

## BLOCK GRAPHS WITH LARGE PAIRED DOMINATION MULTISUBDIVISION NUMBER

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### Abstract

The paired domination multisubdivision number of a nonempty graph  $G$ , denoted by  $\text{msd}_{\text{pr}}(G)$ , is the smallest positive integer  $k$  such that there exists an edge which must be subdivided  $k$  times to increase the paired domination number of  $G$ . It is known that  $\text{msd}_{\text{pr}}(G) \leq 4$  for all graphs  $G$ . We characterize block graphs with  $\text{msd}_{\text{pr}}(G) = 4$ .

**Keywords:** paired domination, domination subdivision number, domination multisubdivision number, block graph.

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### 1. INTRODUCTION

The study of changes that occur in domination-related parameters of a graph when its edges are subdivided<sup>1</sup> was initiated in [11]. If  $\pi$  is a domination-type parameter of  $G$ , the smallest number of edges that must be subdivided, where each edge of  $G$  can be subdivided at most once, in order to increase  $\pi$  is called

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<sup>1</sup>See Section 2 for definitions of terms used in this section.

the  $\pi$ -subdivision number, denoted by  $\text{sd}_\pi(G)$ . Subdivision numbers have been studied for the domination number [6, 11], as well as for connected [4], double [1], Roman [10], total [7, 9] and paired domination numbers [5].

Instead of subdividing multiple edges once each, one may wish to subdivide a single edge multiple times. The smallest number of times that a single edge of  $G$  must be subdivided to increase  $\pi$  is called the  $\pi$ -multisubdivision number, denoted by  $\text{msd}_\pi(G)$ . Domination and paired domination multisubdivision numbers were studied in [3] and [2], respectively. In particular, it was shown in [2] that the paired domination multisubdivision number  $\text{msd}_{\text{pr}}(G)$  of any graph  $G$  is at most four. For brevity we refer to a graph  $G$  with  $\text{msd}_{\text{pr}}(G) = 4$  as an msd-4 graph. Msd-4 trees were characterized in [2].

We discuss methods of combining msd-4 graphs to yield new msd-4 graphs and use our results, combined with results from [2], to characterize msd-4 block graphs. Definitions and previous results are given in Section 2. We state the characterization of msd-4 block graphs in Section 3, but defer its proof to Section 6 to allow us to prove a number of results used in the proof; results that apply to general msd-4 graphs are given in Section 4, while results specific to block graphs can be found in Section 5.

## 2. DEFINITIONS AND PREVIOUS RESULTS

We refer the reader to [8] for domination parameters not defined here. A set  $S$  of vertices of a graph  $G = (V, E)$  without isolated vertices is a *paired dominating set* of  $G$  if every vertex of  $G$  is adjacent to a vertex in  $S$ , and the subgraph  $G[S]$  of  $G$  induced by  $S$  has a perfect matching. If  $u, v \in S$  and there exists a perfect matching  $M$  of  $G[S]$  such that  $uv \in M$ , we say that  $u$  and  $v$  are *paired* in  $S$ . The smallest cardinality of a paired dominating set of  $G$  is the *paired domination number* of  $G$ , denoted by  $\gamma_{\text{pr}}(G)$ . If  $S$  is a paired dominating set of  $G$  such that  $|S| = \gamma_{\text{pr}}(G)$ , we call  $S$  a  $\gamma_{\text{pr}}(G)$ -set, or simply a  $\gamma_{\text{pr}}$ -set if the graph is clear from the context. If  $u$  is a vertex of  $G$  such that  $G - u$  has no isolated vertices and  $\gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G)$  (in which case  $\gamma_{\text{pr}}(G - u) = \gamma_{\text{pr}}(G) - 2$ ), we say that  $u$  is a  $\gamma_{\text{pr}}(G)$ -critical vertex, or simply a  $\gamma_{\text{pr}}$ -critical vertex, and define  $\text{Cr}(G) = \{u \in V(G) : u \text{ is a } \gamma_{\text{pr}}\text{-critical vertex}\}$ .

A *neighbour* of a vertex  $u \in V(G)$  is a vertex adjacent to  $u$ . The (*open*) *neighbourhood*  $N(u)$  of a vertex  $u$  is the set of all vertices adjacent to  $u$ , and its *closed neighbourhood* is  $N[u] = N(u) \cup \{u\}$ . For a set  $S \subseteq V(G)$ , the (*open*) *neighbourhood* of  $S$  is  $N(S) = \bigcup_{u \in S} N(u)$ , and its *closed neighbourhood* is  $N[S] = N(S) \cup S$ . For a vertex  $u \in S$ , the *private neighbourhood of  $u$  with respect to  $S$*  is the set  $\text{PN}(u, S) = N[u] \setminus N[S \setminus \{u\}]$ . It is possible that  $u \in \text{PN}(u, S)$ , but if  $S$  is a paired dominating set, then  $u$  is adjacent to the vertex it is paired with,



so  $u \notin \text{PN}(u, S)$  in this case.

An edge  $uv$  of a graph  $G$  is *subdivided* if it is replaced by a path  $(u, x, v)$ , where  $x$  is a new vertex, and *multisubdivided* if it is replaced by a path  $(u, x_1, \dots, x_k, v)$ ,  $k \geq 2$ , where  $x_1, \dots, x_k$  are new vertices; we also say that  $uv$  is *subdivided  $k$  times*. Let  $G_{uv,k}$  denote the graph obtained from  $G$  by subdividing the edge  $uv$   $k$  times. The *paired domination multisubdivision number*  $\text{msd}_{\text{pr}}(G)$  of a graph  $G$  without isolated vertices is the smallest positive integer  $k$  such that there exists an edge  $uv$  which must be subdivided  $k$  times for  $\gamma_{\text{pr}}(G_{uv,k})$  to exceed  $\gamma_{\text{pr}}(G)$ . As mentioned above,  $\text{msd}_{\text{pr}}(G) \leq 4$  for all graphs. The three graphs in Figure 1 are all  $\text{msd}$ -4 graphs; the red vertices form  $\gamma_{\text{pr}}$ -sets.

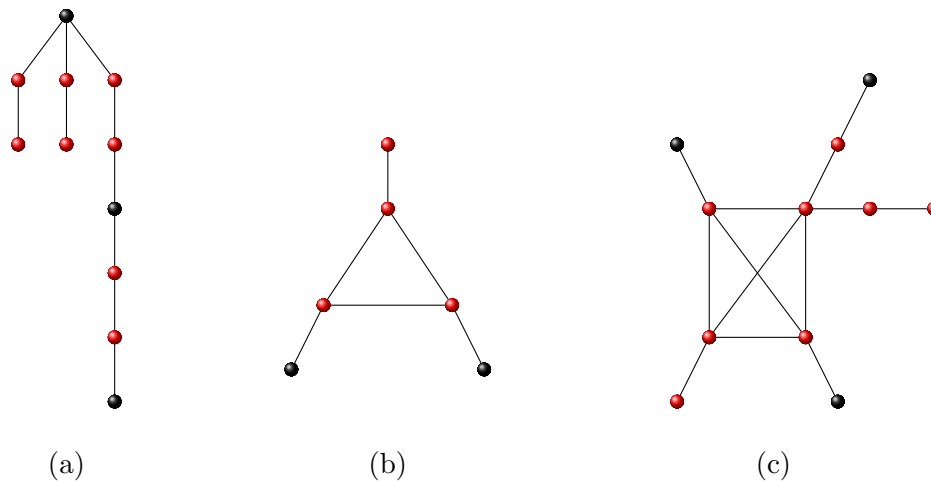


Figure 1. (a) The spider  $S(2, 2, 6)$  (b) the corona  $K_3 \circ K_1$  (c) a flared corona  $K_4 \circ^{*2} K_1$ .

A *leaf* of a graph is a vertex of degree one, and its neighbour is called a *stem*. The following properties of  $\text{msd}$ -4 graphs were proved in [2].

**Theorem 1** [2]. *Let  $G$  be an  $\text{msd}$ -4 graph. Then*

- (i) *each edge of  $G$  belongs to a matching of a minimum paired dominating set of  $G$ ;*
- (ii) *any leaf of  $G$  is a  $\gamma_{\text{pr}}$ -critical vertex;*
- (iii) *each stem is adjacent to exactly one leaf.*

The complete bipartite graph  $K_{1,k}$ ,  $k \geq 2$ , is called a *star*. Let  $K_{1,k}$  have partite sets  $\{u\}$  and  $\{v_1, \dots, v_k\}$ . The *spider*  $S(\ell_1, \dots, \ell_k)$ ,  $\ell_i \geq 1$ ,  $k \geq 2$ , is a tree obtained from  $K_{1,k}$  by subdividing the edge  $uv_i$   $\ell_i - 1$  times,  $i = 1, \dots, k$ . Note that  $S(2, 2) \cong P_5$ . See Figure 1(a) for  $S(2, 2, 6)$ . The characterization of  $\text{msd}$ -4 trees in [2] immediately gives the following result.

**Proposition 2** [2]. *The spider  $T = S(2, \dots, 2)$  satisfies  $\text{msd}_{\text{pr}}(T) = 4$ , and  $\text{Cr}(T)$  consists of the leaves of  $T$ .*

The *corona*  $G \circ K_1$  of a graph  $G$  is the graph obtained by joining each vertex of  $G$  to a new leaf;  $K_3 \circ K_1$  is illustrated in Figure 1(b). A *flared corona*  $G \circ^{*t} K_1$  of  $G$  is a graph obtained by joining each vertex of  $G$ , except one vertex  $w$ , to a new leaf, while  $w$  is joined to a single vertex of each of  $t \geq 1$  copies of  $K_2$ . The flared corona  $K_4 \circ^{*2} K_1$  is depicted in Figure 1(c). The following facts can be verified easily and are stated without proof.

**Remark 3.**

- (i) A corona  $K_n \circ K_1$ ,  $n \geq 2$ , is an msd-4 graph if and only if  $n$  is odd.
- (ii) A flared corona  $K_n \circ^{*t} K_1$ ,  $n \geq 2$ , is an msd-4 graph if and only if  $n$  is even.
- (iii) A vertex of  $K_{2n+1} \circ K_1$  or  $K_{2n} \circ^{*t} K_1$  is  $\gamma_{\text{pr}}$ -critical if and only if it is a leaf (see Theorem 1).

A *block* of a graph is a maximal connected subgraph with no cut-vertex, and a *block graph* is a graph, each of whose blocks is a complete graph. Thus, trees are block graphs since each block of a nontrivial tree is a  $K_2$ . Evidently, coronas and flared coronas are also block graphs. To characterize msd-4 block graphs, we use spiders  $S(2, \dots, 2)$ , coronas  $K_{2n+1} \circ K_1$  and flared coronas  $K_{2n} \circ^{*t} K_1$ , combining them by identifying vertices and edges in a prescribed way.

We begin by describing two operations, collectively known as  $\oplus$ -operations, for joining disjoint graphs; since the operations can be performed on any graphs, we state them in their most general form. (The operations are well known but we need to define our notation.)

$G_1 \oplus^{u_1 u_2} G_2$ : Let  $G_1$  and  $G_2$  be vertex disjoint graphs and  $u_i \in V(G_i)$  for  $i \in \{1, 2\}$ . We denote the graph obtained from  $G_1$  and  $G_2$  by identifying  $u_1$  and  $u_2$  into one vertex  $u = u_1 = u_2$  by  $G_1 \oplus_u^{u_1 u_2} G_2$  (or by  $G_1 \oplus^{u_1 u_2} G_2$  if the label  $u$  is unimportant).

$G_1 \oplus_e^{e_1 e_2} G_2$ : Let  $G_1$  and  $G_2$  be vertex disjoint graphs and  $e_i = u_i v_i \in E(G_i)$ . We denote the graph obtained from  $G_1$  and  $G_2$  by identifying  $u_1$  and  $u_2$  into one vertex  $u = u_1 = u_2$ ,  $v_1$  and  $v_2$  into one vertex  $v = v_1 = v_2$ , and  $e_1$  and  $e_2$  into one edge  $e = uv$  by  $G_1 \oplus_e^{e_1 e_2} G_2$  (or by  $G_1 \oplus^{e_1 e_2} G_2$  if the label  $e$  is unimportant).

The graph  $G_1 \oplus_e^{e_1 e_2} G_2$ , where  $G_1 = S(2, 2, 6)$ ,  $G_2 = K_3 \circ K_1$ , and  $e_i = u_i v_i$  for  $i = 1, 2$ , is illustrated in Figure 2. Note that  $u_i$  is  $\gamma_{\text{pr}}(G_i)$ -critical for  $i = 1, 2$ , and  $u_1 = u_2$  is  $\gamma_{\text{pr}}$ -critical in  $G_1 \oplus_e^{e_1 e_2} G_2$ . The spider  $S(2, 2, 6)$ , in turn, is obtained as  $H_1 \oplus^{u_1 u_2} H_2$ , where  $H_1 = S(2, 2, 2)$ ,  $H_2 = P_5 = S(2, 2)$ , and  $u_i$  is a leaf of  $H_i$ ,  $i = 1, 2$ .

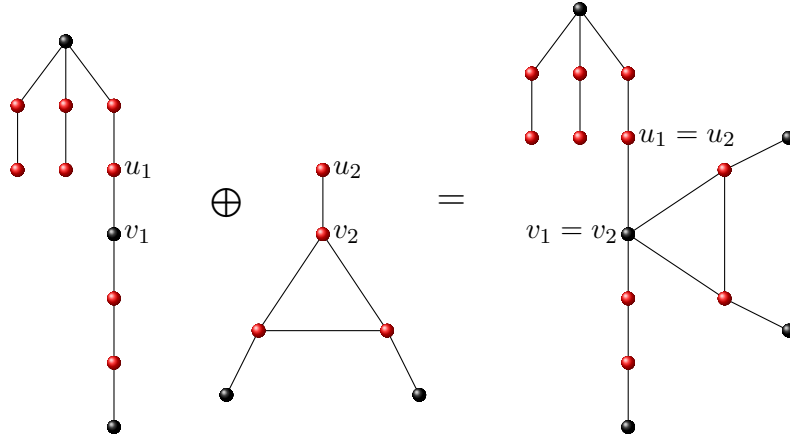


Figure 2. The graph  $S(2, 2, 6) \oplus^{u_1 v_1 u_2 v_2} K_3 \circ K_1$ .

### 3. CHARACTERIZATION OF MSD-4 BLOCK GRAPHS

We now state our main result — the characterization of msd-4 block graphs. The proof is deferred to Section 6.

Let  $\mathcal{U}$  be the collection of all spiders  $S(2, \dots, 2)$ , coronas  $K_{2n+1} \circ K_1$  and flared coronas  $K_{2n} \circ^{*t} K_1$ ,  $n \geq 1$ . Define  $\mathcal{B}$  to be the family of all block graphs  $G$  that can be obtained as a graph  $G_j$ ,  $j \geq 1$ , from a sequence  $G_1, \dots, G_j$  of graphs, where  $H_1 = G_1 \in \mathcal{U}$ , and, if  $j > 1$ ,  $G_{i+1}$  can be constructed recursively from  $G_i$  by

- adding a graph  $H_{i+1} \in \mathcal{U}$ ,
- choosing vertices  $u_1 \in \text{Cr}(G_i)$ ,  $u_2 \in \text{Cr}(H_{i+1})$ , and if necessary,  $v_1 \in N(u_1)$ ,  $v_2 \in N(u_2)$ ,
- performing the operation  $G_i \oplus^{u_1 u_2} H_{i+1}$  or  $G_i \oplus^{u_1 v_1 u_2 v_2} H_{i+1}$ .

**Theorem 4.** *Let  $G$  be a connected block graph. Then  $G$  is an msd-4 graph if and only if  $G \in \mathcal{B}$ . Moreover, if  $G$  is an msd-4 graph constructed from the graphs  $H_1, \dots, H_j \in \mathcal{U}$ , then  $\text{Cr}(G) = \bigcup_{i=1}^j \text{Cr}(H_i)$ .*

The second statement of Theorem 4 implies that any  $\gamma_{\text{pr}}$ -critical vertex  $v$  of an msd-4 block graph remains  $\gamma_{\text{pr}}$ -critical after the  $\oplus$ -operations have been performed any number of times, whether  $v$  was identified with another vertex or not. The following corollary of Theorem 4 was proved in [2].

**Corollary 5.** *A tree  $T$  is an msd-4 graph if and only if  $T \in \mathcal{B}$ , that is, if and only if  $T$  can be constructed as described, using only spiders  $S(2, \dots, 2)$ .*

## 4. GENERAL RESULTS

In this section we discuss ways of constructing larger  $\text{msd-4}$  graphs from smaller ones. We first prove a useful lemma.

**Lemma 6.** *Let  $G$  be a graph with  $\text{msd}_{\text{pr}}(G) = 4$ . For any edge  $uv$  of  $G$ , subdivide  $uv$  by replacing it with the path  $(u, x_1, x_2, x_3, v)$ . If  $D$  is any  $\gamma_{\text{pr}}(G_{uv,3})$ -set, then  $D \cap \{u, x_1, x_2, x_3, v\} =$*

- (i)  $\{x_1, x_2\}$  or  $\{x_2, x_3\}$ , or
- (ii)  $\{u, x_1, v\}$  or  $\{u, x_3, v\}$ .

*If the first part of (i) holds, then  $u$  is  $\gamma_{\text{pr}}$ -critical, and if the second part of (i) holds, then  $v$  is  $\gamma_{\text{pr}}$ -critical.*

**Proof.** Let  $X = \{x_1, x_2, x_3\}$ . To dominate  $x_2$ ,  $X \cap D \neq \emptyset$ . We consider three cases.

*Case 1.*  $X \cap D = X$ . Without loss of generality assume that  $x_1$  is paired with  $u \in D$ , and  $x_2$  and  $x_3$  are paired. Then  $v \notin D$ , otherwise  $D \setminus \{x_2, x_3\}$  is also a paired dominating set of  $G_{uv,3}$ , contradicting the minimality of  $D$ . But now  $D' = (D \setminus X) \cup \{v\}$  is a paired dominating set of  $G$ , which is impossible because  $\text{msd}_{\text{pr}}(G) = 4$ .

*Case 2.*  $|X \cap D| = 2$ . If  $X \cap D = \{x_1, x_3\}$ , then  $\{u, v\} \subseteq D$  with  $u$  paired with  $x_1$ , and  $v$  with  $x_3$ . However, then  $D \setminus \{x_1, x_3\}$  is a paired dominating set of  $G$ , contradicting  $\text{msd}_{\text{pr}}(G) = 4$ . Suppose  $X \cap D = \{x_1, x_2\}$ . Then  $x_1$  and  $x_2$  are paired in  $D$ . If  $\{u, v\} \cap D \neq \emptyset$ , then  $D \setminus \{x_1, x_2\}$  is a paired dominating set of  $G$ , which is a contradiction. Hence  $D \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\}$ . Now  $D \setminus \{x_1, x_2\}$  is a paired dominating set of  $G - u$ , so  $\gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G_{uv,3}) = \gamma_{\text{pr}}(G)$ . We conclude that  $u$  is  $\gamma_{\text{pr}}$ -critical. Arguing similarly if  $X \cap D = \{x_2, x_3\}$ , we conclude that (i) and the last part of the statement of the lemma hold.

*Case 3.*  $|X \cap D| = 1$ . Then  $x_2 \notin D$ . If  $x_1 \in D$ , then  $x_1$  is paired with  $u \in D$ , while  $v \in D$  to dominate  $x_3$ . Consequently,  $D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_1, v\}$ . Similarly, if  $x_3 \in D$ , then  $D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_3, v\}$ . ■

Our first result regarding the construction of  $\text{msd-4}$  graphs from smaller graphs shows that subdividing any edge of an  $\text{msd-4}$  graph four times produces another  $\text{msd-4}$  graph. Repeatedly subdividing edges of an  $\text{msd-4}$  graph thus yields, for example,  $\text{msd-4}$  graphs of arbitrary large girth. In fact, we prove a stronger result: subdividing any edge of any graph  $G$  without isolated vertices four times produces a graph that has the same multisubdivision number as  $G$ .

**Proposition 7.** *For any graph  $G$  and any edge  $e$  of  $G$ ,  $\text{msd}_{\text{pr}}(G_{e,4}) = \text{msd}_{\text{pr}}(G)$ .*

**Proof.** Say  $\text{msd}_{\text{pr}}(G) = t \leq 4$  and  $e = uv$  has been subdivided by replacing it with the path  $(u, x_1, x_2, x_3, x_4, v)$ . Then  $\gamma_{\text{pr}}(G_{e,4}) = \gamma_{\text{pr}}(G) + 2$  and there exists an edge  $e'$  of  $G$  such that  $\gamma_{\text{pr}}(G_{e',t}) = \gamma_{\text{pr}}(G) + 2$ . If  $e \neq e'$ , then subdividing  $e \in E(G_{e',t})$  four times yields the graph  $(G_{e',t})_{e,4}$ . Since  $\text{msd}_{\text{pr}}(G_{e',t}) \leq 4$ ,  $\gamma_{\text{pr}}((G_{e',t})_{e,4}) = \gamma_{\text{pr}}(G_{e',t}) + 2 = \gamma_{\text{pr}}(G) + 4$ . But  $(G_{e',t})_{e,4} = (G_{e,4})_{e',t}$ , hence  $\gamma_{\text{pr}}((G_{e,4})_{e',t}) = \gamma_{\text{pr}}(G) + 4 = \gamma_{\text{pr}}(G_{e,4}) + 2$ . If  $e = e'$ , say  $uv$  has been subdivided, in  $G$ , by replacing it with  $(u, x_1, \dots, x_t, v)$ . Subdividing (without loss of generality) the edge  $x_tv$  four times by replacing it with  $(x_t, x_{t+1}, \dots, x_{t+4}, v)$ , we obtain the graph  $(G_{e,t})_{x_tv,4} = (G_{e,4})_{x_tv,t}$  with  $\gamma_{\text{pr}}((G_{e,4})_{x_tv,t}) = \gamma_{\text{pr}}(G_{e,4}) + 2$ . It follows that  $\text{msd}_{\text{pr}}(G_{e,4}) \leq t$ .

We show that  $\text{msd}_{\text{pr}}(G_{e,4}) \geq t$ . If  $t = 1$ , this is obvious, hence assume  $t \geq 2$ . Consider any  $e' \in E(G)$ . Suppose first that  $e' \neq e$ . Since  $\text{msd}_{\text{pr}}(G) = t$ ,  $\gamma_{\text{pr}}(G_{e',t-1}) = \gamma_{\text{pr}}(G)$ . If  $D'$  is any  $\gamma_{\text{pr}}(G_{e',t-1})$ -set, then  $D = D' \cup \{x_1, x_4\}$  (if  $u$  and  $v$  are paired in  $D'$ ) or  $D = D' \cup \{x_2, x_3\}$  (otherwise) is a paired dominating set of  $(G_{e,4})_{e',t-1}$  of cardinality  $|D| = \gamma_{\text{pr}}(G_{e',t-1}) + 2 = \gamma_{\text{pr}}(G) + 2 = \gamma_{\text{pr}}(G_{e,4})$ .

Assume  $e' = e$ . Without loss of generality subdivide the edge  $x_4v$  of  $G_{e,4}$   $t-1$  times by replacing it with the path  $(x_4, \dots, x_{3+t}, v)$  and denote the resulting graph  $(G_{e,4})_{x_4v,t-1}$  by  $G_{e,3+t}$  for simplicity. Also consider the graph  $G_{e,t-1}$  obtained from  $G$  by subdividing  $e = uv$  by replacing it with  $(u, x_1, \dots, x_{t-1}, v)$ . Since  $\text{msd}_{\text{pr}}(G) = t$ ,  $\gamma_{\text{pr}}(G_{e,t-1}) = \gamma_{\text{pr}}(G)$ . Let  $S'$  be any  $\gamma_{\text{pr}}(G_{e,t-1})$ -set. We consider three cases. In each case we construct a paired dominating set  $S$  of  $G_{e,3+t}$  such that  $|S| = |S'| + 2 = \gamma_{\text{pr}}(G_{e,4})$ ; this shows that  $\text{msd}_{\text{pr}}(G_{e,4}) \geq t$ .

*Case 1.*  $t = 2$ . If  $x_1 \notin S'$ , then without loss of generality  $u \in S'$  to dominate  $x_1$ , and  $S' \setminus \{u\}$  dominates  $v$ . Let  $S = S' \cup \{x_3, x_4\}$ . If  $x_1 \in S'$ , then again without loss of generality  $x_1$  is paired with  $u$ . Let  $S = S' \cup \{x_4, x_5\}$ .

*Case 2.*  $t = 3$ . If  $S' \cap \{x_1, x_2\} = \emptyset$ , then  $u$  dominates  $x_1$  while  $v$  dominates  $x_2$ ; let  $S = S' \cup \{x_3, x_4\}$  (so  $v$  dominates  $x_6$ ). If (without loss of generality)  $S' \cap \{x_1, x_2\} = \{x_1\}$ , then  $u$  and  $x_1$  are paired, and  $S' \setminus \{u, x_1\}$  dominates  $v$ . Let  $S = S' \cup \{x_4, x_5\}$ . If  $\{x_1, x_2\} \subseteq S'$ , then  $x_1$  and  $x_2$  are paired (otherwise  $S' \setminus \{x_1, x_2\}$  is a paired dominating set of  $G$ , which is not the case). Let  $S = S' \cup \{x_5, x_6\}$ .

*Case 3.*  $t = 4$ . By Lemma 6, without loss of generality  $S' \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\}$  or  $\{u, x_1, v\}$ . In the former case, let  $S = S' \cup \{x_5, x_6\}$ , and in the latter case, let  $S = S' \cup \{x_4, x_5\}$ .

In all cases,  $S$  is a paired dominating set of  $G_{e,3+t}$  of cardinality  $\gamma_{\text{pr}}(G) + 2 = \gamma_{\text{pr}}(G_{e,4})$ , and  $\text{msd}_{\text{pr}}(G_{e,4}) \geq t$ . It follows that  $\text{msd}_{\text{pr}}(G_{e,4}) = t$ , as required. ■

We next prove results pertaining to the  $\oplus$ -operations defined above that hold for general  $\text{msd}$ -4 graphs, not only block graphs. We show that the  $\oplus$ -operations can be used to construct new connected  $\text{msd}$ -4 graphs from smaller ones.

Our next result shows that performing the operation  $G_1 \oplus_u^{u_1 u_2} G_2$  on msd-4 graphs  $G_1$  and  $G_2$  with  $\gamma_{\text{pr}}$ -critical vertices  $u_1$  and  $u_2$ , respectively, results in an msd-4 graph in which each  $\gamma_{\text{pr}}(G_i)$ -critical vertex is  $\gamma_{\text{pr}}(G)$ -critical.

**Proposition 8.** *Let  $G_1$  and  $G_2$  be disjoint msd-4 graphs with  $\gamma_{\text{pr}}(G_i)$ -critical vertices  $u_i$ ,  $i = 1, 2$ . Then for the graph  $G = G_1 \oplus_u^{u_1 u_2} G_2$ ,  $\gamma_{\text{pr}}(G) = \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 2$ , any  $\gamma_{\text{pr}}(G_i)$ -critical vertex (including  $u$ ) is  $\gamma_{\text{pr}}(G)$ -critical and*

$$\text{msd}_{\text{pr}}(G) = 4.$$

**Proof.** Since  $u_i \in V(G_i)$  is  $\gamma_{\text{pr}}(G_i)$ -critical,  $\gamma_{\text{pr}}(G_1 - u_1) + \gamma_{\text{pr}}(G_2 - u_2) = \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 4$ , and at most two more vertices are needed to pairwise dominate  $G$ . Therefore  $\gamma_{\text{pr}}(G) \leq \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 2$ .

Suppose there exists a paired dominating set  $S$  of  $G$  such that  $|S| < \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 2$  and let  $S_i = S \cap V(G_i)$ . First suppose that  $u \notin S$ . Assume without loss of generality that  $S_1$  dominates  $u$ . Then  $S_1$  is a paired dominating set of  $G_1$  and  $S_2$  is a paired dominating set of  $G_2 - u_2$ . Hence  $|S_1| \geq \gamma_{\text{pr}}(G_1)$  and  $|S_2| \geq \gamma_{\text{pr}}(G_2) - 2$ . But then  $|S| = |S_1| + |S_2| \geq \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 2$ , which is not the case. Therefore we may assume that  $u \in S$  (in this case  $u_i \in S_i$ ,  $i = 1, 2$ ) and  $|S_1| + |S_2| = |S| + 1$ . Without loss of generality,  $u$  is paired with  $v \in V(G_1)$ , hence  $S_1$  is a paired dominating set of  $G_1$ . Therefore  $|S_1| \geq \gamma_{\text{pr}}(G_1)$  so that  $|S_2| \leq \gamma_{\text{pr}}(G_2) - 3$ . If  $N_{G_2}(u_2) \subseteq S_2$ , then  $S_2 \setminus \{u_2\}$  is a paired dominating set of  $G_2$ , and if there exists  $w \in N_{G_2}(u_2) \setminus S_2$ , then  $S_2 \cup \{w\}$  is a paired dominating set of  $G_2$ . This is impossible because  $|S_2 \cup \{w\}| \leq \gamma_{\text{pr}}(G_2) - 2$ . Hence

$$\gamma_{\text{pr}}(G) = \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 2.$$

If  $w_i$  is  $\gamma_{\text{pr}}(G_i)$ -critical, then, for  $j \neq i$ , the union of any  $\gamma_{\text{pr}}(G_i - w_i)$ -set and any  $\gamma_{\text{pr}}(G_j - u_j)$ -set is a paired dominating set of  $G - w_i$  (this holds for  $w_i = u_i = u$  also), so

$$\gamma_{\text{pr}}(G - w_i) \leq \gamma_{\text{pr}}(G_i - w_i) + \gamma_{\text{pr}}(G_j - u_j) = \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 4 < \gamma_{\text{pr}}(G).$$

Therefore  $w_i$  is  $\gamma_{\text{pr}}(G)$ -critical.

Without loss of generality consider  $e \in E(G_1)$  and subdivide  $e$  three times. Then, since  $\text{msd}_{\text{pr}}(G_1) = 4$  and  $u_2$  is  $\gamma_{\text{pr}}(G_2)$ -critical, we obtain

$$\gamma_{\text{pr}}(G_{e,3}) \leq \gamma_{\text{pr}}(G_{1e,3}) + \gamma_{\text{pr}}(G_2 - u_2) = \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) - 2 = \gamma_{\text{pr}}(G).$$

Therefore  $\text{msd}_{\text{pr}}(G) = 4$ . ■

We show next that performing the operation  $G_1 \oplus^{e_1 e_2} G_2$  on msd-4 graphs  $G_i$ ,  $i = 1, 2$ , with edges  $e_i = x_i y_i$ , where  $x_i$  is a  $\gamma_{\text{pr}}(G_i)$ -critical vertex, results in an msd-4 graph in which each  $\gamma_{\text{pr}}(G_i)$ -critical vertex is  $\gamma_{\text{pr}}(G)$ -critical.



**Proposition 9.** *Let  $G_i, i = 1, 2$ , be disjoint msd-4 graphs with  $e_i = x_i y_i \in E(G_i)$ , where  $x_i \in Cr(G_i)$ . Then for the graph  $G = G_1 \oplus^{e_1 e_2} G_2$ ,  $\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$ , any  $\gamma_{pr}(G_i)$ -critical vertex (including  $x = x_1 = x_2$ ) is  $\gamma_{pr}(G)$ -critical and  $msd_{pr}(G) = 4$ .*

**Proof.** By Theorem 1, there exists a  $\gamma_{pr}(G_i)$ -set in which  $x_i$  and  $y_i$  are matched. Therefore

$$(1) \quad \gamma_{pr}(G) \leq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2.$$

On the other hand, it suffices to add two vertices to a  $\gamma_{pr}(G)$ -set when splitting it into paired dominating sets of  $G_1$  and  $G_2$ . Hence we have equality in (1). As in the proof of Proposition 8, any  $\gamma_{pr}(G_i)$ -critical vertex is  $\gamma_{pr}(G)$ -critical.

Let  $e \in E(G)$  be any edge. If  $e \in E(G_1) \setminus \{e_1\}$ , then

$$\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1e,3}) + \gamma_{pr}(G_2 - x_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 = \gamma_{pr}(G).$$

The case when  $e \in E(G_2) \setminus \{e_2\}$  is analogous. Thus assume  $e = xy$  and subdivide  $e$  by replacing it with the path  $(x, u, v, w, y)$ . Let  $S$  be any  $\gamma_{pr}(G - x)$ -set. As shown above,  $|S| = \gamma_{pr}(G) - 2$ . Now  $S \cup \{u, v\}$  is a paired dominating set of  $G_{e,3}$  of cardinality  $\gamma_{pr}(G)$ . It follows that  $G$  is an msd-4 graph. ■

We now describe a type of “reverse” operation, called a *split operation*, for each of the  $\oplus$ -operations.

$G \ominus u$ . Let  $G$  be a connected graph with a cut-vertex  $u$ . Denote the components of  $G - u$  by  $F_1, F_2, \dots, F_k$ . For each  $i$ , let  $G_i$  be the graph obtained from  $F_i$  by adding a new vertex  $u_i$ , joining  $u_i$  to  $v_i \in V(F_i)$  if and only if  $uv_i \in E(G)$ . Denote the disjoint union  $G_1 + \dots + G_k$  by  $G \ominus u$ .

$G \ominus xy$ . Let  $G$  be a connected graph containing a vertex-cut  $\{x, y\}$ , where  $xy \in E(G)$ . Denote the components of  $G - \{x, y\}$  by  $F_1, F_2, \dots, F_k$ . For each  $i$ , let  $G_i$  be the graph obtained from  $F_i$  by adding the edge  $x_i y_i$ , joining  $x_i$  ( $y_i$ , respectively) to  $v_i \in V(F_i)$  if and only if  $xv_i \in E(G)$  ( $yv_i \in E(G)$ , respectively). Denote the disjoint union  $G_1 + \dots + G_k$  by  $G \ominus xy$ .

The next proposition shows that if an msd-4 graph  $G$  is split at a  $\gamma_{pr}$ -critical cut-vertex  $u$ , the components of  $G \ominus u$  are msd-4 graphs having the copies of  $u$  as  $\gamma_{pr}$ -critical vertices.

**Proposition 10.** *Let  $G$  be an msd-4 graph with a  $\gamma_{pr}$ -critical cut-vertex  $u$ . Denote the components of  $G \ominus u$  by  $G_1, \dots, G_k$ . Then for each  $i = 1, \dots, k$ ,  $u_i$  is a  $\gamma_{pr}(G_i)$ -critical vertex and  $msd_{pr}(G_i) = 4$ .*

**Proof.** Since  $u$  is  $\gamma_{\text{pr}}(G)$ -critical and  $G - u$  is the disjoint union of  $G_i - u_i$ ,  $i = 1, \dots, k$ ,

$$\gamma_{\text{pr}}(G) - 2 = \gamma_{\text{pr}}(G - u) = \sum_{i=1}^k \gamma_{\text{pr}}(G_i - u_i).$$

Suppose  $\gamma_{\text{pr}}(G_1 - u_1) \geq \gamma_{\text{pr}}(G_1)$ . Let  $R_1$  be a  $\gamma_{\text{pr}}(G_1)$ -set and, for  $i \geq 2$ , let  $R_i$  be a  $\gamma_{\text{pr}}(G_i - u_i)$ -set. Since  $R_1$  dominates  $u_1$ ,  $R = \bigcup_{i=1}^k R_i$  is a paired dominating set of  $G$ . But then

$$\gamma_{\text{pr}}(G) \leq |R| \leq \gamma_{\text{pr}}(G_1) + \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i) \leq \sum_{i=1}^k \gamma_{\text{pr}}(G_i - u_i) = \gamma_{\text{pr}}(G) - 2,$$

which is impossible. Thus  $u_1$  is  $\gamma_{\text{pr}}(G_1)$ -critical. The same argument works for each  $i \in \{2, \dots, k\}$ .

Consider an arbitrary edge  $e \in E(G_1)$  and subdivide  $e$  three times. Then

$$(2) \quad \gamma_{\text{pr}}(G_{e,3}) \leq \gamma_{\text{pr}}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i).$$

We show that equality holds in (2). Let  $S$  be any  $\gamma_{\text{pr}}(G_{e,3})$ -set and define  $S_1 = S \cap V(G_{1_{e,3}})$  and  $S_i = S \cap V(G_i)$  for  $i = 2, \dots, k$  (if  $u \in S$ , then  $u_i \in S_i$  for each  $i$ ). First suppose that  $u \notin S$ . If  $S_1$  dominates  $u$ , then  $S_1$  is a paired dominating set of  $G_{1_{e,3}}$  and  $S_i$ ,  $i \geq 2$ , is a paired dominating set of  $G_i - u_i$ . Hence  $|S_1| \geq \gamma_{\text{pr}}(G_{1_{e,3}})$  and  $|S_i| \geq \gamma_{\text{pr}}(G_i - u_i)$ , so that  $\gamma_{\text{pr}}(G_{e,3}) = |S| = \sum_{i=1}^k |S_i| \geq \gamma_{\text{pr}}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i)$  as required. On the other hand, if  $S_1$  does not dominate  $u$ , then  $S_j$  is a paired dominating set of  $G_j$  for some  $j \geq 2$ , so that  $|S_j| \geq \gamma_{\text{pr}}(G_j) = \gamma_{\text{pr}}(G_j - u_j) + 2$  (since  $u_j$  is  $\gamma_{\text{pr}}(G_j)$ -critical). Let  $S'_j$  be a  $\gamma_{\text{pr}}(G_j - u_j)$ -set,  $S'_1 = S_1 \cup \{u, u'\}$  for some  $u' \in N_{G_1}(u)$ , and  $S' = (S \setminus S_1 \setminus S_j) \cup S'_1 \cup S'_j$ . Then  $|S'| = |S|$ ,  $S'_1$  is a paired dominating set of  $G_{1_{e,3}}$  and the result follows as before.

Now suppose that  $u \in S$ . Then  $|S_1| + \sum_{i=2}^k |S_i| = |S| + k - 1$  and  $u$  is paired with a vertex in exactly one of the graphs  $G_{1_{e,3}}$  or  $G_i$ ,  $i \geq 2$ . For each of the  $k - 1$  other graphs, either  $S_i \cup \{w_i\}$ , for some neighbour  $w_i \notin S_i$  of  $u_i$ , or  $S_i \setminus \{u_i\}$  (if all neighbours of  $u_i$  in  $G_i$  belong to  $S_i$ ) is a paired dominating set. Hence

$$\gamma_{\text{pr}}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{\text{pr}}(G_i) \leq |S| + 2(k - 1).$$

Since  $u_i$  is  $\gamma_{\text{pr}}(G_i)$ -critical for each  $i = 2, 3, \dots, k$ ,  $\gamma_{\text{pr}}(G_i - u_i) = \gamma_{\text{pr}}(G_i) - 2$ . This gives

$$\gamma_{\text{pr}}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i) \leq |S| = \gamma_{\text{pr}}(G_{e,3}).$$

Therefore we have equality (2). Now

$$\begin{aligned} \gamma_{\text{pr}}(G_{1e,3}) &= \gamma_{\text{pr}}(G_{e,3}) - \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i) = \gamma_{\text{pr}}(G) - \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i) \\ &= \gamma_{\text{pr}}(G_1) + \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i) - \sum_{i=2}^k \gamma_{\text{pr}}(G_i - u_i) = \gamma_{\text{pr}}(G_1). \end{aligned}$$

Hence, for any edge  $e \in E(G_1)$ ,  $\gamma_{\text{pr}}(G_{1e,3}) = \gamma_{\text{pr}}(G)$ . Thus  $\text{msd}_{\text{pr}}(G_1) = 4$ . Similar reasoning may be applied to  $G_i$  for  $i \in \{2, 3, \dots, k\}$ . ■

### 5. MSD-4 BLOCK GRAPHS

The last three results we need for the proof of Theorem 4 concern block graphs. In the first result we prove that every non-leaf vertex of an msd-4 block graph is a cut-vertex.

**Theorem 11.** *Let  $G$  be a graph containing a block  $B \cong K_n$ , where  $n \geq 3$ , such that some vertex of  $B$  is not adjacent to any vertex of  $G - B$ . Then*

$$\text{msd}_{\text{pr}}(G) < 4.$$

**Proof.** Suppose the hypothesis of the theorem holds but  $\text{msd}_{\text{pr}}(G) = 4$ . Let  $V(B) = \{v_0, \dots, v_{n-1}\}$  and say  $u = v_0$  is not adjacent to any vertex of  $G - B$ . Subdivide the edge  $uv_2$  by replacing it with the path  $(u, x_3, x_2, x_1, v_2)$  (see Figure 3). Denote  $X = \{x_1, x_2, x_3\}$  and let  $D$  be a  $\gamma_{\text{pr}}$ -set of  $G_{uv_2,3}$ . By Lemma 6 we only have to consider the cases  $D \cap \{u, x_1, x_2, x_3, v_2\} \in \{\{u, x_1, v_2\}, \{u, x_3, v_2\}, \{x_1, x_2\}, \{x_2, x_3\}\}$ .

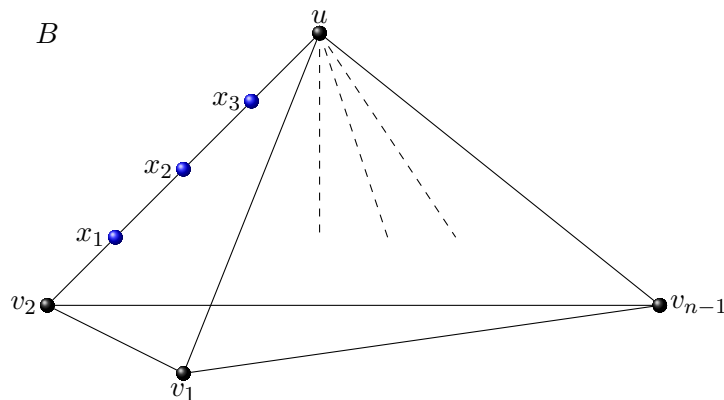


Figure 3. The block  $B$  with the edge  $uv_2$  subdivided with vertices  $x_1, x_2, x_3$ .

*Case 1.*  $|X \cap D| = 1$ . If  $D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_1, v_2\}$ , then  $x_1$  and  $v_2$  are paired in  $D$ , while  $u$  is paired with  $v_i$  for some  $i \neq 0, 2$ . However, then  $D \setminus \{x_1, u\}$ , with  $v_2$  and  $v_i$  paired, is a smaller paired dominating set of  $G$ . If  $D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_3, v_2\}$ , then  $D \setminus \{x_3, u\}$  is a smaller paired dominating set of  $G$ . In either case  $\text{msd}_{\text{pr}}(G) < 4$ , contrary to our assumption.

*Case 2.*  $|X \cap D| = 2$ . If  $D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_1, x_2\}$ , then  $x_1$  and  $x_2$  are paired in  $D$ . To pairwise dominate  $u$ ,  $v_i \in D$  for some  $i \neq 0, 2$ . But then  $D \setminus \{x_1, x_2\}$  is a paired dominating set of  $G$  (with  $v_i$  paired as in  $D$ ) and  $\text{msd}_{\text{pr}}(G) < 4$ , contrary to our assumption. Hence assume  $D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_2, x_3\}$ . Then  $x_2$  and  $x_3$  are matched in  $D$ . If  $v_i \in D$  for some  $i$ , then  $D \setminus \{x_2, x_3\}$  is a paired dominating set of  $G$  (again with  $v_i$  paired as in  $D$ ), a contradiction.

We therefore assume henceforth that

- (i)  $D$  contains  $x_2$  and  $x_3$ , but neither  $x_1$  nor any  $v_0, \dots, v_{n-1}$ .

By Lemma 6,  $u$  is  $\gamma_{\text{pr}}$ -critical, that is,

- (ii)  $\gamma_{\text{pr}}(G - u) = \gamma_{\text{pr}}(G) - 2$ .

For each  $i = 1, \dots, n-1$ , let  $G_i$  be the component of  $G - E(B)$  that contains  $v_i$ . Since  $B$  is a block of  $G$ , the subgraphs  $G_i$  are distinct and pairwise vertex-disjoint. Let  $D_i = D \cap V(G_i)$ . Then  $|\bigcup_{i=1}^{n-1} D_i| = |D \setminus \{x_2, x_3\}| = \gamma_{\text{pr}}(G) - 2$ . By (i), each  $D_i$  is a  $\gamma_{\text{pr}}(G_i)$ -set that does not contain  $v_i$ .

We next show that

- (iii) no  $\gamma_{\text{pr}}(G)$ -set contains  $u = v_0$  and at least two  $v_i$ ,  $i \geq 1$ .

Suppose there exists such a set  $Z$ ; assume without loss of generality that  $\{u, v_1, v_2, \dots, v_k\} \subseteq Z$ ,  $k \geq 2$ . Necessarily,  $u$  is paired with some  $v_i$ ,  $i = 1, \dots, k$ , in  $Z$ . Assume (again without loss of generality)  $u$  is paired with  $v_1$ . Let  $Z_1 = Z \cap V(G_1) \setminus \{v_1\}$  and, for  $i \geq 2$ , let  $Z_i = Z \cap V(G_i)$ . Then  $\bigcup_{i=1}^{n-1} Z_i \subseteq V(G - u)$  and  $|\bigcup_{i=1}^{n-1} Z_i| = |Z| - 2 = \gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G)$ , by (ii). Since  $v_1$  and  $u$  are paired,  $G_1[Z_1]$  contains a perfect matching, as does  $G[\bigcup_{i=2}^{n-1} Z_i]$ . Since  $v_1$  is not adjacent to any vertex of  $G_i - v_i$ ,  $i \geq 2$ , and  $v_2$  dominates  $B$  in  $G$ ,  $\bigcup_{i=2}^{n-1} Z_i$  is a paired dominating set of  $G - G_1$ .

Suppose  $|Z_1| < |D_1|$ . Since both  $Z_1$  and  $D_1$  have even cardinality,  $|Z_1| \leq |D_1| - 2$ . Then  $Z_1$  does not dominate  $G_1 - v_1$ , otherwise  $\bigcup_{i=1}^{n-1} Z_i$  is a paired dominating set of  $G$  of cardinality less than  $\gamma_{\text{pr}}(G)$ , which is impossible. Since  $Z_1 \cup \{v_1\}$  dominates  $G_1$ , there exists a vertex  $w \in N_{G_1}(v_1)$  that is undominated by  $Z_1$ . Then  $W_1 = Z_1 \cup \{w, v_1\}$  is a paired dominating set of  $G_1$  of cardinality at most  $|D_1|$  that contains  $v_1$ . But now  $W_1 \cup D_2 \cup D_3 \cup \dots \cup D_{n-1}$  is a paired dominating set of  $G$  of cardinality at most  $|D \setminus \{x_2, x_3\}| = \gamma_{\text{pr}}(G) - 2$ , which is impossible. We conclude that  $|Z_1| = |D_1|$ .

Let  $Z' = D_1 \cup (\bigcup_{i=2}^{n-1} Z_i)$ . Since  $\bigcup_{i=2}^{n-1} Z_i$  is a paired dominating set of  $G - G_1$  and  $D_1$  is a paired dominating set of  $G_1$ ,  $Z'$  is a paired dominating set of  $G$ .

Moreover,

$$|Z'| = \left| \bigcup_{i=2}^{n-1} Z_i \right| + |D_1| = \left| \bigcup_{i=1}^{n-1} Z_i \right| = |Z| - 2 = \gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G),$$

which is impossible. This concludes the proof of (iii).

Subdivide the edge  $v_1v_2$  with vertices  $y_1, y_2, y_3$ , where  $y_1$  is adjacent to  $v_1$  and  $y_3$  is adjacent to  $v_2$  (see Figure 4). Denote  $Y = \{y_1, y_2, y_3\}$  and let  $Q$  be a  $\gamma_{\text{pr}}$ -set of  $G_{v_1v_2,3}$ . Without loss of generality, by Lemma 6 we only have to consider the cases  $Q \cap \{v_1, v_2, y_1, y_2, y_3\} \in \{\{y_1, y_2\}, \{v_1, v_2, y_1\}\}$ .

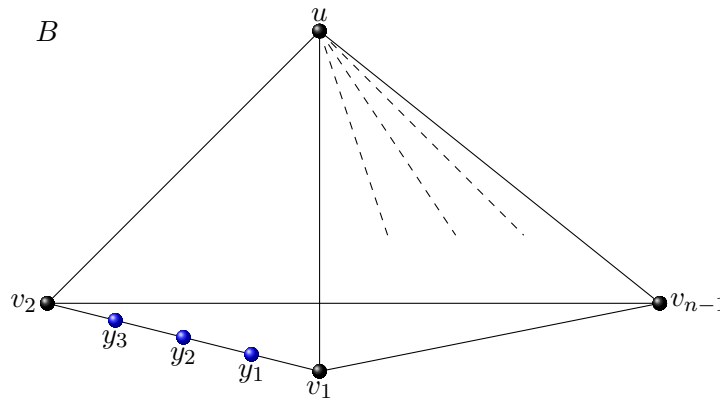


Figure 4. The block  $B$  with the edge  $v_1v_2$  subdivided with vertices  $y_1, y_2, y_3$ .

*Case 3a.*  $Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{y_1, y_2\}$ . Then these two vertices are paired in  $Q$ . To pairwise dominate  $u, v_i \in Q$  for some  $i$ . It follows that  $Q \setminus \{y_1, y_2\}$  is a paired dominating set of  $G$ , so  $\text{msd}_{\text{pr}}(G) < 4$ , contrary to our assumption.

*Case 3b.*  $Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{v_1, v_2, y_1\}$ . Then  $y_1$  is paired with  $v_1$ . If  $u \notin Q$ , then  $Q' = (Q \setminus \{y_1\}) \cup \{u\}$  is a paired dominating set of  $G$  containing  $u, v_1, v_2$ . By (iii),  $Q'$  is not a  $\gamma_{\text{pr}}$ -set of  $G$ , from which it follows that  $\gamma_{\text{pr}}(G) < |Q|$  and  $\text{msd}_{\text{pr}}(G) < 4$ . Assume therefore that  $u \in Q$ . Then  $u$  is paired in  $Q$  with  $v_i$  for some  $i > 1$ . Now  $Q'' = Q \setminus \{y_1, u\}$  is a paired dominating set of  $G$  in which  $v_1$  and  $v_i$  are paired. In both cases we again have a contradiction and the proof is complete. ■

The graph in Figure 5 shows that the statement of Theorem 11 is false if the complete subgraph  $B$  is not a block of  $G$ .

The next result in this section shows that  $\text{msd}$ -4 block graphs have many  $\gamma_{\text{pr}}$ -critical vertices.

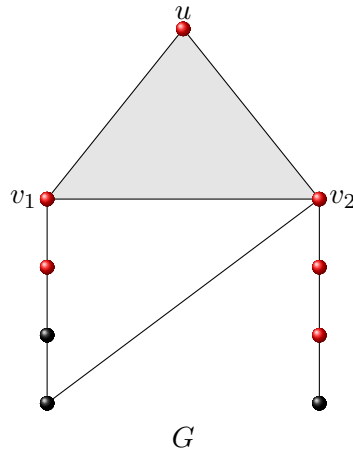


Figure 5. A graph  $G$  with  $\text{msd}_{\text{pr}}(G) = 4$  and a subgraph  $K_3$  that is not a block of  $G$ .

**Theorem 12.** *If  $G$  is a block graph with  $\text{msd}_{\text{pr}}(G) = 4$ , then for any edge  $uv \in E(G)$ ,*

$$(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) \neq \emptyset.$$

**Proof.** Suppose there exists an edge  $uv \in E(G)$  such that  $(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) = \emptyset$ . By Theorem 1, no vertex in  $N_G[u] \cup N_G[v]$  is a leaf. We subdivide the edge  $uv$  by replacing it with the path  $(u, x_1, x_2, x_3, v)$  to obtain the graph  $G_{uv,3}$ . By Lemma 6, for any  $\gamma_{\text{pr}}$ -set  $S$  of  $G_{uv,3}$ ,  $S \cap \{u, v, x_1, x_2, x_3\} \in \{\{u, v, x_1\}, \{u, v, x_3\}\}$ . Without loss of generality assume there exists such a set  $S$  such that  $S \cap \{u, v, x_1, x_2, x_3\} = \{u, v, x_1\}$ , and among all such sets  $S$ , let  $D$  be one for which  $\text{PN}(u, D)$  is as small as possible. Then  $x_1$  and  $u$  are paired in  $D$ .

Say  $v$  is paired with  $v'$  and let  $B$  be the block of  $G$  that contains  $uv$ . If  $v' \in V(G) \setminus V(B)$ , let  $G_v$  be the subgraph of  $G - E(B)$  that contains  $v$ , and if  $v' \in V(B)$ , let  $G_v$  be the subgraph of  $G - (E(B) - \{vv'\})$  that contains  $v$ . In either case,  $v' \in V(G_v)$ . Let  $D_v = D \cap V(G_v)$  and  $D' = D \setminus \{x_1, u\}$ . Then  $G[D']$  has a perfect matching and  $D_v$  is a paired dominating set of  $G_v$  containing  $v$  and  $v'$ . In fact,  $D_v$  is a  $\gamma_{\text{pr}}(G_v)$ -set, for if not, let  $D''$  be a smaller paired dominating set of  $G_v$ . Consider  $N_G(u) \setminus V(B)$ . If  $B \cong K_2$ , then  $N_G(u) \setminus V(B) = N_G(u) \setminus \{v\}$  is nonempty because  $u$  is not a leaf, and if  $B \cong K_n$  for  $n \geq 3$ , then  $N_G(u) \setminus V(B)$  is nonempty by Theorem 11. If  $N_G(u) \setminus V(B) \subseteq D$ , then  $D'$  is a paired dominating set of  $G$ , and if there exists  $w \in N_G(u) \setminus V(B) \setminus D$ , then  $(D \setminus \{x_1\} \setminus D_v) \cup D'' \cup \{w\}$  is a smaller paired dominating set of  $G$  than  $D$ . In both cases we have a contradiction to  $\text{msd}_{\text{pr}}(G) = 4$ .

Since  $\text{msd}_{\text{pr}}(G) = 4$ ,  $|D'| = \gamma_{\text{pr}}(G_{uv,3}) - 2 = \gamma_{\text{pr}}(G) - 2$ . Consequently,  $D'$  does not dominate  $G$ . Since  $v \in D'$  dominates  $B$  in  $G$ , there exist vertices  $w_1, \dots, w_k \in N_G(u) \setminus N_G[v] \subseteq N_G(u) \setminus B$  that are undominated by  $D'$ , that is,

$\{w_1, \dots, w_k\} = \text{PN}(u, D)$ . For  $i = 1, \dots, k$ , let  $G_i$  be the component of  $G - u$  that contains  $w_i$ . Possibly,  $G_i = G_j$  for  $i \neq j$ ; this happens exactly when  $w_i w_j \in E(G)$ , and then  $w_i$  and  $w_j$  also belong to the same (complete) block of  $G_i$ . Since no  $w_i$  is adjacent to  $v$  or  $v'$ ,  $V(G_i) \cap V(G_v) = \emptyset$  for each  $i$ . Define  $D_i = D \cap V(G_i)$ . Then  $G_i[D_i]$  has a perfect matching, but does not dominate  $w_i$ . If it is nevertheless true that  $\gamma_{\text{pr}}(G_i) = |D_i|$  for some  $i$ , let  $Q_i$  be a  $\gamma_{\text{pr}}(G_i)$  set. Then  $D^* = (D \setminus D_i) \cup Q_i$  is a  $\gamma_{\text{pr}}(G_{uv,3})$ -set such that  $\text{PN}(u, D^*) \subseteq \text{PN}(u, D) \setminus \{w_i\}$ , contrary to the choice of  $D$ . Therefore  $\gamma_{\text{pr}}(G_i) \geq |D_i| + 2$  for each  $i$ .

Since each stem belongs to all paired dominating sets, no  $w_i$  is a stem, and by our initial assumption, no  $w_i$  is a leaf. Subdivide the edge  $uw_1$  by replacing it with the path  $(u, y_1, y_2, y_3, w_1)$ . Consider a  $\gamma_{\text{pr}}(G_{uw_1,3})$ -set  $S$ . Since  $u, w_1 \notin \text{Cr}(G)$ , Lemma 6 states that  $S \cap \{u, y_1, y_2, y_3, w_1\} \in \{\{u, y_1, w_1\}, \{u, y_3, w_1\}\}$ .

- In the former case,  $y_1$  is paired with  $u$  and  $S_1 = S \cap V(G_1)$  is a paired dominating set of  $G_1$ ; hence  $|S_1| \geq \gamma_{\text{pr}}(G_1) \geq |D_1| + 2$ . Since  $w_1$  is adjacent to all  $w_i \in V(G_1)$ ,  $D_1 \cup \{w_1\}$  dominates  $G_1$  (but not pairwise). Now  $S' = (S \setminus S_1) \cup D_1 \cup \{w_1, y_3\}$  is a paired dominating set of  $G_{uw_1,3}$  such that  $|S'| \leq |S|$ , hence  $S'$  is a  $\gamma_{\text{pr}}(G_{uw_1,3})$ -set. Moreover,  $S' \cap \{u, y_1, y_2, y_3, w_1\} = \{u, y_1, y_3, w_1\}$ , contrary to Lemma 6.

- In the latter case,  $y_3$  is paired with  $w_1$ . Then  $S_2 = (S \cap V(G_1)) \cup \{y_3\}$  is a paired dominating set of the graph obtained from  $G_1$  by joining  $y_3$  to  $w_1$ . If all neighbours of  $w_1$  in  $G_1$  belong to  $S_2$ , then  $S_2 \setminus \{w_1, y_3\}$  is a paired dominating set of  $G_1$ . But then  $S'' = S \setminus \{w_1, y_3\}$  is a paired dominating set of  $G$  such that  $|S''| < |S|$ , contradicting  $\text{msd}_{\text{pr}}(G) = 4$ . Assume some neighbour  $z$  of  $w_1$  in  $G_1$  does not belong to  $S_2$ . Then  $S_3 = (S_2 \setminus \{y_3\}) \cup \{z\}$  is a paired dominating set of  $G_1$ , so that  $|S_2| = |S_3| \geq |D_1| + 2$ . Since  $u \in S$  and  $\{w_1, \dots, w_k\} \subseteq N(u)$ ,  $S^* = (S \setminus S_2) \cup D_1$  is a paired dominating set of  $G$  such that  $|S^*| < |S|$ , again a contradiction.

This completes the proof of the theorem. ■

Although the graph  $G$  in Figure 5 satisfies  $\text{msd}_{\text{pr}}(G) = 4$  without being a block graph, Theorem 12 holds for  $G$  as well.

Our final result in this section concerns the reverse operation  $G \ominus xy$  for certain  $\text{msd}$ -4 block graphs.

**Proposition 13.** *Let  $G$  be a connected  $\text{msd}$ -4 block graph such that the only  $\gamma_{\text{pr}}(G)$ -critical vertices are leaves. Let  $x$  be a leaf adjacent to the stem  $y$ , where  $\{x, y\}$  is a vertex-cut, and denote the components of  $G \ominus xy$  by  $G_1, \dots, G_k$ . Then for each  $i = 1, \dots, k$ ,  $G_i$  is an  $\text{msd}$ -4 graph and  $x_i \in \text{Cr}(G_i)$ .*

**Proof.** If  $G_i$  is an  $\text{msd}$ -4 graph, it will follow from Theorem 1(ii) that  $x_i \in \text{Cr}(G_i)$ . However, we need the fact that  $x_i$  is  $\gamma_{\text{pr}}(G_i)$ -critical to show that  $\text{msd}_{\text{pr}}(G_i) = 4$ , hence this is what we prove first.

Since  $G$  is a block graph,  $N_{G_i-x_i}(y_i)$  induces a clique for each  $i = 1, \dots, k$ . Since  $x$  is a leaf,  $y$  belongs to every paired dominating set of  $G$ , and by Theorem 1(ii),  $x \in \text{Cr}(G)$ . Hence  $y$  belongs to no  $\gamma_{\text{pr}}(G-x)$ -set (for such a set would dominate  $x$  and thus  $G$ , contradicting  $x \in \text{Cr}(G)$ ).

Let  $D$  be a  $\gamma_{\text{pr}}(G-x)$  set such that  $|D \cap N(y)|$  is maximum and let  $D_i = D \cap V(G_i)$ ,  $i = 1, \dots, k$ . Since  $x \in \text{Cr}(G)$  and  $y \notin D$ ,  $|D| = \sum_{i=1}^k |D_i| = \gamma_{\text{pr}}(G) - 2$ . Also,  $D_i$  is a paired dominating set of  $G_i - \{x_i, y_i\}$  for each  $i$ , and a paired dominating set of  $G_i - x_i$  for at least one  $i$ . We show that, in fact,

(A)  $D_i$  is a paired dominating set of  $G_i - x_i$  for each  $i$ .

First suppose  $|N_{G_i-x_i}(y_i)| \geq 2$ ; say  $z_1, z_2 \in N_{G_i-x_i}(y_i)$ . Since  $N_{G_i-x_i}(y_i)$  induces a clique,  $z_1 z_2 \in E(G)$ . By Theorem 12,  $(N_G[z_1] \cup N_G[z_2]) \cap \text{Cr}(G) \neq \emptyset$ . Since  $N_G[z_i] = N_{G_i-x_i}[z_i]$  and  $z_i$  is not a leaf (and thus, by the hypothesis, not  $\gamma_{\text{pr}}(G)$ -critical),  $z_1$  or  $z_2$  is adjacent to a  $\gamma_{\text{pr}}(G)$ -critical vertex, i.e., a leaf. Say  $z_1$  is adjacent to a leaf  $z'$ . Then  $z_1$  belongs to any paired dominating set of any subgraph of  $G$  containing both  $z_1$  and  $z'$ , so  $z_1 \in D$ . Therefore  $D_i$  dominates  $y_i$  and (A) holds.

Assume therefore that  $|N_{G_i-x_i}(y_i)| = 1$ , say  $N_{G_i-x_i}(y_i) = \{z\}$ . If  $z \in D$ , we are done, hence assume  $z \notin D$ . By Theorem 1(iii),  $z$  is not a leaf, hence there exists a vertex  $z' \in N_{G_i-x_i}(z) \setminus \{y_i\}$ . By Theorem 1(i),  $G$  has a  $\gamma_{\text{pr}}$ -set  $X$  such that  $zz'$  belongs to a matching of  $G[X]$ . Now  $y \in X$ , but  $y$  is not paired with any vertex of  $G_i - x_i$ , since  $N_{G_i-x_i}(y_i) = \{z\}$ . Therefore  $X_i = (X \setminus \{x, y\}) \cap V(G_i)$  is a paired dominating set of  $G_i - x_i$ . Moreover,  $|X_i| \leq |D_i|$ , otherwise  $(X - X_i) \cup D_i$  is a smaller paired dominating set of  $G$ , which is impossible. However, now  $D' = (D \setminus D_i) \cup X_i$  is a paired dominating set of  $G - x$ , hence a  $\gamma_{\text{pr}}(G-x)$ -set, containing more neighbours of  $y$  than  $D$ , contrary to the choice of  $D$ . Hence (A) holds in this case as well.

Therefore  $\gamma_{\text{pr}}(G_i - x_i) \leq |D_i|$  for each  $i$ , so that

$$(3) \quad \sum_{i=1}^k \gamma_{\text{pr}}(G_i - x_i) \leq \sum_{i=1}^k |D_i| = |D| = \gamma_{\text{pr}}(G - x).$$

Suppose there exists a  $\gamma_{\text{pr}}(G_i - x_i)$ -set  $Y_i$  containing  $y_i$ . Since no  $D_j$  contains  $y_j$ ,  $D' = (D \setminus D_i) \cup Y_i$  is a paired dominating set of  $G - x$  such that  $|D'| \leq |D| = \gamma_{\text{pr}}(G) - 2$  and  $D'$  dominates  $x$ . Then  $D'$  is a paired dominating set of  $G$ , which is impossible. Therefore no  $\gamma_{\text{pr}}(G_i - x_i)$ -set contains  $y_i$ . Similarly, if  $\gamma_{\text{pr}}(G_i - x_i) < |D_i|$  for some  $i$  and  $Z_i$  is a  $\gamma_{\text{pr}}(G_i - x_i)$ -set, then  $D'' = (D \setminus D_i) \cup Z_i$  is a paired dominating set of  $G - x$  such that  $|D''| < |D|$ , which is also impossible. From these two facts we deduce that  $D_i$  is a  $\gamma_{\text{pr}}(G_i - x_i)$ -set, equality holds in (3) and  $\gamma_{\text{pr}}(G_i) = \gamma_{\text{pr}}(G_i - x_i) + 2$ , that is,  $x_i$  is  $\gamma_{\text{pr}}(G_i)$ -critical for each  $i$ .

We show that  $\text{msd}_{\text{pr}}(G_1) = 4$ ; it will follow similarly that  $\text{msd}_{\text{pr}}(G_i) = 4$  for each  $i$ . Since  $D_1$  is a  $\gamma_{\text{pr}}(G_1 - x_1)$ -set, it is easy to see that we can pairwise



dominate  $G_{1_{xy,3}}$  by  $|D_1| + 2 = \gamma_{pr}(G_1)$  vertices. Hence consider any edge  $e \in E(G_1 - x_1)$  and the graphs  $G_{e,3}$  and  $G_{1_{e,3}}$ . Since combining any  $\gamma_{pr}(G_{1_{e,3}})$ -set with the sets  $D_j, j = 2, \dots, k$ , produces a paired dominating set of  $G_{e,3}$ ,

$$(4) \quad \gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{pr}(G_i - x_i).$$

We show that equality holds in (4). For convenience of notation, define  $H_1 = G_{1_{e,3}}$  and  $H_i = G_i, i \geq 2$ . Let  $S$  be a  $\gamma_{pr}(G_{e,3})$ -set and define  $S_i = S \cap V(H_i)$  for  $i = 1, \dots, k$  (since  $y \in S, y_i \in S_i$  for each  $i$ , and if  $x \in S$ , then  $x_i \in S_i$  for each  $i$ ). We consider two cases, depending on whether  $x \in S$  or not.

*Case 1.*  $x \notin S$ . Then  $\sum_{i=1}^k |S_i| = |S| + k - 1$ . Note that  $y$  is paired with  $w \in V(H_i) \setminus \{x_i, y_i\}$  for exactly one  $i$ . Then  $S_i$  is a paired dominating set of  $H_i$ . For  $j \neq i, S_j \cup \{x_j\}$  is a paired dominating set of  $H_j$ . Therefore  $\gamma_{pr}(H_i) \leq |S_i|$  and  $\gamma_{pr}(H_j) \leq |S_j| + 1$  for  $j \neq i$ . For  $\ell \geq 2, x_\ell$  is  $\gamma_{pr}(H_\ell)$ -critical, hence  $\gamma_{pr}(H_\ell - x_\ell) \leq \gamma_{pr}(H_\ell) - 2$ . Therefore

$$\gamma_{pr}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^k |S_i| - 2(k-1) + (k-1) = \sum_{i=1}^k |S_i| - (k-1) = |S|$$

and equality holds in (4).

*Case 2.*  $\{x, y\} \subseteq S$ . Then  $x$  and  $y$  are paired in  $S, \{x_i, y_i\} \subseteq S_i$  for each  $i$ , and  $S_i$  is a paired dominating set of  $H_i$ . Also,  $\sum_{i=2}^k |S_i| = |S| + 2(k-1) - |S_1|$ . Since  $x_i$  is  $\gamma_{pr}(G_i)$ -critical,

$$\gamma_{pr}(G_{1_{e,3}}) + \sum_{i=2}^k \gamma_{pr}(G_i - x_i) \leq |S_1| + \sum_{i=2}^k |S_i| - 2(k-1) = |S| = \gamma_{pr}(G_{e,3}),$$

giving equality in (4).

It now follows as in the proof of Proposition 10 that  $\text{msd}(G_1) = 4$ . Similarly,  $\text{msd}(G_i) = 4$  for  $i \geq 2$ . ■

### 6. PROOF OF THEOREM 4

We are now ready to prove our main theorem, the characterization of  $\text{msd}$ -4 block graphs. We restate the theorem here for convenience.

**Theorem 4** (again). *Let  $G$  be a connected block graph. Then  $G$  is an  $\text{msd}$ -4 graph if and only if  $G \in \mathcal{B}$ . Moreover, if  $G$  is an  $\text{msd}$ -4 graph constructed from the graphs  $H_1, \dots, H_j \in \mathcal{U}$ , then  $\text{Cr}(G) = \bigcup_{i=1}^j \text{Cr}(H_i)$ .*

**Proof.** If  $G \in \mathcal{B}$ , it follows immediately from Propositions 8 and 9 that  $G$  is an msd-4 graph and  $\text{Cr}(G) = \bigcup_{i=1}^j \text{Cr}(H_i)$ .

For the converse, let  $G$  be an msd-4 block graph. If  $G$  is a tree, the result follows from Corollary 5, hence we assume that  $B \cong K_n$ ,  $n \geq 3$ , is a block of  $G$ . By (the contrapositive of) Theorem 11, each vertex of  $B$  is a cut-vertex, so  $\deg(v) \geq n$  for each  $v \in V(B)$ . Since each non-leaf vertex of a  $K_2$ -block is a cut-vertex, we deduce that each vertex of  $G$  is either a leaf or a cut-vertex.

Suppose  $v \in V(B)$  is  $\gamma_{\text{pr}}$ -critical. Applying Proposition 10 to  $v$  we obtain an msd-4 graph  $G_1$  with  $v_1 = v$  and  $N_{G_1}[v_1] = B$ , which contradicts Theorem 11. Thus every  $\gamma_{\text{pr}}(G)$ -critical vertex belongs only to  $K_2$ -blocks.

We say that a vertex  $u$  is a *type-A vertex* if it is a  $\gamma_{\text{pr}}(G)$ -critical cut-vertex, and an edge  $uv$  is a *type-A edge* if  $u$  is a leaf (hence  $\gamma_{\text{pr}}(G)$ -critical) and  $G - \{u, v\}$  is disconnected. Denote the number of type-A elements (vertices and edges together) of  $G$  by  $a(G)$ . First we show that

(B) if  $a(G) = 0$ , then  $G \in \mathcal{U}$ .

Suppose  $a(G) = 0$ . Then every  $\gamma_{\text{pr}}(G)$ -critical vertex is a leaf. Say  $V(B) = \{v_1, \dots, v_n\}$ . Since no vertex of  $B$  is  $\gamma_{\text{pr}}(G)$ -critical, Theorem 12 implies that  $v_1$  or  $v_n$  is adjacent to a  $\gamma_{\text{pr}}(G)$ -critical vertex. Without loss of generality we assume that  $v_1 u_1 \in E(G)$ ,  $u_1 \notin V(B)$ , and  $u_1$  is  $\gamma_{\text{pr}}(G)$ -critical. Similarly, without loss of generality,  $v_i$  is adjacent to a  $\gamma_{\text{pr}}(G)$ -critical vertex  $u_i \notin V(B)$  for  $i = 2, \dots, n-1$ . Since  $a(G) = 0$  and each vertex of  $G$  is either a leaf or a cut-vertex,  $\deg_G(u_i) = 1$  for each  $i = 1, \dots, n-1$  and  $G - \{v_i, u_i\}$  is connected. Thus,  $v_i$  belongs to only the two blocks  $B$  and  $v_i u_i$ , so  $\deg_G(v_i) = n$  for each  $i = 1, \dots, n-1$ .

Since  $v_n$  is a cut-vertex,  $N(v_n) \setminus V(B) \neq \emptyset$ . If  $v_n$  is adjacent to a  $\gamma_{\text{pr}}(G)$ -critical vertex, say  $u_n$ , then, arguing as above,  $\deg(u_n) = 1$ ,  $\deg(v_n) = n$  and  $G = K_n \circ K_1$ . By Remark 3(i),  $n$  is odd, hence  $G$  belongs to the family  $\mathcal{U} \subseteq \mathcal{B}$ . If no vertex in  $N(v_n) \setminus V(B)$  is critical, let  $N(v_n) \setminus V(B) = \{w_1, \dots, w_t\}$  for  $t \geq 1$ . By Theorem 12, each  $w_i$  is adjacent to a critical vertex  $w'_i \neq v_n$ , and since  $a(G) = 0$ ,  $w'_i$  is a leaf. We show that

(C)  $\{w_1, \dots, w_t\}$  is an independent set of  $G$ .

Suppose (without loss of generality) that  $w_1 w_2 \in E(G)$  and consider  $G_{w_1 w_2, 3}$ . Let  $w_1, x_1, x_2, x_3, w_2$  be the  $w_1 - w_2$  path in  $G_{w_1 w_2, 3}$  and let  $D$  be a  $\gamma_{\text{pr}}(G_{w_1 w_2, 3})$ -set. Since  $w'_1$  and  $w'_2$  are leaves,  $w_1, w_2 \in D$ . To dominate  $x_2$ ,  $\{x_1, x_2, x_3\} \cap D \neq \emptyset$ . If  $|\{x_1, x_2, x_3\} \cap D| = 2$ , then  $D \setminus \{x_1, x_2, x_3\}$  is a paired dominating set (with  $w_1$  and  $w_2$  paired) of  $G$  of smaller cardinality than  $D$ , contrary to  $\text{msd}(G) = 4$ . Hence assume without loss of generality that  $\{x_1, x_2, x_3\} \cap D = \{x_1\}$ , so  $w_1$  and  $x_1$  are paired (and  $w'_1 \notin D$ ), while  $w_2$  is paired with either  $w'_2$  or  $v_n$ . However, each vertex in  $N_G(v_n)$  is adjacent to a leaf and belongs to  $D$ , thus  $D \setminus \{v_n\}$  dominates  $G$ . Therefore, either  $D \setminus \{x_1, w'_2\}$  or  $D \setminus \{x_1, v_n\}$  is a paired dominating set of  $G$  in which  $w_1$  and  $w_2$  are paired, contrary to  $\text{msd}(G) = 4$ . It follows that (C) holds.

Since  $G$  is a block graph,  $w_i$  and  $w_j$  belong to different components of  $G - v_n$  for all  $i \neq j$ .

Consequently, if there exists a vertex  $z \notin \{v_n, w'_i\}$  adjacent to  $w_i$ , then  $z$  and  $v_n$  belong to different components of  $G - \{w_i, w'_i\}$ . But now  $w_i w'_i$  is a type-A edge, which is not the case as  $a(G) = 0$ . Hence  $\deg(w_i) = 2$  and  $G \cong K_n \circ^{*t} K_1$ . Since  $\text{msd}(G) = 4$ ,  $n$  is even, by Remark 3(ii). Therefore  $G \in \mathcal{U} \subseteq \mathcal{B}$ . Thus (B) holds.

Now suppose  $a(G) \geq 1$ . If  $G$  has a type-A critical cut-vertex  $u$ , perform the operation  $G \ominus u$ ; each resulting graph is an  $\text{msd-4}$  graph by Proposition 10, and clearly a block graph. Moreover, the copies of  $u$  in each graph are  $\gamma_{\text{pr}}$ -critical. Repeat this process until no resulting  $\text{msd-4}$  block graph has a type-A critical cut-vertex. Let  $G_1, \dots, G_k$  be the resulting graphs. Then each critical vertex of each  $G_i$  is a leaf. If any  $G_i$  has a type-A critical edge  $uv$ , where  $u$  is a leaf, perform the operation  $G \ominus uv$ . Each resulting graph is an  $\text{msd-4}$  block graph by Proposition 13. Repeat this process until all resulting graphs  $H_j$  satisfy  $a(H_j) = 0$ . If  $H_j$  is a tree, then  $H_j \cong S(2, \dots, 2) \in \mathcal{U}$  by Corollary 5, otherwise  $H_j \in \mathcal{U}$  by (B). Now  $G$  can be reconstructed by performing the  $\oplus$ -operations on the  $H_j$ , hence  $G \in \mathcal{B}$ , as required. ■

7. OPEN PROBLEMS

We conclude with a short list of open problems for future consideration.

**Question 1.** *Does Theorem 12 hold for all  $\text{msd-4}$  graphs?*

Define another  $\oplus$ -operation as follows.

$\oplus_{u,Q}^{u_1 Q_1, u_2 Q_2}$ : Let  $G_1$  and  $G_2$  be vertex disjoint graphs containing (not necessarily maximal) cliques  $Q_1$  and  $Q_2$  of equal size, and vertices  $u_i \in V(Q_i)$  for  $i \in \{1, 2\}$ . We denote a graph obtained from  $G_1$  and  $G_2$  by identifying  $Q_1$  and  $Q_2$  into one clique  $Q$ , and  $u_1$  and  $u_2$  into one vertex  $u = u_1 = u_2$ , by  $G_1 \oplus_{u,Q}^{u_1 Q_1, u_2 Q_2} G_2$  (or by  $G_1 \oplus^{u_1 Q_1, u_2 Q_2} G_2$  if  $u$  and  $Q$  are unimportant).

Note that if the cliques  $Q_i$  have order at least three, then identifying the vertices of  $Q_i - u_i$  in different ways may yield different graphs. Both operations  $\oplus_u^{u_1 u_2}$  and  $\oplus_e^{e_1 e_2}$  are special cases of  $\oplus_{u,Q}^{u_1 Q_1, u_2 Q_2}$ .

**Question 2.** *Let  $G_1$  and  $G_2$  be disjoint  $\text{msd-4}$  graphs containing cliques  $Q_1$  and  $Q_2$  of equal size and  $\gamma_{\text{pr}}(G_i)$ -critical vertices  $u_i \in V(Q_i)$ ,  $i = 1, 2$ . Is it true that for any graph  $G = G_1 \oplus_{u,Q}^{u_1 Q_1, u_2 Q_2} G_2$ ,  $u$  is  $\gamma_{\text{pr}}(G)$ -critical and  $\text{msd}_{\text{pr}}(G) = 4$ ?*

If  $G_1$  and  $G_2$  are copies of the  $\text{msd-4}$  graph in Figure 5, with  $u_i = u$ , which is  $\gamma_{\text{pr}}$ -critical, and  $Q_i$  is the triangle containing  $u$ , then both graphs obtainable as  $G_1 \oplus_{u,Q}^{u_1 Q_1, u_2 Q_2} G_2$  are  $\text{msd-4}$  graphs having  $u$  as critical vertex.

**Question 3.** *Let  $G$  be a graph with  $\text{msd}_{\text{pr}}(G) = 4$ . What is the largest number of edges of  $G$  that can be subdivided three times before the paired domination number increases? If this number can be arbitrarily high, what is its ratio to the number of edges of  $G$ ?*

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