



# Connected components of the space of proper gradient vector fields

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**Abstract.** We show that there exist two proper gradient vector fields on  $\mathbb{R}^n$  which are homotopic in the category of proper maps but not homotopic in the category of proper gradient maps.

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## 1. Introduction

Two vector fields on a closed unit ball  $B^n \subset \mathbb{R}^n$  which do not vanish on  $S^{n-1}$  are homotopic if and only if they have the same degree. In 1990, A. Parusinski asked and answered the following question: can we get a better invariant if we restrict the class of vector fields to gradient vector fields? The answer is negative. Namely (Theorem 1 in [5]), two gradient vector fields on  $B^n$ , which do not vanish on  $S^{n-1}$ , are gradient homotopic if and only if they are homotopic, i.e. if and only if they have the same degree.

The aim of this note is to show that we get a different answer if instead of vector fields on a ball we consider proper vector fields on  $\mathbb{R}^n$ . Again, two proper vector fields are homotopic if and only if they have the same degree. However, there exist proper gradient vector fields having the same degree, which are not gradient homotopic.

*Remark 1.1.* The motivation for considering this problem comes from the study of invariants of 3 and 4-dimensional manifolds. The Bauer–Furuta type invariant is a homotopy class of a map which is a compact perturbation of a Fredholm operator and extends to the one point compactification between Hilbert spaces. Such classes are in one-to-one correspondence with elements of some stable homotopy groups (see [1]). On the other hand, there are invariants defined by Manolescu via the Conley index theory ([4]). Our work suggests that although the invariant defined via the Conley index theory is stronger than the corresponding Bauer–Furuta type invariant, this should not

longer be the case if one restricts the class of homotopies to the category of gradient maps.

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## 2. Main theorem

Denote by  $B(R) \subset \mathbb{R}^n$  a closed ball of radius  $R$  centered at the origin. We identify vector fields on  $\mathbb{R}^n$  with maps from  $\mathbb{R}^n$  into itself. Therefore, a proper vector field extends to a continuous map between one point compactifications  $S^n := S^{\mathbb{R}^n}$ . Let  $\mathcal{V}$  be the space of proper vector fields on  $\mathbb{R}^n$  with a topology induced from  $C^0(S^n, S^n)$ . Let  $F$  and  $G$  be proper. In particular, we can choose  $R > 0$  such that  $F^{-1}(0), G^{-1}(0) \subset B(R)$ . Then,  $F$  and  $G$  are homotopic as proper maps, i.e. lie in the same connected component of  $\mathcal{V}$ , if and only if they have the same degree:

$$\deg(F, B(R)) = \deg(G, B(R)).$$

See [6] for a more detailed discussion. Now we would like to ask a question, whether the degree characterises also connected components of proper gradient vector fields.

**Definition 2.1.** Denote by  $\mathcal{V}^\nabla$  subspace of  $\mathcal{V}$  of proper gradient vector fields. We say that  $F$  and  $G \in \mathcal{V}^\nabla$  are *gradient homotopic* if they lie in the same connected component of  $\mathcal{V}^\nabla$ .

Our aim is to show the following.

### Main theorem

If  $n \geq 2$ , then there exist proper gradient vector fields on  $\mathbb{R}^n$  which are homotopic but not gradient homotopic.

#### 2.1. Local Morse cohomology

Let  $\eta$  be a local flow on  $\mathbb{R}^n$ .

**Definition 2.2.** We say that a closed and bounded set  $U$  is an *isolating neighbourhood* for  $\eta$  if  $\text{inv}(U, \eta) \subset \text{int } U$ , where  $\text{inv}(U, \eta) = \{x \in U \mid \eta(\mathbb{R}, x) \subset U\}$  is the maximal invariant subset.

The following Proposition is well known (see [7] or [3] for more general setting).

**Proposition 2.3.** *Let  $f$  be a Morse–Smale function and let  $U$  be an isolating neighbourhood for the gradient flow of  $f$ . Then the local Morse cohomology groups  $H^*(f, U)$  are well-defined. Moreover, let  $\{\eta_\lambda\}_{\lambda \in [0,1]}$  be a continuous family of flows such that  $U$  is an isolating neighbourhood for  $\eta_\lambda$  for every  $\lambda$  and  $\eta_0, \eta_1$  are gradient flows for the Morse–Smale functions  $f$  and  $g$ , respectively. Then*

$$H^*(f, U) \simeq H^*(g, U).$$



We will now show that a ball of big enough radius is an isolating neighbourhood for a family of flows generated by proper gradient vector fields (compare [8, pp. 56-57]).

**Proposition 2.4.** *Let  $\{f_\lambda\}_{\lambda \in [0,1]}$  be a family of functions on  $\mathbb{R}^n$  such that the corresponding family  $\{\nabla f_\lambda\}_{\lambda \in [0,1]}$  is a continuous family of proper vector fields. Denote by  $\eta_\lambda$  the flow generated by  $\nabla f_\lambda$ . Then there exists  $R > 0$  such that  $B(R)$  is an isolating neighbourhood for every  $\eta_\lambda$  and*

$$\text{inv}(B(R'), \eta_\lambda) = \text{inv}(B(R), \eta_\lambda)$$

for every  $\lambda$  and every  $R' > R$ .

*Proof.* By the properness, we can find an  $r_1$  such that

$$\bigcup_{\lambda \in [0,1]} (\nabla f_\lambda)^{-1}(B(1)) \subset B(r_1).$$

Put

$$r_2 = \max\{|f_\lambda(x)| : x \in B(r_1), \lambda \in [0, 1]\}.$$

We will show that  $R = 2(r_1 + r_2)$  satisfies the statement. Let  $u : \mathbb{R} \rightarrow \mathbb{R}^n$  be a bounded trajectory of the flow  $\eta_\lambda$  for some  $\lambda \in [0, 1]$ . We have to show that  $|u(0)| < R$ . Suppose that  $|u(0)| > r_1$ . Recall that both the  $\alpha$ -limit and  $\omega$ -limit of a bounded trajectory of a gradient flow are contained in the rest point set ([2, p.13]). Therefore, the  $\alpha$ -limit of  $u$  is contained in the interior of  $B(r_1)$  and we can choose  $t_0 < 0$  such that  $|u(t)| \geq r_1$  for every  $t \in [t_0, 0]$  and  $|u(t_0)| = r_1$ . Let  $x$  and  $y$  be points in  $\alpha$  and  $\omega$  limit of  $u$ , respectively. We have

$$|x - u(0)| \leq |x - u(t_0)| + |u(t_0) - u(0)| \leq 2r_1 + \int_{t_0}^0 |\dot{u}(s)| ds$$

On the other hand,

$$|\dot{u}(s)| \leq |\dot{u}(s)|^2 = \langle \nabla f_\lambda(u(s)), \dot{u}(s) \rangle = \frac{d}{ds}(f_\lambda \circ u)(s)$$

for  $s \in [t_0, 0]$ . Finally,

$$\int_{t_0}^0 |\dot{u}(s)| ds \leq \int_{t_0}^0 \frac{d}{ds}(f_\lambda \circ u)(s) ds \leq f_\lambda(y) - f_\lambda(x) \leq 2r_2.$$

By Propositions 2.3 and 2.4, we get the following corollary.

**Corollary 2.5.** *Let  $f$  and  $g$  be two Morse–Smale functions with proper gradient vector fields, such that  $\nabla f, \nabla g$  are proper gradient homotopic. Then for  $R \gg 0$ , the Morse cohomology groups  $H^*(f, B(R))$  and  $H^*(g, B(R))$  are isomorphic.*



## 2.2. Proof of the main theorem

Put

$$f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2, \quad g(x_1, \dots, x_n) = -x_1^2 - x_2^2 + \dots + x_n^2.$$

Then  $\deg(\nabla f, B(R)) = 1 = \deg(\nabla g, B(R))$ , so  $\nabla f$  and  $\nabla g$  are homotopic in the category of proper maps. However,  $H^q(f, B(R))$  is non-zero only if  $q = 0$  and  $H^q(g, B(R))$  is non-zero only if  $q = 2$  so by Corollary 2.5 they are not gradient homotopic.

## 3. Final remarks

### 3.1. Infinite dimensional case

Note first that instead of using Morse cohomology groups one can use Conley index theory instead. If instead of  $\mathbb{R}^n$  we take a separable Hilbert space and assume that the vector fields are compact perturbations of a fixed bounded self-adjoint Fredholm operator then the cohomological invariant, called  $E$ -cohomological Conley index, is still well defined (see [3, 9] or [10]). The proof of Proposition 2.4 remain unchanged in this case. Since the  $E$ -cohomological Conley index is invariant under homotopies (Theorem 2.12 in [3]), the conclusion of this note is also true for Hilbert spaces.

### 3.2. Further questions

We have proved that if two Morse–Smale functions have different Morse cohomology groups, then their gradients have to belong to different connected components of  $\mathcal{V}^\nabla$ . This leads to the question whether the map assigning a Morse–Smale function its Morse cohomology groups is injective on connected components of  $\mathcal{V}^\nabla$ . If the answer is affirmative, then one would like to know what are possible Morse cohomology groups for functions whose gradient is in  $\mathcal{V}^\nabla$ .

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