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Signal Propagation in Electromagnetic Media Modelled by the Two-Sided Fractional Derivative

Jacek Gulgowski [1,*], Dariusz Kwiatkowski [1] and Tomasz P. Stefański [2]

[1] The Faculty of Mathematics, Physics and Informatics, University of Gdansk, 80-308 Gdansk, Poland; ktoztam@gmail.com
[2] The Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, 80-233 Gdansk, Poland; tomasz.stefanski@pg.edu.pl
* Correspondence: jacek.gulgowski@ug.edu.pl

Abstract: In this paper, wave propagation is considered in a medium described by a fractional-order model, which is formulated with the use of the two-sided fractional derivative of Ortigueira and Machado. Although the relation of the derivative to causality is clearly specified in its definition, there is no obvious relation between causality of the derivative and causality of the transfer function induced by this derivative. Hence, causality of the system is investigated; its output is an electromagnetic signal propagating in media described by the time-domain two-sided fractional derivative. It is demonstrated that, for the derivative order in the range $[1, +\infty)$, the transfer function describing attenuated signal propagation is not causal for any value of the asymmetry parameter of the derivative. On the other hand, it is shown that, for derivative orders in the range $(0, 1)$, the transfer function is causal if and only if the asymmetry parameter is equal to certain specific values corresponding to the left-sided Grünwald–Letnikov derivative. The results are illustrated by numerical simulations and analyses. Some comments on the Kramers–Krönig relations for logarithm of the transfer function are presented as well.

Keywords: fractional derivatives; Ortigueira–Machado fractional derivative; Grünwald–Letnikov fractional derivative; Maxwell's equations; causality

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1. Introduction

Fractional-order (FO) Maxwell's equations [1–4] represent a generalization of classical electromagnetism with the use of FO derivatives, which provides new interesting solutions constituting intermediate cases between the ones already existing in physics. However, the main advantage of FO modelling stems from the possibility to describe the evolution of electromagnetic systems with memory, which are usually dissipative and very complex [5].

Unfortunately, although several attempts have been made [6–9], it is not clear which definition of the FO derivative should be used in electrical sciences. Recent discussion in literature [10] suggests that pointing out which definition of the FO derivative can be applied in electrical sciences is of the utmost importance. Therefore, in this paper, we employ a very general definition of the FO derivative, i.e., the two-sided Ortigueira–Machado derivative [6,11], which unites the ideas of forward and backward differentiations, and employs two parameters, i.e., the derivative order and the asymmetry parameter. Therefore, this definition of the FO derivative covers the cases of the left- and right-sided Grünwald–Letnikov derivatives, the Liouville and Liouville–Caputo derivatives (both left- and right-sided), the symmetric two-sided derivative and the anti-symmetric two-sided derivative, see [6] (Table 1). Hence, the application of this derivative to the analysis of physical problems allows one to select the FO derivative definition which is the most suitable one for the considered physical problem. For this purpose, we consider the classical problem of plane-wave propagation in the media described by FO model (FOM) [12], employing the two-sided Ortigueira–Machado derivative. Assuming that the definition of the FO derivative should satisfy the semigroup property and the trigonometric functions invariance [9], we are able to demonstrate that causal solutions to this problem are obtained only for the derivative parameters corresponding to the left-sided Grünwald–Letnikov fractional derivative (or equivalently to the Marchaud derivative).

2. Basic Notations

Let us introduce the notation used in the paper. The imaginary unit is denoted as $j = \sqrt{-1}$. The real part of the complex number $s \in \mathbb{C}$ is denoted as $\Re s$, whereas its imaginary part is denoted as $\Im s$. The right half-plane is denoted as $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re s > 0\}$. The branch of the complex logarithm is selected so $\ln(Ae^{j\phi}) = \ln A + j\phi$, where $A \in (0, +\infty)$ and $\phi \in (-\pi, \pi)$. The power of the complex number s^α for $\alpha > 0$ is defined on the complex right half-plane $\mathbb{C}_+$ as

$$s^\alpha = |s|^\alpha e^{j\phi\alpha} \tag{1}$$

where $s = |s|e^{j\phi}$ and $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Consequently, one obtains

$$(j\omega)^\alpha = |\omega|^\alpha e^{j\alpha \mathrm{sgn}(\omega)\frac{\pi}{2}}. \tag{2}$$

We refer to the Fourier and Laplace transformations of the real function $f : \mathbb{R} \to \mathbb{R}$. Because both definitions appear in various versions in the literature, it is necessary to be very precise here. The employed definition of the Fourier transformation of the integrable function $f(t)$ follows the one given in [6,11]

$$\mathcal{F}(f)(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\mathrm{d}t \tag{3}$$

and

$$\mathcal{F}^{-1}(F)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}\mathrm{d}t. \tag{4}$$

Consequently, the (two-sided) Laplace transformation for the locally integrable function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-st}dt. \tag{5}$$

Hence, if an imaginary axis lies in the region of convergence of the Laplace transform, one obtains (3) from (5) when $s = j\omega$.

3. Fractional Calculus

The concept of fractional integrals and derivatives has a very long history and many different approaches. Classical attitudes are presented in well-known and widely-cited monographs [13–16]. Among the most important definitions, we should mention the Riemann–Liouville derivative (with a finite or an infinite base point), the Caputo derivative (with a finite or an infinite base point), and the Grünwald–Letnikov derivative (with the equivalent representation known as the Marchaud derivative). There are plenty of other definitions—for further details, we refer the reader to the review papers [17–20]. Furthermore, various definitions of the fractional derivative and integral are applied to electromagnetism and electrical circuits. Therefore, we have recently decided to put this situation in order by analysing the FO derivatives existing in literature from the point of view of electromagnetism and circuit theory [7–9]. As a result of our investigations, advantages of the Grünwald–Letnikov and Marchaud derivatives presented below have been demonstrated in applications related to electrical sciences. It stems mainly from their properties, which are presented below.

3.1. Grünwald–Letnikov and Marchaud Derivatives

The Grünwald–Letnikov derivative of the order $\alpha > 0$ is given by the discrete formula (refer to [15] (Formula (20.7)))

$$D^{\alpha} f(t) = \lim_{h \to 0^{+}} \frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^{m} \binom{\alpha}{m} f(t - mh) \tag{6}$$

where $\binom{\alpha}{m} = \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m!}$. The formula above presents the so-called left-sided version of this derivative (also referred to as backward differentiation).

The corresponding right-sided version of the Grünwald–Letnikov derivative (also called forward differentiation) is given by

$$D^{\alpha} f(t) = \lim_{h \to 0^{+}} \frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^{m} \binom{\alpha}{m} f(t + mh). \tag{7}$$

The left-sided definition looks at past times, whereas the right-sided version looks into the future.

On the other hand, the Marchaud definition for $\alpha \in (n-1, n)$ is given by

$$D^{\alpha} f(t) = \frac{\{\alpha\}}{\Gamma(1 - \{\alpha\})} \int_{0}^{+\infty} \frac{f^{(n-1)}(t) - f^{(n-1)}(t - \tau)}{\tau^{1+\{\alpha\}}} \mathrm{d}\tau \tag{8}$$

where $\{\alpha\} = \alpha - (n-1)$ and f is assumed to be sufficiently smooth, e.g., $f \in C^{n-1}(\mathbb{R})$ with $f^{(n-1)}$ bounded. The definition of the Marchaud derivative is equivalent to the Grünwald–Letnikov definition (for a broad class of functions, covering periodic functions and $L^{p}(\mathbb{R})$ functions for $p \in [1, +\infty)$, please refer to Theorems 20.2 and 20.4 in [15]). The recent survey paper [21] discusses both approaches in detail. For a historical perspective, one is referred to [22].

The most important properties of the (left-sided) Grünwald–Letnikov and Marchaud derivatives are as follows:

1. Compatibility with IO Derivative (see [16] (Formula (2.28)))

$$D^{\alpha} f(t) = \frac{d^{\alpha}}{dt^{\alpha}} f(t), \qquad \alpha \in \mathbb{N}. \tag{9}$$

2. Linearity

$$D^{\alpha}(af(t) + bg(t)) = aD^{\alpha} f(t) + bD^{\alpha} g(t). \tag{10}$$

3. Semigroup Property (see [16] (Section 2.6.1))

$$D^{\alpha} D^{\beta} f(t) = D^{\beta} D^{\alpha} f(t) = D^{\alpha+\beta} f(t), \quad \alpha, \beta > 0. \tag{11}$$

4. Trigonometric Functions Invariance (see [16] (Formula (2.65)))

$$D^{\alpha} e^{j\omega t} = (j\omega)^{\alpha} e^{j\omega t}. \tag{12}$$

5. Laplace Transform (see [16] (Sections 2.7.3 and 2.8))

$$\mathcal{L}\{D^{\alpha} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\}, \qquad \Re s > 0. \tag{13}$$

6. Fourier Transform (see [16] (Section 2.7.4))

$$\mathcal{F}\{D^{\alpha} f\} = (j\omega)^{\alpha} \mathcal{F}\{f\}. \tag{14}$$

3.2. Two-Sided Ortigueira–Machado Derivative

In recently published papers [6,11], the concept of a two-sided unified derivative is introduced. It joins the ideas of the forward and backward differentiations. This definition contains, as special cases, the left- and right-sided Grünwald–Letnikov derivatives. Furthermore, it is introduced in the frequency domain through the properties of the Fourier transformation of the derivative. The definition may also be formulated in the time domain (for different attitudes, please refer to Formulas (35) and (39) in [6]), but the frequency-domain definition may be applied directly to numerical simulations that are presented below.

The definition employs parameters α (derivative order) and θ (asymmetry parameter), and is formulated by means of an appropriate behaviour of the Fourier transform. Formula (28) in [11] (see also Definition 2 in [6]) defines the derivative $D^{\alpha}_{\theta}f(t)$ of the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{F}(D^{\alpha}_{\theta}f)(\omega) = |\omega|^{\alpha} e^{j\frac{\pi}{2}\theta \mathrm{sgn}(\omega)} \mathcal{F}(f)(\omega) = (j\omega)^{\alpha} e^{j\frac{\pi}{2}(\theta-\alpha)\mathrm{sgn}(\omega)} \mathcal{F}(f)(\omega). \tag{15}$$

According to Table 1 in [6], the definition of the Ortigueira–Machado derivative covers the cases of the left-sided Grünwald–Letnikov derivative ($\theta = \alpha$), the right-sided Grünwald–Letnikov derivative ($\theta = -\alpha$), the Liouville and Liouville–Caputo derivatives (both left- and right-sided), the symmetric two-sided derivative ($\theta = 0$), and the anti-symmetric two-sided derivative ($\theta = \pm 1$). Hence, the application of this derivative to the analysis of physical problems allows one to select the FO derivative definition which is the most suitable one for the considered physical problem. The most important properties of the two-sided Ortigueira–Machado derivative may be summarized as follows:

1. Linearity
$$D^{\alpha}_{\theta}(af(t) + bg(t)) = aD^{\alpha}_{\theta}f(t) + bD^{\alpha}_{\theta}g(t). \tag{16}$$
2. Semigroup Property (see Property 3 following Definition 3.1. in [11])
$$D^{\alpha}_{\theta}D^{\beta}_{\eta}f(t) = D^{\alpha+\beta}_{\theta+\eta}f(t). \tag{17}$$
3. Trigonometric Functions Invariance (see Property 1 following Definition 3.1 and Formula (29) in [11])
$$D^{\alpha}_{\theta}e^{j\omega t} = |\omega|^{\alpha} e^{j\frac{\pi}{2}\theta \mathrm{sgn}(\omega)} e^{j\omega t} = e^{j\frac{\pi}{2}(\theta-\alpha)\mathrm{sgn}(\omega)} (j\omega)^{\alpha} e^{j\omega t}. \tag{18}$$
4. Fourier Transform
$$\mathcal{F}\{D^{\alpha}_{\theta}f\} = (j\omega)^{\alpha} e^{j\frac{\pi}{2}(\theta-\alpha)\mathrm{sgn}(\omega)} \mathcal{F}\{f\}. \tag{19}$$

In our considerations of FO Maxwell's equations, we assume that the asymmetry parameter θ depends on α in a specific way, i.e.,

$$\theta = \theta(\alpha) = \Theta \cdot \alpha \tag{20}$$

where $\Theta \in \mathbb{R}$. First of all, one should notice that no generality is lost for $\alpha \neq 0$ because, for a fixed $\alpha > 0$, any θ may be represented as $\theta = \Theta\alpha$ for an appropriate selection of $\Theta \in \mathbb{R}$. The other motivation for this idea is the fact that we cannot see any natural interpretation of the asymmetry parameter (as an independent parameter) in the time-derivative in Maxwell's equations. Hence, it is natural to relate the two parameters by some (linear) relationship. In addition, last but not least, one of the important properties of the fractional-derivative operators used in electromagnetism and circuit theory is the semigroup property (we refer the reader again to [7–9]). For the fixed $\Theta \in \mathbb{R}$, one may look at the semigroup property in a more direct way, i.e.,

$$D^{\beta}_{\theta(\beta)}D^{\alpha}_{\theta(\alpha)}f(t) = D^{\beta+\alpha}_{\theta(\beta)+\theta(\alpha)}f(t) = D^{\beta+\alpha}_{\Theta(\beta+\alpha)}f(t). \tag{21}$$

Having the parameter Θ fixed, we can use the notation

$$D_{\Theta}^{\alpha} f(t) = D_{\Theta\alpha}^{\alpha} f(t). \tag{22}$$

Following the convention used in (22), the semigroup property is satisfied in its pure form

$$D_{\Theta}^{\alpha} D_{\Theta}^{\beta} f(t) = D_{\Theta}^{\beta} D_{\Theta}^{\alpha} f(t) = D_{\Theta}^{\alpha+\beta} f(t), \quad \alpha, \beta > 0. \tag{23}$$

In this convention, the two important properties may be formulated as follows:

- Trigonometric Functions Invariance

$$D_{\Theta}^{\alpha} e^{j\omega t} = |\omega|^{\alpha} e^{j\frac{\pi}{2}\Theta\alpha \mathrm{sgn}(\omega)} e^{j\omega t} = e^{j\frac{\pi}{2}(\Theta-1)\alpha \mathrm{sgn}(\omega)} (j\omega)^{\alpha} e^{j\omega t}. \tag{24}$$

- Fourier Transform

$$\mathcal{F}\{D_{\Theta}^{\alpha} f\} = (j\omega)^{\alpha} e^{j\frac{\pi}{2}(\Theta-1)\alpha \mathrm{sgn}(\omega)} \mathcal{F}\{f\}. \tag{25}$$

4. Propagation of Electromagnetic Waves in Media Described by FOM

In [12], the model of propagation of a monochromatic plane wave is presented for isotropic and homogeneous media described by FOM and the Marchaud derivative in the time domain. The property of the trigonometric functions' invariance (12) is required to obtain the solution in the phasor domain. Our aim is to follow a similar idea employing the two-sided Ortigueira–Machado derivative.

In this section, propagation of the plane wave is analysed for the media described by FOM. We assume that the considered medium is isotropic and homogeneous. For the sake of brevity, we assume that the current density is related to the electric-field intensity by the classical Ohm's law, with the conductivity $\sigma_1 = 0$, and there is no current or charge sources in the considered space. Therefore, one can formulate Maxwell's equations based on $\mathbf{E}$ and $\mathbf{H}$ fields only as

$$\nabla \cdot \mathbf{E} = 0 \tag{26}$$

$$\nabla \times \mathbf{E} = -\mu_{\gamma} D_t^{\gamma} \mathbf{H} \tag{27}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{28}$$

$$\nabla \times \mathbf{H} = \epsilon_{\beta} D_t^{\beta} \mathbf{E}. \tag{29}$$

Let us consider the monochromatic plane wave propagating along the z direction with the frequency ω, refer to Figure 1.

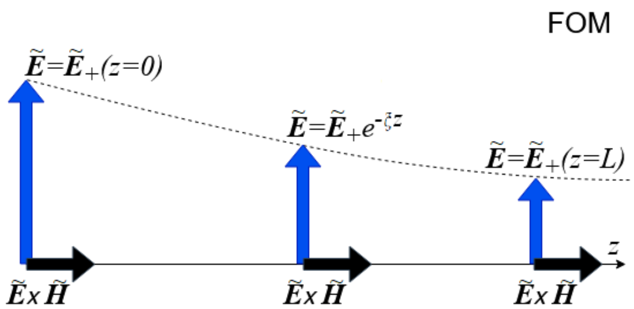

Figure 1. Considered plane wave propagating in medium described by FOM.

In this case, we can use phasor representation for the electromagnetic field, i.e.,

$$\mathbf{E} = \Re(\tilde{\mathbf{E}} e^{j\omega t}) \tag{30}$$

$$\mathbf{H} = \Re(\tilde{\mathbf{H}} e^{j\omega t}) \tag{31}$$

where $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}(x, y, z)$ and $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}(x, y, z)$ are electric and magnetic field phasors which are functions of spatial variables (x, y, z) only. Then, taking the complex representation of electric and magnetic fields $\mathbf{E} = \tilde{\mathbf{E}}e^{j\omega t}$ and $\mathbf{H} = \tilde{\mathbf{H}}e^{j\omega t}$, one can write (26)–(29) as

$$\nabla \cdot \widetilde{\mathbf{E}} = 0 \tag{32}$$

$$\nabla \times \widetilde{\mathbf{E}} = -\mu_\gamma e^{j\frac{\pi}{2}(\Theta-1)\gamma\,\mathrm{sgn}(\omega)}(j\omega)^\gamma \widetilde{\mathbf{H}} \tag{33}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0 \tag{34}$$

$$\nabla \times \widetilde{\mathbf{H}} = \epsilon_\beta e^{j\frac{\pi}{2}(\Theta-1)\beta\,\mathrm{sgn}(\omega)}(j\omega)^\beta \widetilde{\mathbf{E}}. \tag{35}$$

Because $\nabla \times \nabla \times \widetilde{E} = \nabla(\nabla \cdot \widetilde{E}) - \nabla^2 \widetilde{E}$, one obtains the following diffusion-wave equation in the frequency domain:

$$\nabla^2 \widetilde{E} - \mu_\gamma \epsilon_\beta (j\omega)^{2\nu} e^{j\pi\nu(\Theta-1)\mathrm{sgn}(\omega)} = 0 \tag{36}$$

where $\nu = \frac{\beta+\gamma}{2}$. Because $(j\omega)^{2\nu} = |\omega|^{2\nu} e^{j\pi\nu\mathrm{sgn}(\omega)}$, one can denote $\xi^2 = \mu_\gamma \epsilon_\beta |\omega|^{2\nu} e^{j\pi\nu\Theta\mathrm{sgn}(\omega)}$, and the previous equation can be written as

$$\frac{d^2 \widetilde{E}}{dz^2} - \xi^2 \widetilde{E} = 0. \tag{37}$$

The general solution to (37) is given by

$$\widetilde{E} = \widetilde{E}_+ e^{-\xi z} + \widetilde{E}_- e^{\xi z}. \tag{38}$$

Considering wave propagation in the $z+$ direction only, the propagation constant ξ depends on the choice of ν and Θ parameters, and is selected as the one with a positive real part. This leads to the solution with attenuated propagation of the signal in the direction of increasing z. Hence, one obtains

- if $\cos(\frac{\pi}{2}\nu\Theta) > 0$, then $\xi = \frac{|\omega|^\nu}{c_{\mu\epsilon}} e^{j\frac{\pi}{2}\nu\Theta\mathrm{sgn}(\omega)}$is taken
- if $\cos(\frac{\pi}{2}\nu\Theta) < 0$, then $\xi = -\frac{|\omega|^\nu}{c_{\mu\epsilon}} e^{j\frac{\pi}{2}\nu\Theta\mathrm{sgn}(\omega)}$ is taken.

The case when the real part is equal to zero, i.e., $\cos(\frac{\pi}{2}\nu\Theta) = 0$, is not considered below because it does not lead to attenuated signal propagation. Hence, we further assume that $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$.

Finally, the formula in the time domain can be written as

$$e(z, t) = \Re\{\mathcal{F}^{-1}\{\widetilde{E}_+ e^{-\xi z}\}\}. \tag{39}$$

Moreover, one can notice that the transfer function in the frequency domain for the considered system is given by

$$G_{\Theta,\nu}(\omega) = e^{-\xi z} \tag{40}$$

where

$$\xi(\omega) = \mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{1}{c_{\mu\epsilon}}(j\omega)^\nu e^{j\frac{\pi}{2}\nu(\Theta-1)\mathrm{sgn}(\omega)} = \mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{1}{c_{\mu\epsilon}}|\omega|^\nu e^{j\frac{\pi}{2}\nu\Theta\mathrm{sgn}(\omega)}. \tag{41}$$

Eventually, $G_{\Theta,\nu}$ may be written as

$$G_{\Theta,\nu}(\omega) = e^{-\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{z}{c_{\mu\epsilon}}(j\omega)^\nu e^{j\frac{\pi}{2}\nu(\Theta-1)\mathrm{sgn}(\omega)}} = e^{-\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{z}{c_{\mu\epsilon}}|\omega|^\nu e^{j\frac{\pi}{2}\nu\Theta\mathrm{sgn}(\omega)}}. \tag{42}$$

5. Causality

As it has been thoroughly explained in [6], the definition of the two-sided derivative introduced by Ortigueira and Machado starts from a certain mixture of the forward (also called causal) and backward (also called anti-causal) derivatives. This mixture is actually neither causal nor anti-causal, hence the name acausal is suggested by the authors. Therefore, from the time-domain perspective, the two-sided derivative of the function f in the time point t_0 looks both in the past and in the future relative to t_0.

Below, we ask about causality of the transfer function $G_{\Theta,\nu}$ in the frequency domain, which strongly depends on the definition of the derivative (including the parameters ν and Θ). However, one should note that this is not the question about causality of the derivative. There is no obvious relation between causality of the derivative and causality of the transfer function induced by the derivative definition. Hence, the influence of the parameter Θ on causality of the transfer function is surely worth investigating. This is the issue addressed in this section.

Let us formulate basic definitions. The function $f\colon \mathbb{R} \to \mathbb{R}$ is *causal* if its support $\operatorname{supp}(f) \subset [0, +\infty)$, i.e., if $f(t) = 0$ for $t \in (-\infty, 0)$. The Fourier transform $F = \mathcal{F}(f)$ is called a *causal transform* if the function f is causal.

The first observation is related to periodicity of the transfer function $G_{\Theta,\nu}(\omega)$ with respect to Θ. It is obvious that

$$G_{\Theta,\nu}(\omega) = G_{\Theta+4k/\nu,\nu}(\omega) \tag{43}$$

for any $k \in \mathbb{Z}$. The next observation is related to another symmetry with respect to Θ. This symmetry shows that, if $G_{\Theta,\nu}(\omega)$ is a causal transform, then surely $G_{-\Theta,\nu}(\omega)$ is not.

Lemma 1. *Let $z > 0$ and $\nu, \Theta > 0$. Then,*

$$g_{\Theta,\nu}(t) = g_{-\Theta,\nu}(-t) \tag{44}$$

where $g_{\Theta,\nu} = \mathcal{F}^{-1}(G_{\Theta,\nu})$.

Proof. Let $z > 0$, $\nu \in (0, 1)$ and $g_{\Theta,\nu} = \mathcal{F}^{-1}(G_{\Theta,\nu})$. As one can notice,

$$G_{-\Theta,\nu}(\omega) = G_{\Theta,\nu}(-\omega). \tag{45}$$

By the well-known property of the Fourier transformation, one obtains

$$\mathcal{F}^{-1}(G_{-\Theta,\nu})(t) = \mathcal{F}^{-1}(G_{\Theta,\nu})(-t), \tag{46}$$

which completes the proof. □

The concept of causality and causal transforms is generally well-understood for L^2 functions. Two classical theorems are used as the main mathematical tools for the analysis of causality in the frequency domain, i.e., the Titchmarsh theorem and the Paley–Wiener theorem.

The Titchmarsh theorem (originally proven in [23]) is formulated below as in Nussenzveig's book [24] (Theorem 1.6.1), with slight modifications related to the change in the Fourier-transformation definition. For more information on the history of this theorem and its background, one is referred to [25].

Theorem 1. *If a square-integrable function $G(\omega)$ fulfills one of the four conditions below, then it fulfills all four of them:*

(i) *The inverse Fourier transform $g(t)$ of $G(\omega)$ vanishes for $t < 0$:*

$$g(t) = 0 \qquad (t < 0).$$

(ii) *$G(v)$ is, for almost all v, the limit as $u \to 0^+$ of an analytic function $\tilde{G}(u + jv)$ that is holomorphic in the right half-plane and square integrable over any line parallel to the imaginary axis:*

$$\int_{-\infty}^{\infty} |\tilde{G}(u + jv)|^2 \mathrm{d}v < C \qquad (u > 0).$$

(iii) *$\Re G$ and $\Im G$ verify the first Plemelj formula:*

$$\Re G(\omega) = -\frac{1}{\pi} \fint_{-\infty}^{+\infty} \frac{\Im G(\omega')}{\omega' - \omega} \mathrm{d}\omega'. \tag{47}$$

(iv) *$\Re G$ and $\Im G$ verify the second Plemelj formula:*

$$\Im G(\omega) = \frac{1}{\pi} \fint_{-\infty}^{+\infty} \frac{\Re G(\omega')}{\omega' - \omega} \mathrm{d}\omega'. \tag{48}$$

The integrals in (47) and (48) should be understood in the *principal value* sense, i.e.,

$$\fint_{-\infty}^{+\infty} \frac{f(t)}{x-t} \mathrm{d}t = \lim_{\varepsilon \to 0^+} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \mathrm{d}t. \tag{49}$$

Formulas (47) and (48) are usually referred to as the *dispersion relations* or the *Kramers–Krönig* (K–K) relations. Formally, if one deals with functions from $L^2(\mathbb{R})$, these relations should be considered as valid for almost all $\omega \in \mathbb{R}$. In practical terms, when continuous functions are considered, one may often replace the almost everywhere equality with the equality for all $\omega \in \mathbb{R}$.

If the function $G(\omega)$ is the Fourier transform of the real-valued function $g(t)$ (it is hermitian, i.e., it has an even real part and an odd imaginary part), then the K–K relations (47) and (48) can be represented for almost all $\omega \in \mathbb{R}$ by the following integrals on $(0, +\infty)$:

$$\Re G(\omega) = \frac{2}{\pi} \fint_{0}^{+\infty} \frac{\tau \Im G(\tau)}{\omega^2 - \tau^2} \mathrm{d}\tau \tag{50}$$

$$\Im G(\omega) = -\frac{2\omega}{\pi} \fint_{0}^{+\infty} \frac{\Re G(\tau)}{\omega^2 - \tau^2} \mathrm{d}\tau. \tag{51}$$

One should note that, in Nussenzveig's book, the procedure with subtractions for not L^2–integrable functions (or even for distributions) is also described. The idea behind this method (as described in Section 1.7 of [24]) is that, if $F(\omega) = G(\omega)/(j\omega)$, $F \in L^2(\mathbb{R})$ satisfies the K–K relations, then not only F is causal but G (which should be treated as a tempered distribution, not necessarily as an L^2 function) is causal as well.

The next theorem is characterization of the *modulus* of the complex-valued L^2 function, which may be a causal Fourier transform.

Theorem 2 (Paley–Wiener, [26] (Theorem XII))**.** *Let $\phi(\omega)$ be a real nonnegative function, not equivalent to 0 and belonging to $L^2(\mathbb{R})$. A necessary and sufficient condition that there should exist a real- or complex-valued function $g(t)$, vanishing for $t \leq t_0$, for some number t_0, and such that the Fourier transform $G(\omega) = \mathcal{F}(g(t))(\omega)$ should satisfy $|G(\omega)| = \phi(\omega)$, is that*

$$\int_{-\infty}^{+\infty} \frac{|\ln(\phi(\omega))|}{1+\omega^2} \mathrm{d}\omega < +\infty. \tag{52}$$

One should note that the Paley–Wiener theorem does not state that the complex-valued function $G(\omega)$ is a causal transform. It states that, for the modulus $\phi(\omega)$ satisfying (52), the causal transform $G(\omega)$ exists with the same modulus. It also states that, if $\phi(\omega) = |G(\omega)|$ does not satisfy (52), then $G(\omega)$ is surely not a causal transform. This theorem is a valuable tool which may be used to prove that the transfer function is not a causal transform. This is the case of $\nu \geq 1$ for the considered transfer function in the frequency domain.

Theorem 3. *If $\nu \geq 1$ and $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$, then $G_{\Theta,\nu}(\omega)$ is not a causal transform.*

Proof. Let us notice that

$$\phi_{\Theta,\nu}(\omega) = |G_{\Theta,\nu}(\omega)| = e^{-\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{z}{c_{\mu\epsilon}}|\omega|^{\nu}\cos(\frac{\pi}{2}\nu\Theta)}. \tag{53}$$

Then,

$$|\ln\phi_{\Theta,\nu}(\omega)| = \frac{z}{c_{\mu\epsilon}}|\omega|^{\nu}|\cos(\frac{\pi}{2}\nu\Theta)|. \tag{54}$$

Hence, if only $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$, then there is

$$\int_{-\infty}^{+\infty}\frac{|\ln(\phi_{\Theta,\nu}(\omega))|}{1+\omega^2}\mathrm{d}\omega = +\infty. \tag{55}$$

□

Let us assume now that $\nu \in (0,1)$. First, we are going to prove that, in some cases, the transfer function is causal.

Theorem 4. *If $\nu \in (0,1)$, $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$ and $\frac{1}{2}(\Theta-1)\nu \in \mathbb{Z}$, then the transfer function is a causal transform.*

Proof. Let us observe that, if

$$\tfrac{1}{2}(\Theta-1)\nu = k \in \mathbb{Z} \tag{56}$$

then $\cos(\frac{\pi}{2}\Theta\nu) = \cos(\frac{\pi}{2}(\Theta-1)\nu + \frac{\pi}{2}\nu) = \cos(k\pi + \frac{\pi}{2}\nu)$. Because $\frac{\pi}{2}\nu < \frac{\pi}{2}$, one can notice that $\mathrm{sgn}(\cos(\frac{\pi}{2}\Theta\nu)) = (-1)^k$. One can also notice that

$$e^{j\frac{\pi}{2}\nu(\Theta-1)\mathrm{sgn}(\omega)} = e^{jk\pi\mathrm{sgn}(\omega)} = (-1)^k. \tag{57}$$

Then, the function

$$\tilde{G}_{\Theta,\nu}(s) = e^{-\frac{z}{c_{\mu\epsilon}}s^{\nu}} \tag{58}$$

defined for $s \in \mathbb{C}_+$ is the holomorphic extension of $G_{\Theta,\nu}(\omega)$ and such that

$$\lim_{u\to 0^+}\tilde{G}_{\Theta,\nu}(u+j\omega) = G_{\Theta,\nu}(\omega). \tag{59}$$

Moreover, one can notice that, for the fixed $u > 0$

$$\int_{-\infty}^{\infty}|\tilde{G}_{\Theta,\nu}(u+j\omega)|^2\mathrm{d}\omega = \int_{-\infty}^{\infty}e^{-2\frac{z}{c_{\mu\epsilon}}\sqrt{u^2+\omega^2}^{\nu}\cos(\nu\,\mathrm{atan}\frac{\omega}{u})}\mathrm{d}\omega. \tag{60}$$

This integral is bounded, hence the condition (ii) of Theorem 1 is satisfied. It means that $G_{\Theta,\nu}(\omega)$ is a causal transform, which completes the proof. □

Now, we are going to state a certain non-causality result. We are going to show that the K–K relation (50) is not satisfied for certain values of $\nu \in (0,1)$ and $\Theta \in \mathbb{R}$. Before that, one should notice that

$$\Re G_{\Theta,\nu}(\omega) = e^{-c|\cos(\frac{\pi}{2}\nu\Theta)||\omega|^{\nu}}\cos\Big(-\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))c|\omega|^{\nu}\sin(\frac{\pi}{2}\nu\Theta)\Big) \tag{61}$$

$$\Im G_{\Theta,\nu}(\omega) = e^{-c|\cos(\frac{\pi}{2}\nu\Theta)||\omega|^{\nu}}\sin\Big(-\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))c|\omega|^{\nu}\sin(\frac{\pi}{2}\nu\Theta)\Big) \tag{62}$$

where $c = \frac{z}{c_{\mu\epsilon}}$.

Lemma 2. *If $\nu \in (0,1)$, $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$ and $\frac{1}{2}(\Theta-1)\nu \notin \mathbb{Z}$, then the relation (50) is not satisfied for $\omega = 0$.*

Proof. It is clear that $\Re G_{\Theta,\nu}(0) = 1$. The left side of the Equation (50) is equal to

$$\frac{2}{\pi}\int_0^\infty \frac{e^{-c|\cos(\frac{\pi}{2}\nu\Theta)|\tau^\nu}\sin\big(\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))c\tau^\nu\sin(\frac{\pi}{2}\nu\Theta)\big)}{\tau}\mathrm{d}\tau. \tag{63}$$

Substituting $x = c|\cos(\frac{\pi}{2}\nu\Theta)|\tau^\nu$, one obtains

$$\frac{2}{\pi\nu}\int_0^\infty \frac{e^{-x}\sin\Big(\mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))x\frac{\sin(\frac{\pi}{2}\nu\Theta)}{|\cos(\frac{\pi}{2}\nu\Theta)|}\Big)}{x}\mathrm{d}x = \frac{2}{\pi\nu}\int_0^\infty \frac{e^{-x}\sin\big(x\tan(\frac{\pi}{2}\nu\Theta)\big)}{x}\mathrm{d}x = \tag{64}$$
$$\frac{2}{\pi\nu}\mathrm{atan}(\tan(\frac{\pi}{2}\nu\Theta)).$$

The last equality is the consequence of $\int_0^\infty \frac{e^{-x}\sin(ax)}{x}\mathrm{d}x = \mathrm{atan}(a)$ (see Formula 3.941 in [27]).

In particular, if $\frac{2}{\pi\nu}\mathrm{atan}(\tan(\frac{\pi}{2}\nu\Theta)) \neq 1$, then the Equation (50) is not satisfied for $\omega = 0$. Hence, one obtains

$$\tan(\frac{\pi}{2}\nu\Theta) \neq \tan(\frac{\pi\nu}{2}) \tag{65}$$
$$\frac{\pi}{2}\nu\Theta \neq \frac{\pi\nu}{2} + k\pi$$
$$\frac{\nu(\Theta-1)}{2} \neq k$$

for any $k \in \mathbb{Z}$. □

Theorem 5. *If $\nu \in (0,1)$, $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$ and $\frac{1}{2}(\Theta-1)\nu \notin \mathbb{Z}$, then the transform $G_{\Theta,\nu}(\omega)$ is not causal.*

Proof. In general, violation of any of the conditions (47) and (48) for the transform $G_{\Theta,\nu}$ at a single point does not prove that the transform is not causal. This is because the equalities in (47) and (48) are in L^2 sense; hence, such equalities are valid almost everywhere. Fortunately, it appears that, in certain cases, it may be shown that the relations (47) and (48) are valid for all $\omega \in \mathbb{R}$.

First, let us notice that $G_{\Theta,\nu}$ is a locally Hölder continuous function, as the superposition of the Hölder function $\omega \mapsto |\omega|^\nu$ with a locally Lipschitz function. The result of Wood [28] (Theorem I) (see also [29] (Section 3.4.1)) says that when $\Im G_{\Theta,\nu}$ is a locally Hölder function, such that the integrals $\int_M^{+\infty}\frac{\Im G_{\Theta,\nu}(\omega)}{\omega}\mathrm{d}\omega$ and $\int_{-\infty}^{-M}\frac{\Im G_{\Theta,\nu}(\omega)}{\omega}\mathrm{d}\omega$ exist for certain $M > 0$, then also the integral

$$⨍_{-\infty}^{+\infty}\frac{\Im G_{\Theta,\nu}(\tau)}{\tau-\omega}\mathrm{d}\tau \tag{66}$$

is a Hölder continuous function. When both $\Re G_{\Theta,\nu}(\omega)$ and the function

$$H_{\Theta,\nu}(\omega) = ⨍_{-\infty}^{+\infty}\frac{\Im G_{\Theta,\nu}(\tau)}{\tau-\omega}\mathrm{d}\tau \tag{67}$$

are continuous, then the relation (47) is satisfied for all $\omega \in \mathbb{R}$. The same is true for (50); hence, the violation of (50) in one point proves that $G_{\Theta,\nu}$ is not a causal transform. □

5.1. K–K Relations for Logarithm

The idea to check the K–K relations for the logarithm of the transfer function was introduced as a concept of *the logarithmic Hilbert transform* [30]. In practical terms, these relations may be given as a certain integral equality between the phase velocity v_{ph} and the attenuation constant ζ (see [31] (Egn. (3)))

$$\frac{1}{v_{ph}} = \frac{2}{\pi} \int_0^\infty \left(\zeta(\omega') - \zeta(\omega)\right) \frac{d\omega'}{\omega'^2 - \omega^2}. \tag{68}$$

One should note that the above formula is a representation of (47) and (48) for the hermitian function $L(\omega)/(j\omega)$. The same relations between the phase velocity and the attenuation constant are concluded in [32] directly from physical properties. In [12] (Appendix B), this type of relation is also verified for the media described by FOM with the Marchaud derivative of the order $\nu \in [1/2, 1]$.

Nevertheless, the K–K relations for the logarithm (68) are not based on the if and only if the relationship with causality (as is the case for the transfer function itself by the Titchmarsh Theorem 1). The situation is more complicated. Let us (not in a very formal way) review these two possibilities.

When the transfer function $G(\omega)$ has no zeros, then its logarithm is well-defined and one may ask if the K–K relations (with subtractions) are valid for $L(\omega) = \ln G(\omega)$. If this is the case, then one may show that $L(\omega)$ may be extended to the function $\tilde{L}(s)$ holomorphic in $\mathbb{C}_+$, as in (ii) of Theorem 1. Then, $\tilde{G}(s) = e^{\tilde{L}(s)}$ is a holomorphic extension of the function $G(\omega)$ to the half–plane $\mathbb{C}_+$ (extension in the sense of (ii) in Theorem 1). However, it does not necessarily mean that $G(\omega)$ is a causal transform. One may not forget about the important assumption of having the function $\tilde{G}(s)$ square integrable on vertical lines $\{\sigma + j\omega : \omega \in \mathbb{R}\}$ for fixed $\sigma > 0$.

Hence, the K–K relations for the logarithm of the transfer function imply causality of the transfer function $G(\omega)$ only, when the holomorphic extension of the logarithm of the transfer function satisfies appropriate growth conditions. Formally, these conditions alone do not allow us to draw any causality conclusions.

Let us now assume that $G(\omega)$ is a causal transform, and $\ln G(\omega)$ exists. Then, one should note that the K–K relations for the logarithm of the transfer function are not necessarily satisfied. The K–K relations for the logarithm are satisfied when the holomorphic extension of the causal transform (in the sense of (ii) in Theorem 1) has no zeros in the right half-plane. In this case, if $L(\omega)/(j\omega)$ is an L^2 function, one has a natural candidate for the holomorphic extension of $L(\omega)/(j\omega)$ into the right half-plane. If one may show that this extension satisfies all the assumptions of (ii) in Theorem 1, the K–K relations for the logarithm are surely satisfied. Hence, violation of the K–K relations for the logarithm of the transfer function, as given by (68), does not seem to be a sufficient condition for non-causality. In other words, showing that (68) is violated does not imply lack of causality.

In general, there is no direct way to conclude that if the K–K relations for the logarithm (in the form (68)) are not satisfied, then $G(\omega)$ is not a causal transform. Anyway, the next Theorem shows an interesting property, even if it is not a formal proof of non-causality.

Theorem 6. *Let us assume that* $\nu \in (0,1)$ *and* $\cos(\frac{\pi}{2}\nu\Theta) \neq 0$*. For the transfer function* $G_{\Theta,\nu}$ *given by* (42)*, the relation* (68) *is satisfied if and only if* $\frac{1}{2}\nu(\Theta - 1) \in \mathbb{Z}$*.*

Proof. The attenuation constant and the phase velocity are given by

$$\zeta(\omega) = \Re(\xi(\omega)) \tag{69}$$

$$v_{ph} = \frac{\omega}{\kappa(\omega)} \tag{70}$$

where $\kappa(\omega) = \Im(\xi(\omega))$ and $\xi(\omega)$ are given by (41). In the integration range, the function ζ can be written as

$$\zeta(\omega) = \mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta)) \frac{1}{c_{\mu\epsilon}} \cos(\frac{\pi}{2}\nu\Theta)\omega^\nu, \tag{71}$$

whereas

$$\kappa(\omega) = \mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{1}{c_{\mu\epsilon}}\sin(\frac{\pi}{2}\nu\Theta)\omega^{\nu}. \tag{72}$$

Hence, one obtains (following the lines of Appendix B in [12])

$$\frac{2}{\pi}\int_0^{\infty}(\zeta(\omega') - \zeta(\omega))\frac{d\omega'}{\omega'^2 - \omega^2} = \mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{1}{c_{\mu\epsilon}}\cos(\frac{\pi}{2}\nu\Theta)\tan(\frac{\pi}{2}\nu)\omega^{\nu-1}. \tag{73}$$

On the other hand,

$$v_{ph} = \frac{\omega}{\kappa(\omega)} = \mathrm{sgn}(\cos(\frac{\pi}{2}\nu\Theta))\frac{c_{\mu\epsilon}\omega^{1-\nu}}{\sin(\frac{\pi}{2}\nu\Theta)}. \tag{74}$$

One can notice that, in general, the relation (68) does not apply. The equality is obtained when

$$\tan(\frac{\pi}{2}\nu\Theta) = \tan(\frac{\pi}{2}\nu), \tag{75}$$

so, when $\frac{1}{2}\nu(\Theta - 1) = k \in \mathbb{Z}$. $\square$

6. Numerical Simulations

Computations of plane-wave waveforms in the distance z are executed in accordance with the same algorithm as described in Section 4.2 in [12] (see the scheme in Figure 6 in the above-mentioned paper). Actually, the algorithm is the direct implementation of (39). The simulation is executed in accordance with the following steps:

- $e_r(t)$ is the time-domain waveform of the propagating signal in $z = 0$
- the analytic signal $e_a(t) = e_r(t) + je_i(t)$ is obtained with the use of the Hilbert transformation $e_i(t) = \mathcal{H}[e_r(t)]$
- $\tilde{E}_+(\omega)$ is the Fourier transformation of the analytic signal $e_a(t)$
- $\tilde{E}_+(\omega)$ is multiplied by $e^{-\xi z}$
- the time-domain waveform $e_r(z,t)$ of the signal in the distance z is the real part of the inverse Fourier transformation of $\tilde{E}_+(\omega)e^{-\xi z}$.

In our simulations, the sampling time is set to $T_s = 1/(50 f_{max})$, where $f_{max} = 720 \times 10^{12}$ Hz. It means that $T_s = 2.78 \times t10^{-17}$ s.

The simulations are executed for square-pulse excitation of the length $\Delta t = 536 \times T_s = 1.49 \times 10^{-14}$ s, starting at the time $t_0 = 4024 \times T_s = 1.12 \times 10^{-13}$ s. The measurement is performed at the observation point $L = 10^{-6}$ m. The length of the entire input signal is equal to $8196 \times T_s$. The results of the simulation are presented in Figures 2 and 3.

Remark 1. *In [12], limitation of the sampling time T_s is considered (see Formulas (48) and (49) therein). One may follow the same idea in this context and observe that the transformation of [12] (Equation (48)) leads to inequality similar to [12] (Equation (49)), i.e.,*

$$\frac{z}{c_{\mu\epsilon}}\left(\frac{2\pi}{T_s}\right)^{\nu}\sin\left(\frac{\pi}{2}\nu\Theta\right) \geq 2\pi. \tag{76}$$

Our computation parameters satisfy the condition (76).

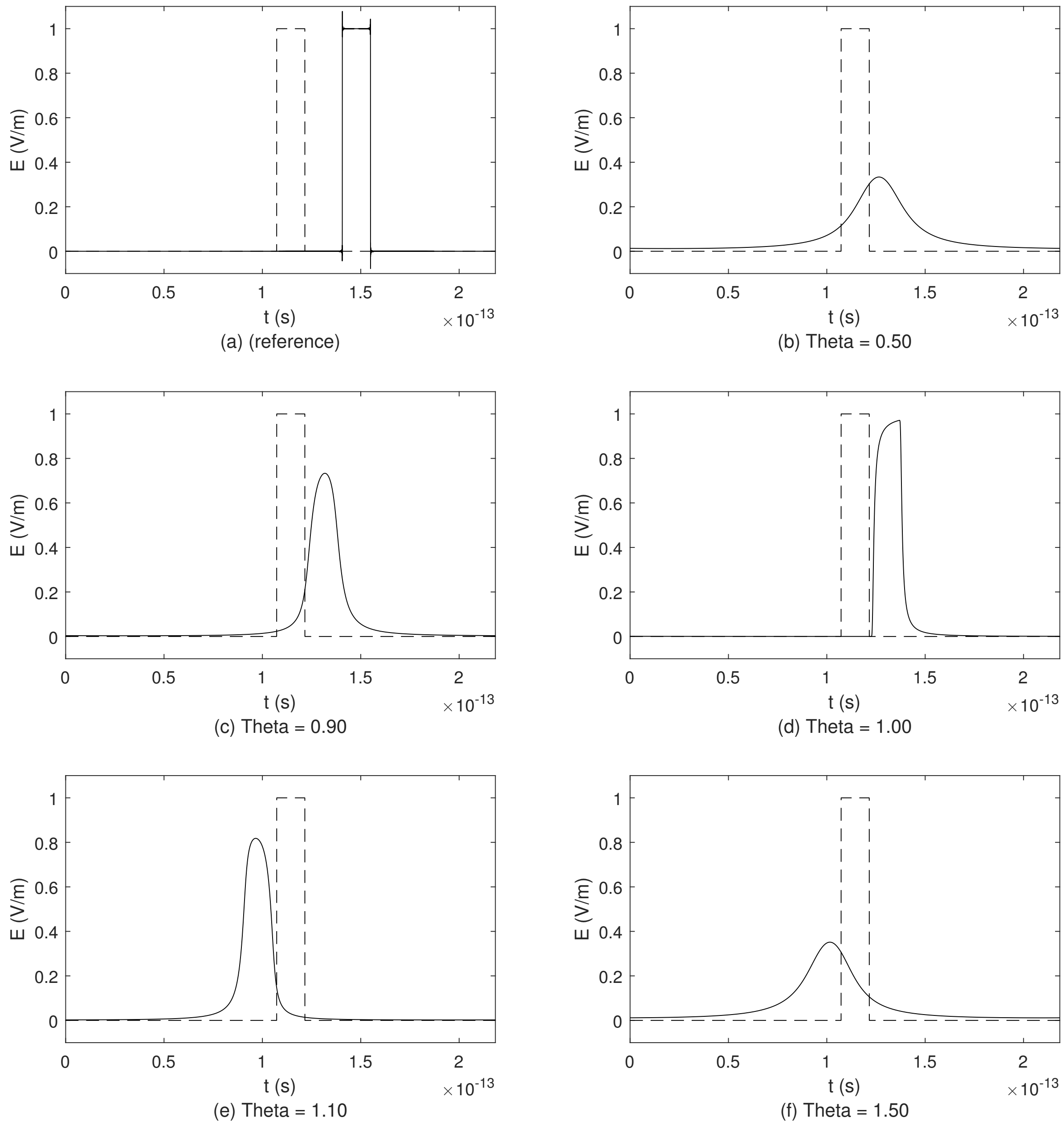

Figure 2. Waveforms of signals propagating in FOM with $\nu = 0.98$. (**a**) reference model ($\nu = 1$ in vacuum). (**b**) $\Theta = 0.5$. (**c**) $\Theta = 0.9$. (**d**) $\Theta = 1$ (i.e., Grünwald–Letnikov derivative). (**e**) $\Theta = 1.1$. (**f**) $\Theta = 1.5$.

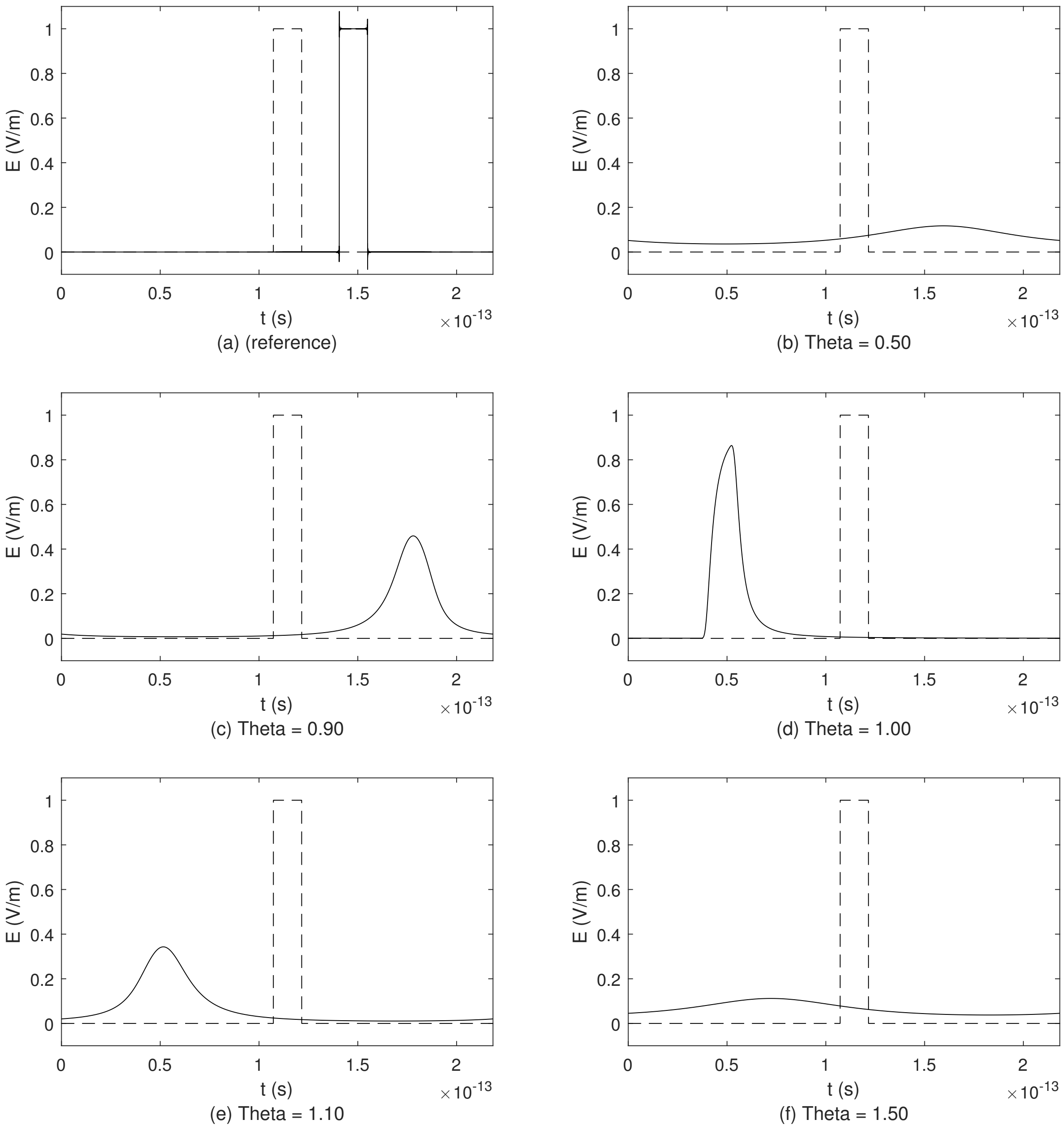

Figure 3. Waveforms of signals propagating in FOM with $\nu = 1.02$. (**a**) reference model ($\nu = 1$ in vacuum). (**b**) $\Theta = 0.5$. (**c**) $\Theta = 0.9$. (**d**) $\Theta = 1$ (i.e., Grünwald–Letnikov derivative). (**e**) $\Theta = 1.1$. (**f**) $\Theta = 1.5$.

7. Discussion

Let us collect the obtained results in terms of causality for the FO derivative alone and the considered system. Figure 4a presents values of the input parameters ν and $\theta = \Theta \cdot \nu$, for which the two-sided derivative is causal, anti-causal, or acausal. For values of the parameters ν and θ between the lines, the derivative is neither causal nor anti-causal (i.e., it is acausal and requires, for derivative computations, values of the input function simultaneously from the past and the future). An analogous presentation of the results for the system response of wave propagation in the media described by FOM is presented in Figure 4b. That is, values of the input parameters ν and θ, for which the system response is causal, are presented. In both cases, dotted lines show the values of parameters not considered in the present paper (i.e., when the propagating signal is not attenuated).

As one can see, the characteristics in Figure 4a,b look the same for $\nu \in (0, 1)$ (please note that, from the perspective of system analysis, the anti-causal system is just non-causal). However, outside this range, although the derivative is causal, the transfer function induced by this derivative is not causal. It demonstrates that one can obtain a frequency response of a system which is not causal, using the FO derivative, which is causal.

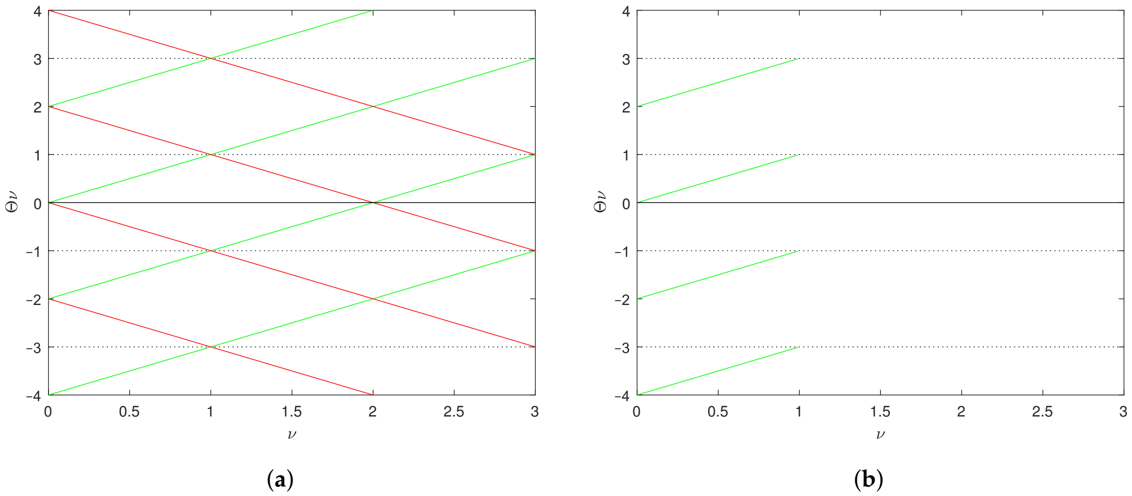

Figure 4. Values of input parameters ν and $\theta = \Theta\nu$ for which (**a**) two-sided derivative is either causal or anti-causal and (**b**) system response is causal. (—) Causal. (—) Anti-causal. Dotted lines (- - -) represent values not considered in the paper (no attenuation of propagated signal). For values of parameters ν and θ between the lines, it is acausal (derivative) or non-causal (system).

As it can be seen, for the fractional-derivative order in the range $(1, +\infty)$, the transfer function describing signal propagation is not causal for any value of the asymmetry parameter of the derivative. However, as proven above, for derivative orders in the range $(0, 1)$, the transfer function is causal if and only if the asymmetry parameter is equal to certain specific values corresponding to the left-sided Grünwald–Letnikov derivative (or equivalently to the Marchaud derivative).

Lack of causality for some parameter values does not mean that the two-sided derivative may not be used in FOMs of electromagnetism. It is just a strong indication that the asymmetry parameter $\theta(\alpha) \neq \alpha$ in fractional *time* derivatives leads to a non-causal transfer function. In the case of *spatial* derivatives in FOMs of electromagnetism (not considered in this paper), there is no physical requirement that the solution support should be within the range of positive values of the spatial variable. Hence, for spatial derivatives, none of the asymmetry patterns supported by the selection of the parameter θ may be a priori excluded.

8. Conclusions

In this paper, signal propagation is analysed in terms of causality for the media described by FOM, based on the two-sided Ortigueira–Machado derivative. For the fractional derivative orders $\nu \geq 1$, it is shown that the transfer function of the system (with attenuated signal propagation) is not causal for any value of the asymmetry parameter. On the other hand, for the derivative orders $\nu \in (0, 1)$, causality of the transfer function is proven for certain values of the asymmetry parameter, corresponding to the left-sided Grünwald–Letnikov derivative (or equivalently to the Marchaud derivative). It is shown that the considered electromagnetic system is not causal for other values of the derivative order ν. Numerical simulations illustrating these results are also presented in the paper.

Finally, assuming that the definition of the FO derivative should satisfy the semigroup property and the trigonometric functions' invariance, we are able to prove that causal solutions to the problem of wave propagation in the media described by FOM are obtained only for the derivative parameters corresponding to the left-sided Grünwald–Letnikov fractional derivative (or equivalently to the Marchaud derivative), demonstrating advantages of these derivatives in electrical sciences.

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Abbreviations

The following abbreviations are used in this manuscript:

FO	Fractional Order
FOM	Fractional Order Model
K–K	Kramers–Krönig

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