \alpha$. We denote the support of a simplex $s \in N(\alpha)$ as the union of the forming sets (vertices). Similarly we define the *star* of a set as the union of all simplices intersecting it. We denote by $N^n(\alpha)$ the n-dimensional skeleton of the nerve. Let X, Y be compact spaces. By $CovX$, $CovY$ we denote the sets of finite open coverings of these spaces, respectively.

Definition 5.7. Let $\alpha, \bar{\alpha} \in CovX, \beta, \bar{\beta} \in CovY$ and let $\Phi : X \multimap Y$.
A chain map $\varphi : C_*(N^n(\bar{\alpha})) \to C_*(N^n(\bar{\beta}))$ is called an (α, β)-**approximation** of Φ provided: for every simplex $s \in N^n(\bar{\alpha})$ there exists a point $p(s) \in X$ such that

$$\mathrm{supp}\, s \subset St(p(s), \alpha), \qquad \mathrm{supp}\, \varphi s \subset St(\Phi(p(s)), \beta).$$

The following basic approximation result is a special case of Theorem 4.3 in [5]:

Theorem 5.8. *Let* $\Phi : X \multimap Y$ *be an acyclic-valued u.s.c. map of compact spaces, and* $\alpha \in CovX, \beta \in CovY$. *Then for every* $n \in \mathbb{N}$, *there exist: a refinement* $\bar{\alpha}$ *of* α *and an* (α, β)- *approximation* $\varphi : C_*(N^n(\bar{\alpha})) \to C_*(N^n(\beta))$ *of* Φ.

Recall that given a morphism $\quad \varphi = \{X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y\} \quad$, the map $p^{-1} : X \multimap Z$ is u.s.c. and acyclic, and Theorem 5.8 applies. Thus, we can formulate

Proposition 5.9. *If* $(K, \tau), (L, \mu)$ *are compact polyhedra and*

$$\varphi = \{K \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } L\}$$

is a morphism, then the compositions of chain approximations of p^{-1} *and* q *form an A-system.*

Proof. Let us recall that given a triangulation τ, we can associate an open covering $\alpha(\tau)$ by the sets $st(v, \tau) = Int(St(v, \tau))$, where v are vertices of the triangulation. The simplicial maps $\theta : (K, \tau) \to N(\alpha(\tau)), \lambda : N(\alpha(\tau)) \to (K, \tau)$ defined on vertices by $\theta(v) := st(v, \tau)$, and $\lambda(st(v, \tau)) := v$, define a natural simplicial isomorphism. The same is true for (L, μ). Considering barycentric subdivisions of the triangulation τ we obtain arbitrarily fine refinements of $\alpha(\tau)$). It has been detailly proved that the compositions of sufficiently close chain approximations: ψ_2 of the acyclic map p^{-1} and ψ_1 of the single-valued map q are chain homotopic (see [5, Prop. 4.8 and 4.14], [8, Prop. 50.30 and 50.39]).

Therefore, the compositions of the form $\lambda \circ \psi_1 \circ \psi_2 \circ \theta \circ b$ form an A-system for the given morphism φ. The map b is the standard subdivision map (see [23]) and λ, θ are the above isomorphisms for barycentric subdivisions τ^k, μ^k. $\qquad \square$

Let now U be an open polyhedral subset of a compact polyhedron (K, τ) (i.e. $\overline{U}$ is a subpolyhedron) and let $\varphi : \overline{U} \to K$ be a morphism such that $x \notin \varphi(x)$ (equivalently $p(y) \neq q(y)$ for $y \in p^{-1}(\partial U)$ for the representatives).

Then, we define

$$i(K, \varphi, U) := i_A(K, \varphi, U).$$

By localization property, we do not need to assume that U is polyhedral.

Now let X be a compact metric ANR, $U \subset X$ open and let $\varphi : \overline{U} \to X$ be a morphism such that Fix $\varphi \cap \partial U = \emptyset$. Then, there exists a finite polyhedron K ε-dominating X with $\varepsilon > 0$ smaller than the distance of Fixφ from ∂U (see [3, p. 41]). Denoting the maps $r : K \to X$, $s : X \to K$ ($r \circ s = id_X, d(s \circ r(x), x) < \varepsilon$), we define the index as follows:

$$i(X, \varphi, U) := i(K, s \circ \varphi \circ r, r^{-1}(U)).$$

Independent of the choice of K, r, s is detailly proved in [5], and [8, Sect. IV.52].

The next step is to extend the definition of the index to the case of arbitrary ANRs. By the Arens–Eells embedding Theorem (see, e.g., [8, Thm 1.6]) we can assume that X is a closed subset of a normed space E. Since X is an ANR, there is an open subset $V \subset E$ and a retraction $r : V \to X$. Consider a compact morphism $\varphi = \{\overline{U} \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } X\}$ such that $\overline{q(Z)}$ is compact, and $p(z) \neq q(z)$ for $p(z) \in \partial U$. Denoting by $i : X \to V$ the inclusion map we define a morphism $\psi = i \circ \varphi \circ r : U' = r^{-1}(\overline{U}) \to V$. One easily checks that $\mathrm{Fix}\,\psi \cap r^{-1}(\partial U) = \emptyset$. By the result of J. Girolo [6] there exists a compact ANR K such that $\overline{\psi(U')} \subset K \subset V$. We let $W = K \cap r^{-1}(U)$ and then the index $i(K, \psi\big|_K, W)$ is well defined as above.

Thus, we define

$$i(X, \psi, U) = i(K, \psi\big|_K, W).$$

Theorem 5.10. ([5, Prop. 7.5]) *The index has the following properties:*

1. ***Additivity.*** *If $U_1, U_2 \subset U$ are open and disjoint and if $\mathrm{Fix}\ \varphi\big|_{\overline{U}} \subset U_1 \cup U_2$ then $i(X, \varphi, U) = i(X, \varphi, U_1) + i(X, \varphi, U_2)$.*
2. ***Homotopy invariance.*** *Assume that the morphisms $\varphi, \psi : X \to X$ are homotopic by a homotopy such that for its representatives holds $x \notin q(p^{-1}(x, t))$ whenever $x \in \partial U$. Then $i(X, \varphi, U) = i(X, \psi, U)$.*
3. ***Normalization.*** *If $\Lambda(\varphi_*)$ denotes the (generalized) Lefschetz number of the induced homology homomorphism then $i(X, \varphi, X) = \Lambda(\varphi_*)$.*
4. ***Commutativity.*** *Let $W \subset X$ be open and $\varphi : X \to Y$, $\psi : Y \to X$ be morphisms. Assume that $x \notin \psi \circ \varphi(x)$ for $x \in \partial W$ and $y \notin \varphi \circ \psi(y)$ for $y \in \partial(\psi^{-1}(W))$ and moreover $\psi(\mathrm{Fix}\ \varphi \circ \psi - \psi^{-1}(W)) \cap \mathrm{Fix}\ \psi \circ \varphi\big|_{\overline{W}} = \emptyset$. Then $i(X, \psi \circ \varphi, W) = i(Y, \varphi \circ \psi, \psi^{-1}(W))$.*

Corollary 5.11. *If $i(X, \varphi, U) \neq 0$ then the determined map has a fixed point in U (or equivalently any representative $X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } X$ has a co-incidence in $p^{-1}(U)$).*

Corollary 5.12. *If $Y \subset X$ is an ANR and $\varphi(\overline{U}) \subset Y$, then*

$$i(X, \varphi, U) = i(Y, \varphi, U \cap Y).$$

The first corollary is a consequence of additivity and the last one of commutativity (see. Thm 5.10).

Now, we prove a generalization of the well-known nonlinear alternative (comp. [11, II.6, Thm 5.2]) from which one can derive many fixed point or coincidence results. We omit the corollaries since they are routine (comp. [4,7,11]).

Theorem 5.13. *Let C be a convex subset of a normed space E and U an open set in C containing 0. Then for any morphism*

$$\varphi = \{C \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } C\} : C \longrightarrow C$$

one of the following conditions is satisfied:

(i) φ has a fixed point in $\overline{U}$,

(ii) for some $x \in \partial U$ and $0 < t < 1$ there is $y \in p^{-1}(x)$ such that $x = t \cdot q(y)$.

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
C & \xleftarrow{\quad p \quad} & Z \\
{\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle f} \quad \searrow^{0} \\
C \times [0,1] & \xleftarrow{\quad \tilde{p} \quad} & Z \times [0,1] \xrightarrow{\tilde{q}} C \\
{\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle g} \quad \nearrow_{q} \\
C & \xleftarrow{\quad p \quad} & Z
\end{array}
$$

where $\bar{p}(x,t) = (p(x),t)$, $\bar{q}(x,t) = t \cdot q(x)$, $0(x) = 0$, $f(x) = (x,0)$ and $g(x) = (x,1)$. Denoting by ψ the morphism defined by $(p,0)$ we observe that $i(C, \psi, U) = 1$ since $0 \in U$ and it is a constant map $x \mapsto 0$. If φ has a fixed point in U then we are done. If not and if moreover (ii) is not satisfied that means exactly that assumptions of homotopy invariance (Thm 5.10) are fulfilled. Therefore, $i(C, \varphi, U) = 1$ and φ has a fixed point in U by Cor. 5.11. $\square$

Remark 5.14. We assumed in this section that the considered morphisms determined compact maps, but one can extend all the results to some non-compact maps. We touched the example of compact attraction maps in [5]. The classes of condensing morphisms mentioned in [15] (comp. [24]) may be also included into the theory.

Remark 5.15. The index is a priori a rational number but there are no examples where it is not an integer. Moreover, in [2], it was proved that in the case of acyclic maps of ENR-s, the index is an integer and the theory is unique.

6. Extending morphisms from a sphere

In this short section, we prove one extremely simple geometric result which analog for single-valued maps is an exercise (see, e.g., [11, Thm II.5.1]). We say a morphism is **nullhomotopic** provided it is homotopic to a morphism

$$\varphi = \{ X \xleftarrow{\ id\ } X \xrightarrow{\ y_0\ } Y \}, \quad \text{where } y_0(x) = y_0 \text{ is any constant map.}$$

Proposition 6.1. *A morphism* $\varphi = \{ S^n \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y \}$ *is nullhomotopic if and only if φ is extendable to a morphism $\Phi : D^{n+1} \to Y$ in the unit disc.*

Proof. Assume φ is nullhomotopic, i.e., we have a diagram

$$
\begin{array}{ccc}
S^n & \xleftarrow{\quad p \quad} & Z \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle f_1} \quad \searrow^{q} \\
S^n \times [0,1] & \xleftarrow{\quad \bar{p} \quad} & \bar{Z} \xrightarrow{\bar{q}} Y \\
{\scriptstyle i_0}\uparrow & & \uparrow{\scriptstyle f_0} \quad \nearrow_{y_0} \\
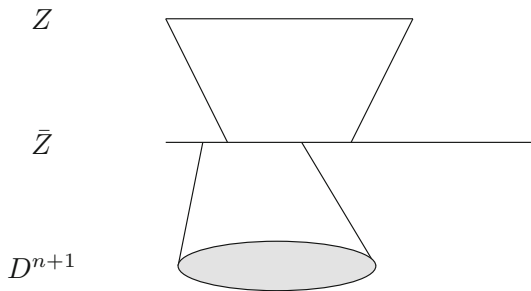S^n & \xleftarrow{\quad id \quad} & S^n
\end{array}
$$

FIGURE 2. A sketch of K

We define a morphism $\Phi : D^{n+1} \to Y$. Let K be a space obtained from the disjoint sum of three spaces $Z \times [0,1], \bar{Z}, S^n \times [0,1] \cup D^{n+1} \times \{0\}$ by the following relations: if $x \in Z$ then $(x,0) \sim f_1(x)$, and if $x \in S^n$ then $(x,1) \sim f_0(x)$ (see Fig. 2 for a sketch).

Define a map $\tilde{p} : K \Longrightarrow D^{n+1}$ by the formula

$$\tilde{p}([x]) = \begin{cases} \frac{1}{4}(3+t)p(x) \ if \ [x] = [(x,t)], x \in Z \\ g(\bar{p}(x)) \qquad if \ x \in \bar{Z} \\ \frac{1}{4}(1+t)x \qquad if \ [x] = [(x,t)], x \in S^n \\ \frac{1}{4}x \qquad\quad if \ x \in D^{n+1} \end{cases},$$

where $g : S^n \times [0,1] \to D^{n+1}, g(x,t) = \frac{1}{4}(2+t)x$.

A map $\tilde{q} : K \to Y$ is defined by the formula

$$\tilde{q}([x]) = \begin{cases} q(x) \ if \ [x] = [(x,t)], x \in Z \\ \bar{q}(x) \ if \ x \in \bar{Z} \\ y_0 \quad in \ other \ case \end{cases}.$$

It is easily seen that p is a Vietoris map, and moreover

$$p^{-1}(S^n) = Z \times \{1\} \approx Z.$$

Hence a morphism $\quad \Phi = \{D^{n+1} \xleftarrow{\tilde{p}} K \xrightarrow{\tilde{q}} Y\} \quad$ is an extension of φ.

Conversely, assume we have a morphism $\quad \{D^{n+1} \xleftarrow{\bar{p}} \bar{Z} \xrightarrow{\bar{q}} Y\}$. There is a map $g : S^n \times [0,1] \to D^{n+1}$ given by

$$g(x,t) = \begin{cases} 0 \qquad\quad if \ 0 \leq t \leq \frac{1}{2} \\ (2t-1)x \ if \ \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Define a space $K = \{(x,t,y,z) \in S^n \times [0,1] \times D^{n+1} \times \bar{Z}; \ y = g(x,t) = \bar{p}(z)\}$. Then we have a commutative diagram

$$
\begin{array}{ccc}
S^n \times \{1\} & \xLeftarrow{\ p\ } & Z \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle f} \ \searrow^{q} \\
S^n \times [0,1] & \xLeftarrow{\ \tilde{p}\ } & K \xrightarrow{\tilde{q}} Y \\
\uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle h} \ \nearrow_{z_0} \\
S^n \times \{0\} & \xLeftarrow{\ id\ } & S^n
\end{array}
\quad .
$$

Here $Z = \bar{p}^{-1}(S^n \times \{1\})$, $\tilde{p}(x, t, y, z) = (x, t)$, $\tilde{q}(x, t, y, z) = \bar{q}(z)$, $f(x) =$
$= (\bar{p}(x), 1, \bar{p}(x), x)$, $h(x) = (x, 0, 0, y_0)$, where y_0 is any point in $\bar{p}^{-1}(0)$
and $z_0 = q(y_0)$, and p, q are restrictions of $\bar{p}, \bar{q}$. The diagram proves that
$\{S^n \xleftarrow{\;\;p\;\;} Z \xrightarrow{\;\;q\;\;} Y\}$ is a morphism homotopic to a constant
map. $\hfill\square$

Combining Propositions 6.1 together with 4.3, we obtain:

Corollary 6.2. *If Y is an acyclic space with $\pi_n(Y) \neq 0$ then there exists a
single-valued map $f : S^n \to Y$ such that f is not extendable to D^{n+1} as a
map but it is extendable as a morphism (there is a multivalued extension).*

The results of this section suggest that one can extend theorems concerning essential maps of [11, Ch. II]. A version of this approach can be found
in [16–18].

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