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Perfect thermal contact of hyperbolic conduction semispaces with an interfacial heat source

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The problem of thermal contact between two bodies with a heat source at their interface presents great scientific and practical interest. On the time scale of a nanosecond or shorter, heat propagation should be considered in the form of thermal waves of finite speeds. This study investigated the thermal behaviour of hyperbolic conduction semispaces in perfect thermal contact subjected to the action of an interfacial heat source. An analytical solution was derived using the Laplace integral transform approach. The contact temperature and heat fluxes were analysed for different ratios of thermal conductivities K_1 and K_2, thermal diffusivities k_1 and k_2, and thermal relaxation times τ_1 and τ_2 of the semispaces. It was shown that the interfacial heat generation results in a step-wise change in the contact temperature. It was also found that the initial partition of heat between the semispaces occurs due to the ratio of $K_1/\sqrt{k_1\tau_1}$ and $K_2/\sqrt{k_2\tau_2}$. The applicability of the obtained solution and its particular expressions was illustrated on the example of microscopic problems of ultra-short laser pulse welding and sliding friction.

Notation

c_v	volumetric heat capacity, J/(m^3 °C)
e	thermal effusivity, $e = K/\sqrt{k}$, W s$^{0.5}$/(m^2 °C)
e_τ	hyperbolic conduction thermal effusivity, $e_\tau = K/\sqrt{k\tau}$, W/(m^2 °C)
$\mathrm{erfc}(\cdot)$	complementary error function
k	thermal diffusivity, $k = K/c_v$, m^2/s
q	heat flux, W/m^2
q_0	specific power of heat source, W/m^2
s	Laplace transform parameter
t	time variable, s
x	spatial coordinate, m
$H(\cdot)$	Heaviside step function
$I_v(\cdot)$	modified Bessel function of the first kind of order v
K	thermal conductivity, W/(m °C)
Q	dimensionless heat flux, $Q = q/q_0$
T	temperature, °C
T_0	initial temperature, °C
α_f	heat partition coefficient
η	dimensionless time variable, $\eta = t/\tau_1$
ϑ	dimensionless temperature, $\vartheta = K_1(T - T_{01})/\left(q_0\sqrt{k_1\tau_1}\right)$
ϑ_0	dimensionless initial temperature of semispace 2, $\vartheta_0 = K_1(T_{02} - T_{01})/\left(q_0\sqrt{k_1\tau_1}\right)$
ξ	dimensionless spatial coordinate, $\xi = x/\sqrt{k_1\tau_1}$
τ	thermal relaxation time, s
χ	thermal diffusivity ratio, $\chi = k_2/k_1$
Θ	thermal relaxation time ratio, $\Theta = \tau_2/\tau_1$
Λ	thermal conductivity ratio, $\Lambda = K_2/K_1$
$\mathcal{L}[\cdot]$	Laplace transform operator
$\tilde{\blacksquare}$	Laplace transform image
$\blacksquare_1$	related to semispace 1
$\blacksquare_2$	related to semispace 2
$\blacksquare_{1,2}$	related to semispaces 1 and 2
$\blacksquare_\mathrm{p}$	related to parabolic conduction equation
$\blacksquare^*$	related to interface $x = 0$

1. Introduction

One of the important problems of mathematical physics is the one of contact heat conduction which aims at finding temperature distributions in two coupled bodies with account of heat transfer between them. In many practical situations, this problem may involve a heat source acting at the interface. The heat source can be of mechanical nature (e.g. sliding friction, impact), electromagnetic nature (e.g. electromagnetic radiation, electrical contact resistance), chemical nature (e.g. thermic reaction, phase transition), etc.

The most natural way to define the problem of contact heat conduction is to assume a *perfect thermal contact*, which implies temperature continuity and heat energy conservation when crossing the interface (Carslaw [1], p.161, 214). Classical heat conduction in two or more bodies coupled by the perfect thermal contact conditions has been comprehensively studied for various domains, boundary conditions and heat sources (Carslaw and Jaeger [2], Luikov [3]). The classical heat conduction equation in the form of a *parabolic* partial differential equation is based on Fourier's law [4] which represents the constitutive relationship $\bar{q} = -K\nabla T$ between the temperature gradient ∇T and heat flux vector $\bar{q}$ with the thermal conductivity K as a proportionality coefficient. Due to this relationship, the propagation of heat occurs with infinite speed.

The development of sciences and technologies poses new heat conduction problems that deal with instantaneous thermal processes in micro-volumes. For instance, short and ultra-short pulses of a laser used for processing of materials have duration from femtoseconds to nanoseconds (Mishra and Yadava [5]). Another example is a mechanical interaction of two roughness asperities located on the sliding surfaces that may last from nanoseconds to microseconds (Kragelskii [6]). Experimental studies suggest that on the time scale of a nanosecond or shorter, heat propagates as thermal waves of finite speeds and, accordingly, Fourier's law is not applicable anymore (Joseph and Preziosi [7], Özişik and Tzou [8]).

Cattaneo [9] and Vernotte [10] modified Fourier's law by introducing a time derivative of the heat flux term, as presented in one-dimensional form below:

$$\tau\frac{\partial q(x,t)}{\partial t} + q(x,t) = -K\frac{\partial T(x,t)}{\partial x} \tag{1}$$

Here x is the spatial coordinate; t is the time variable; T is the temperature; q is the heat flux; τ is the thermal relaxation time which represents the time lag between ∇T and $\bar{q}$. It is generally accepted that the value of τ is of order 10^{-14} to 10^{-12} s for metals and 10^{-12} to 10^{-10} s for dielectric materials (Guillemet and Bardon [11]). Kaminski [12], Mitra et al. [13] and Roetzel et al. [14] reported that non-homogeneous materials, such as sand, sodium bicarbonate, processed meat, have dramatically longer τ of order 10^{-1} to 10 s, although the validity of this range is controversial (Maillet [15]).

Combination of Eq.(1) and the statement of heat energy conservation

$$c_v \frac{\partial T(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

yields the heat conduction equation in the form of *hyperbolic* partial differential equation

$$\tau \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{\partial T(x,t)}{\partial t} = k \frac{\partial^2 T(x,t)}{\partial x^2} \quad (2)$$

Here c_v is the volumetric heat capacity; $k = K/c_v$ is the thermal diffusivity. Eq.(2) implies that heat propagates with the speed equal to $\sqrt{k/\tau}$. It is apparent that this speed increases with decreasing τ. As $\tau \to 0$, Eq.(1) reduces to Fourier's law, while Eq.(2) transforms into the parabolic conduction equation.

It is important to note that in certain cases the hyperbolic conduction equation can lead to nonphysical results (Zhang [16]) such as violation of the second law of thermodynamics (Coleman et al. [17], Bai and Lavine [18], Bright and Zhang [19]), abnormally low or high temperatures (Porrà et al. [20], Körner and Bergmann [21], Yu et al. [22]). Nonetheless, this equation remains one of the basic research tools to study heat propagation of finite speed.

A number of studies have been reported on the problems of hyperbolic heat conduction and, in particular, contact problems. Baumeister and Hamill [23] derived analytical expressions of temperature and heat flux in a semispace due to a step-wise change in its surface temperature. Kazimi and Erdman [24] investigated the contact temperature of coupled semispaces with different initial temperatures. Kao [25] investigated thermally induced stress waves in a semispace caused by a step-wise change in the surface heat flux. Frankel et al. [26] obtained analytical expressions of temperatures and heat fluxes in a multilayer system with volumetric heat sources. Lor and Chu [27] investigated the heating of two coupled layers by an external pulsed heat source with account of a radiation boundary condition at their interface. Duhamel [28] proposed a finite integral transform technique for solution of the problems of heat conduction in heterogeneous media and illustrated its application to a multilayer system. Lewandowska [29] conducted an analytical study of temperature in a semispace due to a time-dependent laser heat source with an exponentially distributed volumetric power. Khadrawi et al. [30] investigated the thermal behaviour of two layers in imperfect thermal contact. Tsai and Hung [31] investigated the thermal behaviour of a bi-layered composite sphere due to a step-wise temperature change at its exterior surface. Ordóñez-Miranda and Alvarado-Gil [32] investigated the propagation of thermal waves in a layer–semispace domain excited by a modulated heat source at the exterior surface. Xue et al. [33] investigated thermal contact problems assuming that both parabolic and hyperbolic types of heat conduction coexist. Nosko [34] investigated temperature and heat flux in a semi-infinite body heated by a surface heat source and an exponentially distributed volumetric heat source taking account of the wear of the surface. Besides, one should mention the studies based on a dual-phase-lag heat conduction concept, e.g. Ho et al. [35], Al-Huniti and Al-Nimr [36], Lee and Tsai [37], Ramadan [38], Akbarzadeh and Pasini [39].

Literature analysis shows that the problem of perfect thermal contact between two hyperbolic conduction bodies has not been systematically studied for the case of interfacial heat source. In particular, the question of how the generated heat is distributed between the hyperbolic conduction bodies remains unanswered. With this in mind, the purpose of the present study was to analytically derive temperatures and heat fluxes in the hyperbolic conduction semispaces coupled by the perfect thermal contact conditions with an interfacial heat source and perform parametric analysis of the contact temperature and heat fluxes.

The paper is organised in the following manner. Section **2** defines the studied hyperbolic conduction problem and represents it in the dimensionless form. Section **3** describes an analytical solution of the problem based on the Laplace integral transform approach. Section **4** presents a parametric analysis of the derived expressions of temperature and heat flux. Finally, Section **5** illustrates the application of the obtained results to simulation of microscopic problems of ultra-short laser pulse welding and sliding friction.

2. Problem definition

Consider semispace 1 and semispace 2 which occupy the respective domains $x > 0$ and $x < 0$ and are interfaced at $x = 0$, as shown in Fig.1. Assume that the semispaces conduct heat according to Eq.(2). Then temperatures $T_{1,2}$ in respective semispaces 1 and 2 satisfy the hyperbolic conduction equations

$$\begin{aligned} &\tau_1 \frac{\partial^2 T_1}{\partial t^2} + \frac{\partial T_1}{\partial t} = k_1 \frac{\partial^2 T_1}{\partial x^2}, \qquad x > 0, \qquad t > 0; \\ &\tau_2 \frac{\partial^2 T_2}{\partial t^2} + \frac{\partial T_2}{\partial t} = k_2 \frac{\partial^2 T_2}{\partial x^2}, \qquad x < 0, \qquad t > 0 \end{aligned} \tag{3}$$

where $k_{1,2}$ are the thermal diffusivities; $\tau_{1,2}$ are the thermal relaxation times.

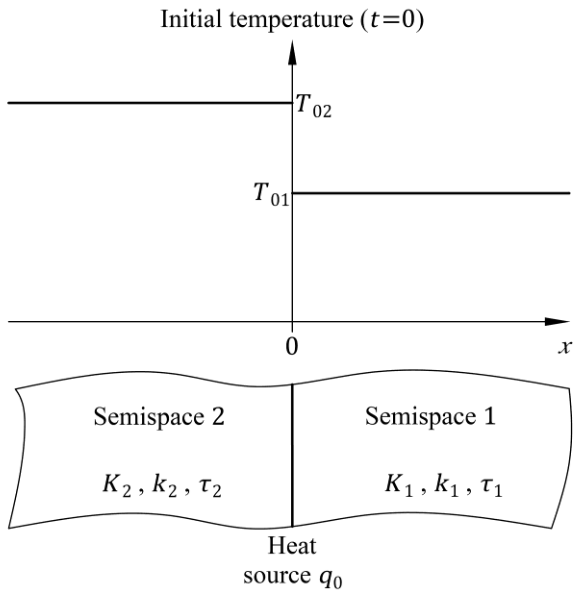

Fig.1. Schematic of the thermal contact of semispaces 1 and 2

At the initial instance of time $t = 0$, semispaces 1 and 2 have different temperatures T_{01} and T_{02}, that is

$$T_1|_{t=0} = T_{01}; \quad T_2|_{t=0} = T_{02} \tag{4}$$

In addition, the initial derivatives of $T_{1,2}$ with respect to t equal zero:

$$\left.\frac{\partial T_1}{\partial t}\right|_{t=0} = \left.\frac{\partial T_2}{\partial t}\right|_{t=0} = 0 \tag{5}$$

The thermal contact between the semispaces is assumed to be perfect, implying temperature continuity

$$T_1|_{x=0} = T_2|_{x=0} \tag{6}$$

A heat source acts at the interface with specific power q_0, resulting in a step-wise change in the heat flux:

$$q_1|_{x=0} - q_2|_{x=0} = q_0 \tag{7}$$

where $q_{1,2}$ are the heat fluxes in respective semispaces 1 and 2. The relationship between $q_{1,2}$ and $T_{1,2}$ is described by Eq.(1):

$$\tau_1 \frac{\partial q_1}{\partial t} + q_1 = -K_1 \frac{\partial T_1}{\partial x}; \quad \tau_2 \frac{\partial q_2}{\partial t} + q_2 = -K_2 \frac{\partial T_2}{\partial x} \tag{8}$$

where $K_{1,2}$ are the thermal conductivities.

The absence of thermal disturbances at infinite distance from the interface yields

$$\left.\frac{\partial T_1}{\partial x}\right|_{x\to+\infty} = \left.\frac{\partial T_2}{\partial x}\right|_{x\to-\infty} = 0 \tag{9}$$

Representation of the problem in the dimensionless form allows to significantly decrease the number of parameters. The dimensionless spatial coordinate ξ, time variable η, temperatures $\vartheta_{1,2}$ and heat fluxes $Q_{1,2}$ are introduced as follows:

$$\xi = \frac{x}{\sqrt{k_1\tau_1}}, \qquad \eta = \frac{t}{\tau_1}, \qquad \vartheta_{1,2} = \frac{K_1\left(T_{1,2} - T_{01}\right)}{q_0\sqrt{k_1\tau_1}}, \qquad Q_{1,2} = \frac{q_{1,2}}{q_0}$$

The dimensionless parameters are the thermal conductivity ratio Λ, thermal diffusivity ratio χ, thermal relaxation time ratio Θ and initial temperature ϑ_0 of semispace 2 given by

$$\Lambda = \frac{K_2}{K_1}, \qquad \chi = \frac{k_2}{k_1}, \qquad \Theta = \frac{\tau_2}{\tau_1}, \qquad \vartheta_0 = \frac{K_1(T_{02} - T_{01})}{q_0\sqrt{k_1\tau_1}}$$

With the introduced quantities, the dimensionless definition of the problem of Eqs.(3)–(9) incorporates the heat conduction equations

$$\begin{gathered} \frac{\partial^2\vartheta_1}{\partial\eta^2} + \frac{\partial\vartheta_1}{\partial\eta} = \frac{\partial^2\vartheta_1}{\partial\xi^2}, \qquad \xi > 0, \qquad \eta > 0; \\ \Theta\frac{\partial^2\vartheta_2}{\partial\eta^2} + \frac{\partial\vartheta_2}{\partial\eta} = \chi\frac{\partial^2\vartheta_2}{\partial\xi^2}, \qquad \xi < 0, \qquad \eta > 0 \end{gathered} \tag{10}$$

initial conditions

$$\begin{gathered} \vartheta_1|_{\eta=0} = 0; \\ \vartheta_2|_{\eta=0} = \vartheta_0 \end{gathered} \tag{11}$$

and

$$\left.\frac{\partial\vartheta_1}{\partial\eta}\right|_{\eta=0} = \left.\frac{\partial\vartheta_2}{\partial\eta}\right|_{\eta=0} = 0 \tag{12}$$

contact conditions

$$\vartheta_1|_{\xi=0} = \vartheta_2|_{\xi=0} \tag{13}$$

and

$$Q_1|_{\xi=0} - Q_2|_{\xi=0} = 1 \tag{14}$$

remembering that

$$\begin{gathered} \frac{\partial Q_1}{\partial\eta} + Q_1 = -\frac{\partial\vartheta_1}{\partial\xi}; \\ \Theta\frac{\partial Q_2}{\partial\eta} + Q_2 = -\Lambda\frac{\partial\vartheta_2}{\partial\xi} \end{gathered} \tag{15}$$

and conditions at infinity

$$\left.\frac{\partial\vartheta_1}{\partial\xi}\right|_{\xi\to+\infty} = \left.\frac{\partial\vartheta_2}{\partial\xi}\right|_{\xi\to-\infty} = 0 \tag{16}$$

The temperatures $\vartheta_{1,2}$ are thus disturbed by two factors: interfacial heat source of Eq.(14) and initial temperature difference of Eq.(11).

3. Analytical solution

Application of the Laplace integral transform $\mathcal{L}$ (Doetsch [40]) to Eq.(10) with respect to η and taking account of Eq.(11) and Eq.(12) yield

$$\begin{aligned} &\frac{\partial^2\tilde{\vartheta}_1}{\partial\xi^2} - s(s+1)\tilde{\vartheta}_1 = 0; \\ &\chi\frac{\partial^2\tilde{\vartheta}_2}{\partial\xi^2} - s(\Theta s+1)\tilde{\vartheta}_2 + \vartheta_0(\Theta s+1) = 0 \end{aligned} \tag{17}$$

where s is the transform parameter; $\tilde{\vartheta}_{1,2}$ are the images of $\vartheta_{1,2}$, i.e. $\tilde{\vartheta}_{1,2}(\xi,s) = \mathcal{L}\left[\vartheta_{1,2}(\xi,\eta)\right]$. The set of solutions of Eq.(17) satisfying Eq.(13) and Eq.(16) reads

$$\begin{aligned} &\tilde{\vartheta}_1(\xi,s) = A(s)\exp\left\{-\xi\sqrt{s(s+1)}\right\}; \\ &\tilde{\vartheta}_2(\xi,s) = \frac{\vartheta_0}{s} + \left(A(s) - \frac{\vartheta_0}{s}\right)\exp\left\{\frac{\xi}{\sqrt{\chi}}\sqrt{s(\Theta s+1)}\right\} \end{aligned} \tag{18}$$

Here $A(s)$ is a yet unknown function.

The heat flux images $\tilde{Q}_{1,2}(\xi,s) = \mathcal{L}\left[Q_{1,2}(\xi,\eta)\right]$ can be determined from Eq.(15) as

$$\begin{aligned} &\tilde{Q}_1(\xi,s) = -\frac{1}{s+1}\frac{\partial\tilde{\vartheta}_1(\xi,s)}{\partial\xi} = A(s)\frac{\sqrt{s}}{\sqrt{s+1}}\exp\left\{-\xi\sqrt{s(s+1)}\right\}; \\ &\tilde{Q}_2(\xi,s) = -\frac{\Lambda}{\Theta s+1}\frac{\partial\tilde{\vartheta}_2(\xi,s)}{\partial\xi} = \left(\frac{\vartheta_0}{s} - A(s)\right)\frac{\Lambda\sqrt{s}}{\sqrt{\chi}\sqrt{\Theta s+1}}\exp\left\{\frac{\xi}{\sqrt{\chi}}\sqrt{s(\Theta s+1)}\right\} \end{aligned} \tag{19}$$

Substitution of Eq.(19) into Eq.(14) in the space of images allows finding $A(s)$ in the form

$$A(s) = \frac{\sqrt{\chi}\sqrt{s+1}\sqrt{\Theta s+1} + \vartheta_0\Lambda\sqrt{s}\sqrt{s+1}}{s\sqrt{s}\left(\Lambda\sqrt{s+1} + \sqrt{\chi}\sqrt{\Theta s+1}\right)} \tag{20}$$

After rationalising the denominator of Eq.(20), Eq.(18) is represented as follows:

$$\begin{aligned} \tilde{\vartheta}_1(\xi,s) = {} & b\exp\left\{-\xi\sqrt{s(s+1)}\right\}\left(c\frac{(s+1)\sqrt{s+\Theta^{-1}}}{s\sqrt{s}(s+a)} - c^2\frac{(s+\Theta^{-1})\sqrt{s+1}}{s\sqrt{s}(s+a)} + \vartheta_0\frac{s+1}{s(s+a)}\right. \\ & \left. - \vartheta_0 c\frac{\sqrt{s+1}\sqrt{s+\Theta^{-1}}}{s(s+a)}\right); \\ \tilde{\vartheta}_2(\xi,s) = {} & \frac{\vartheta_0}{s} + \exp\left\{\frac{\xi\sqrt{\Theta}}{\sqrt{\chi}}\sqrt{s(s+\Theta^{-1})}\right\}\left(bc\frac{(s+1)\sqrt{s+\Theta^{-1}}}{s\sqrt{s}(s+a)} - bc^2\frac{(s+\Theta^{-1})\sqrt{s+1}}{s\sqrt{s}(s+a)}\right. \\ & \left. + \vartheta_0 b\frac{s+1}{s(s+a)} - \vartheta_0 bc\frac{\sqrt{s+1}\sqrt{s+\Theta^{-1}}}{s(s+a)} - \frac{\vartheta_0}{s}\right) \end{aligned} \tag{21}$$

where

$$a = \frac{\Lambda^2 - \chi}{\Lambda^2 - \chi\Theta};$$

$$b = \frac{\Lambda^2}{\Lambda^2 - \chi\Theta};$$

$$c = \frac{\sqrt{\chi\Theta}}{\Lambda}$$

Introduce the functions

$$\begin{aligned}
\tilde{\phi}_1(\xi, s) &= \frac{\exp\left\{-\xi\sqrt{s(s+1)}\right\}}{\sqrt{s(s+1)}};\\
\tilde{\phi}_2(\xi, s) &= \frac{\exp\left\{\xi\sqrt{\Theta/\chi}\sqrt{s(s+\Theta^{-1})}\right\}}{\sqrt{s(s+\Theta^{-1})}};\\
\tilde{\theta}_1(s) &= \frac{a(2+\Theta)-1}{a^2\Theta s} + \frac{1}{a\Theta s^2} + \frac{(1-a)^2(1-a\Theta)}{a^2\Theta(s+a)};\\
\tilde{\theta}_2(s) &= \frac{a(2\Theta+1)-1}{a^2\Theta^2 s} + \frac{1}{a\Theta^2 s^2} + \frac{(1-a)(1-a\Theta)^2}{a^2\Theta^2(s+a)};\\
\tilde{\lambda}_1(s) &= \frac{\sqrt{s+1}}{\sqrt{s}} - 1;\\
\tilde{\lambda}_2(s) &= \frac{\sqrt{s+\Theta^{-1}}}{\sqrt{s}} - 1;\\
\tilde{\psi}(s) &= \frac{s}{\sqrt{s+1}\sqrt{s+\Theta^{-1}}} - 1;\\
\tilde{\varrho}(s) &= \frac{1}{a\Theta s} + \frac{(1-a)(a\Theta-1)}{a\Theta(s+a)};\\
\tilde{\kappa}(s) &= \frac{1}{s+a}
\end{aligned} \tag{22}$$

and express Eq.(21) in the following manner:

$$\begin{aligned}
\tilde{\vartheta}_1(\xi, s) &= b\tilde{\phi}_1(\xi, s)\Big(c\big(\tilde{\theta}_1(s)+1\big)\big(\tilde{\psi}(s)+1\big) - c^2(\tilde{\varrho}(s)+1)\\
&\quad + \vartheta_0\big((1-a)\tilde{\kappa}(s)+1\big)\big(\tilde{\lambda}_1(s)+1\big) - \vartheta_0 c\big((1-a)\tilde{\kappa}(s)+1\big)\big(\tilde{\lambda}_2(s)+1\big)\Big);\\
\tilde{\vartheta}_2(\xi, s) &= \frac{\vartheta_0}{s} + \tilde{\phi}_2(\xi, s)\Big(bc(\tilde{\varrho}(s)+1) - bc^2\big(\tilde{\theta}_2(s)+1\big)\big(\tilde{\psi}(s)+1\big)\\
&\quad + \vartheta_0 b\big((1-a)\tilde{\kappa}(s)+1\big)\big(\tilde{\lambda}_2(s)+1\big)\\
&\quad - \vartheta_0 bc\big((\Theta^{-1}-a)\tilde{\kappa}(s)+1\big)\big(\tilde{\lambda}_1(s)+1\big) - \vartheta_0\big(\tilde{\lambda}_2(s)+1\big)\Big)
\end{aligned} \tag{23}$$

By applying the convolution theorem for the Laplace transform, the originals of Eq.(23) are finally found in the form

$$
\begin{aligned}
&\vartheta_1(\xi,\eta) = b(1-c)(c+\vartheta_0)\phi_1(\xi,\eta)\\
&+b\int_0^\eta \phi_1(\xi,\eta-\varsigma)\Bigg(c\int_0^\varsigma \psi(\epsilon)\theta_1(\varsigma-\epsilon)\,d\epsilon + \vartheta_0(1-a)\int_0^\varsigma \big(\lambda_1(\epsilon)-c\lambda_2(\epsilon)\big)\kappa(\varsigma-\epsilon)\,d\epsilon\\
&\qquad + c\theta_1(\varsigma) + c\psi(\varsigma) - c^2\varrho(\varsigma) + \vartheta_0\lambda_1(\varsigma) - \vartheta_0 c\lambda_2(\varsigma)\\
&\qquad + \vartheta_0(1-a)(1-c)\kappa(\varsigma)\Bigg)d\varsigma\,;\\
&\vartheta_2(\xi,\eta) = \vartheta_0 + (b(1-c)(c+\vartheta_0)-\vartheta_0)\phi_2(\xi,\eta)\\
&+\int_0^\eta \phi_2(\xi,\eta-\varsigma)\Bigg(-bc^2\int_0^\varsigma \psi(\epsilon)\theta_2(\varsigma-\epsilon)\,d\epsilon\\
&\qquad + \vartheta_0 b\int_0^\varsigma \big(c(a-\Theta^{-1})\lambda_1(\epsilon)+(1-a)\lambda_2(\epsilon)\big)\kappa(\varsigma-\epsilon)\,d\epsilon - bc^2\theta_2(\varsigma)\\
&\qquad - bc^2\psi(\varsigma) + bc\varrho(\varsigma) - \vartheta_0 bc\lambda_1(\varsigma) + \vartheta_0(b-1)\lambda_2(\varsigma)\\
&\qquad + \vartheta_0 b\big(1-a+c(a-\Theta^{-1})\big)\kappa(\varsigma)\Bigg)d\varsigma
\end{aligned}
\tag{24}
$$

where $\phi_{1,2}$, $\theta_{1,2}$, $\lambda_{1,2}$, ψ, ϱ and κ are the respective originals of Eq.(22).

The originals $\theta_{1,2}$, ϱ and κ are easily found using the basic Laplace transforms and the property of linearity (Doetsch [40]):

$$
\begin{aligned}
\theta_1(\eta) &= \frac{a(2+\Theta)-1}{a^2\Theta} + \frac{\eta}{a\Theta} + \frac{(1-a)^2(1-a\Theta)}{a^2\Theta}\exp\{-a\eta\}\,;\\
\theta_2(\eta) &= \frac{a(2\Theta+1)-1}{a^2\Theta^2} + \frac{\eta}{a\Theta^2} + \frac{(1-a)(1-a\Theta)^2}{a^2\Theta^2}\exp\{-a\eta\}\,;\\
\varrho(\eta) &= \frac{1}{a\Theta} + \frac{(1-a)(a\Theta-1)}{a\Theta}\exp\{-a\eta\}\,;\\
\kappa(\eta) &= \exp\{-a\eta\}
\end{aligned}
$$

On the other hand, the originals $\phi_{1,2}$, $\lambda_{1,2}$ and ψ are non-elementary functions, and their determination requires application of the special Laplace transforms (Luikov [3], p.587,588):

$$
\begin{aligned}
\phi_1(\xi,\eta) &= \exp\left\{-\frac{\eta}{2}\right\} I_0\left(\frac{\sqrt{\eta^2-\xi^2}}{2}\right) H(\eta-\xi);\\
\phi_2(\xi,\eta) &= \exp\left\{-\frac{\eta}{2\Theta}\right\} I_0\left(\frac{\sqrt{\eta^2-\xi^2\,\Theta/\chi}}{2\Theta}\right) H\left(\eta+\frac{\xi\sqrt{\Theta}}{\sqrt{\chi}}\right);\\
\lambda_1(\eta) &= \frac{1}{2}\exp\left\{-\frac{\eta}{2}\right\}\left(I_0\left(\frac{\eta}{2}\right)+I_1\left(\frac{\eta}{2}\right)\right);\\
\lambda_2(\eta) &= \frac{1}{2\Theta}\exp\left\{-\frac{\eta}{2\Theta}\right\}\left(I_0\left(\frac{\eta}{2\Theta}\right)+I_1\left(\frac{\eta}{2\Theta}\right)\right);
\end{aligned}
$$

$$\psi(\eta) = \frac{1}{2\Theta}\exp\left\{-\frac{1+\Theta}{2\Theta}\eta\right\}\left(-(1+\Theta)I_0\left(\frac{1-\Theta}{2\Theta}\eta\right) + (1-\Theta)I_1\left(\frac{1-\Theta}{2\Theta}\eta\right)\right)$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind of order v; $H(\cdot)$ is the Heaviside step function.

Systematic analysis of the temperatures $\vartheta_{1,2}$ and heat fluxes $Q_{1,2}$ should involve their comparisons with the dimensionless temperatures $\vartheta_{\mathrm{p}1,2}$ and heat fluxes $Q_{\mathrm{p}1,2}$ obtained for the similar parabolic conduction problem defined as

$$\begin{aligned} &\frac{\partial \vartheta_{\mathrm{p1}}}{\partial \eta} = \frac{\partial^2 \vartheta_{\mathrm{p1}}}{\partial \xi^2}, \qquad \xi > 0, \qquad \eta > 0; \\ &\frac{\partial \vartheta_{\mathrm{p2}}}{\partial \eta} = \chi \frac{\partial^2 \vartheta_{\mathrm{p2}}}{\partial \xi^2}, \qquad \xi < 0, \qquad \eta > 0 \end{aligned} \tag{25}$$

The solution of Eq.(25) with the initial conditions, contact conditions and conditions at infinity coinciding with respective Eq.(11), Eqs.(13),(14) and Eq.(16) is built up from the known temperature expressions (Carslaw and Jaeger [2], p.87,88):

$$\begin{aligned} &\vartheta_{\mathrm{p1}}(\xi,\eta) = \frac{\sqrt{\chi}}{\Lambda+\sqrt{\chi}}\left(\frac{2\sqrt{\eta}}{\sqrt{\pi}}\exp\left\{\frac{-\xi^2}{4\eta}\right\} - \xi\,\mathrm{erfc}\left(\frac{\xi}{2\sqrt{\eta}}\right)\right) + \frac{\vartheta_0\Lambda}{\Lambda+\sqrt{\chi}}\mathrm{erfc}\left(\frac{\xi}{2\sqrt{\eta}}\right); \\ &\vartheta_{\mathrm{p2}}(\xi,\eta) = \vartheta_0 + \frac{\sqrt{\chi}}{\Lambda+\sqrt{\chi}}\left(\frac{2\sqrt{\eta}}{\sqrt{\pi}}\exp\left\{\frac{-\xi^2}{4\chi\eta}\right\} + \frac{\xi}{\sqrt{\chi}}\,\mathrm{erfc}\left(\frac{-\xi}{2\sqrt{\chi}\sqrt{\eta}}\right)\right) - \frac{\vartheta_0\sqrt{\chi}}{\Lambda+\sqrt{\chi}}\mathrm{erfc}\left(\frac{-\xi}{2\sqrt{\chi}\sqrt{\eta}}\right) \end{aligned} \tag{26}$$

where $\mathrm{erfc}(\cdot)$ is the complementary error function.

For shorter notation, the temperature distributions are denoted by

$$\vartheta(\xi,\eta) = \begin{cases} \vartheta_1(\xi,\eta), \xi \geq 0; \\ \vartheta_2(\xi,\eta), \xi < 0 \end{cases}$$

and

$$\vartheta_{\mathrm{p}}(\xi,\eta) = \begin{cases} \vartheta_{\mathrm{p1}}(\xi,\eta), \xi \geq 0; \\ \vartheta_{\mathrm{p2}}(\xi,\eta), \xi < 0 \end{cases}$$

The temperature ϑ given by Eq.(24) was validated by comparisons with the known analytical solutions, numerical computation and asymptotic analysis. A small portion of this validation is illustrated in Fig.2. If the semispaces possess the same properties, i.e. $\Lambda = \chi = \Theta = 1$, and there is no initial temperature difference, i.e. $\vartheta_0 = 0$, ϑ is distributed symmetrically and matches the temperature expression by Kao [25]. A numerical algorithm based on the implicit finite-difference approximations validated ϑ for arbitrary combinations of the parameters. It is also seen that the behaviour of ϑ becomes asymptotically equivalent to that of ϑ_{p} given by Eq.(26) with increasing η.

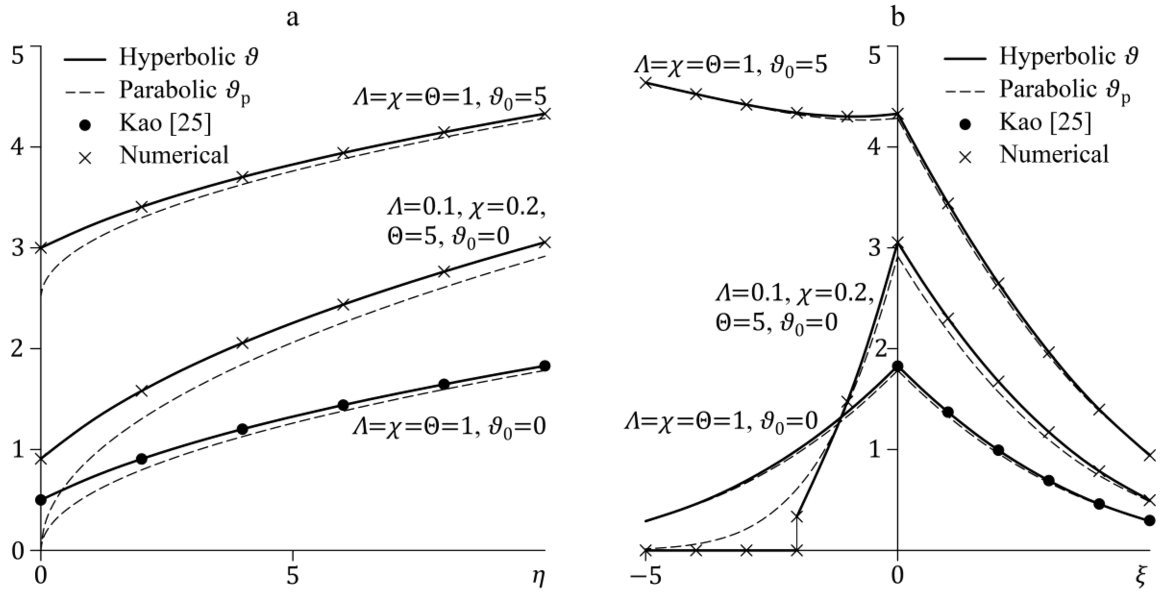

Fig.2. Validation of Eq.(24): (a) contact temperature ($\xi = 0$); (b) temperature distribution at $\eta = 10$

4. Parametric analysis

There are four dimensionless parameters, namely, Λ, χ, Θ and ϑ_0. The thermal conductivity ratio Λ and thermal diffusivity ratio χ are related as

$$\chi = \Lambda \frac{c_{v1}}{c_{v2}}$$

where $c_{v1,2}$ are the volumetric heat capacities. For temperatures above 0 °C, the vast majority of solid materials have volumetric heat capacities lying in a comparatively narrow range of 1–5 J/(cm^3 °C) [41, 42], whereas their thermal conductivities are of order 10^{-1} to 10^2 W/(m °C). Thereby, in general the influence of the ratio c_{v1}/c_{v2} on χ is substantially weaker compared to that of Λ and is neglected in the following graphical illustrations by setting $\chi = \Lambda$.

Equation (24) represents the superposition of the solution of the problem of heating by an interfacial heat source and that of the problem of contact heat transfer due to an initial temperature difference. The solution of the former problem is derived from Eq.(24) by setting $\vartheta_0 = 0$. The latter problem implies zero in the right side of Eq.(14). Its solution can be obtained from Eq.(24) by eliminating the terms not associated with ϑ_0. The mentioned two solutions are analysed further in this section. A special focus is put on the following particular cases: contact of a semispace with a thermal insulator at $\Lambda \to 0$, i.e. $K_2 \to 0$; contact of a semispace with a perfect thermal conductor at $\Lambda \to \infty$, i.e. $K_2 \to \infty$; contact of a semispace with a parabolic conduction semispace (with infinite speed of heat propagation) at $\Theta \to 0$, i.e. $\tau_2 \to 0$; contact between semispaces with the same thermal relaxation time at $\Theta = 1$, i.e. $\tau_1 = \tau_2$; contact of a semispace with a zero heat propagation speed semispace at $\Theta \to \infty$, i.e. $\tau_2 \to \infty$.

4.1. Interfacial heat source

If there is no initial temperature difference, i.e. $\vartheta_0 = 0$, the contact temperature $\vartheta^* = \vartheta_1|_{\xi=0}$, derived from Eq.(24), reads

$$\frac{a\Theta}{bc}\vartheta^*(\eta) = \Theta - c + \big(c(1-a\Theta) - \Theta(1-a)\big)\exp\{-a\eta\}$$
$$-\frac{c}{2}\int_0^\eta (1-(1-a\Theta)\exp\{-a(\eta-\varsigma)\})\exp\left\{-\frac{\varsigma}{2}\right\}\left(I_0\left(\frac{\varsigma}{2}\right) + I_1\left(\frac{\varsigma}{2}\right)\right)d\varsigma \qquad (27)$$
$$+\frac{1}{2}\int_0^\eta (1-(1-a)\exp\{-a(\eta-\varsigma)\})\exp\left\{-\frac{\varsigma}{2\Theta}\right\}\left(I_0\left(\frac{\varsigma}{2\Theta}\right) + I_1\left(\frac{\varsigma}{2\Theta}\right)\right)d\varsigma$$

The partition of the heat generated at the interface $\xi = 0$ is described by the heat flux $Q_1^* = Q_1|_{\xi=0}$ into semispace 1. The inverse transform of Eq.(19) leads to the following expression:

$$\frac{a^2\Theta}{bc}Q_1^*(\eta) = a(1+\Theta-c) - 1 + a\eta + (1-a\Theta)\big(1-a(1-c)\big)\exp\{-a\eta\}$$
$$-\frac{1}{2\Theta}\int_0^\eta \Big(a(1+\Theta) - 1 + a(\eta-\varsigma)$$
$$+(1-a)(1-a\Theta)\exp\{-a(\eta-\varsigma)\}\Big)\exp\left\{-\frac{1+\Theta}{2\Theta}\varsigma\right\}\left((1+\Theta)I_0\left(\frac{1-\Theta}{2\Theta}\varsigma\right)\right. \qquad (28)$$
$$\left.-(1-\Theta)I_1\left(\frac{1-\Theta}{2\Theta}\varsigma\right)\right)d\varsigma$$

Figure 3 illustrates the evolution of ϑ^* at different values of Λ. At $\Lambda \to 0$ the entire heat generated is dissipated in semispace 1. With an increase in Λ, the ability of semispace 2 to remove heat from the interface increases and, accordingly, ϑ^* becomes lower. At $\Lambda \to \infty$ the entire heat generated is immediately dissipated in semispace 2, which yields $\vartheta^* = 0$.

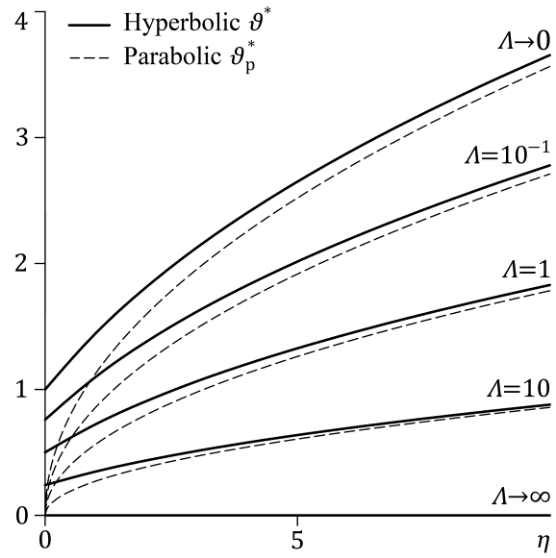

Fig.3. Influence of Λ on the contact temperature ϑ^* at $\Theta = 1$

The influence of Θ on the thermal behaviour is described in Fig.4. As $\Theta \to 0$, heat propagates in semispace 2 with infinite speed, and the curve of ϑ^* is close to that of $\vartheta_{\mathrm{p}}^* = \vartheta_{\mathrm{p1}}\big|_{\xi=0}$ obtained from Eq.(26). With an increase in Θ, both ϑ^* and Q_1^* increase, which is due to a lower heat propagation speed in semispace 2. Furthermore, as $\Theta \to \infty$, the heat is not transferred through semispace 2, resulting in $Q_1^* = 1$, while the curve of ϑ^* coincides with that of '$\Lambda \to 0$' in Fig.3. In the case of $\Theta = 1$, the heat fluxes Q_1^* and $Q_{\mathrm{p1}}^* = Q_{\mathrm{p1}}\big|_{\xi=0}$ are identically equal between each other and are constant in time.

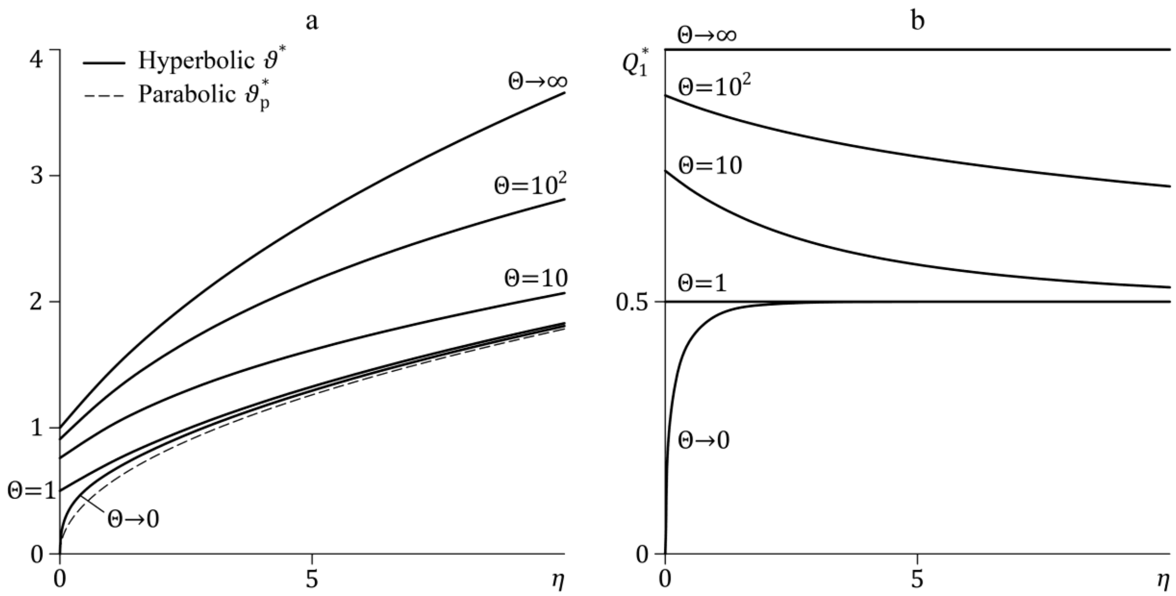

Fig.4. Influence of Θ on the contact temperature ϑ^* (a) and heat flux Q_1^* (b) at $\Lambda = 1$

The qualitative difference between ϑ^* and ϑ_{p}^* lies in a step-wise change of the former at time $\eta = 0$. It follows from Eq.(27) and Eq.(28) that

$$\vartheta^*|_{\eta\to 0} = Q_1^*|_{\eta\to 0} = \frac{\sqrt{\chi\Theta}}{\Lambda + \sqrt{\chi\Theta}} \tag{29}$$

that is, ϑ^* and Q_1^* undergo the same jump. Fig.5 provides the relevant graphical illustration for Θ varying from 10^{-2} to 10^2 in logarithmic scale. It is seen that the magnitude of the jump increases with decreasing Λ or increasing Θ.

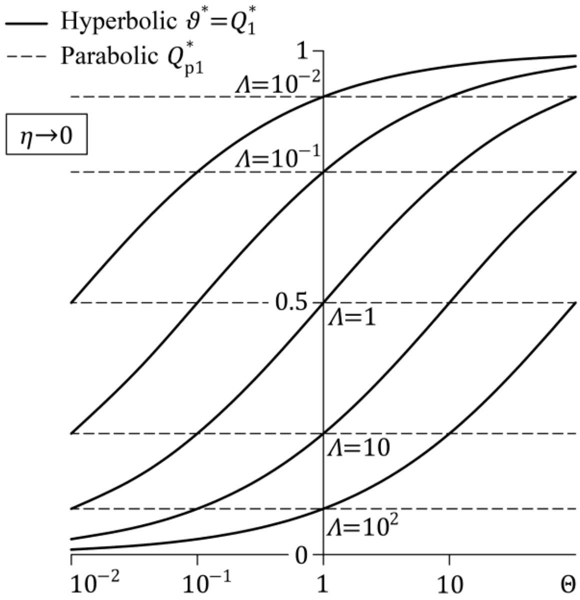

Fig.5. Influence of Λ and Θ on the jump of the contact temperature ϑ^* and heat flux Q_1^*

Of interest is the dimensional representation of Eq.(29) in the form

$$(T^* - T_{01})|_{t\to0} = \frac{q_0}{e_{\tau1} + e_{\tau2}} \tag{30}$$

and

$$q_1^*|_{t\to0} = \frac{e_{\tau1}}{e_{\tau1} + e_{\tau2}} q_0 \tag{31}$$

where $T^* = T_1|_{x=0}$ is the contact temperature; $q_1^* = q_1|_{x=0}$ is the heat flux into semispace 1; $e_{\tau1,2}$ are the coefficients given by

$$e_{\tau1} = \frac{K_1}{\sqrt{k_1\tau_1}};$$

$$e_{\tau2} = \frac{K_2}{\sqrt{k_2\tau_2}}$$

Consequently, the initial heat partition occurs due to the ratio of $e_{\tau1}$ and $e_{\tau2}$. The dimension of $e_{\tau1,2}$ is W/(m^2 °C).

In the case of parabolic conduction, the heat flux q_{p1}^* into semispace 1 equals (Carslaw and Jaeger [2], p.88)

$$q_{p1}^* = \frac{e_1}{e_1 + e_2} q_0 \tag{32}$$

where $e_1 = K_1/\sqrt{k_1}$ and $e_2 = K_2/\sqrt{k_2}$ are the thermal effusivities of respective semispaces 1 and 2. Comparison of Eq.(31) with Eq.(32) yields that the coefficients $e_{\tau1,2}$ play the role of hyperbolic conduction thermal effusivities.

When considering thermal contact problems with interfacial heat sources, a heat partition coefficient α_{f} is often used to indicate the fraction of the generated heat which passes to one of the bodies (first body, for definiteness), while the reminder $(1-\alpha_{\mathrm{f}})$ passes to the other body. In our notation,

$$q_1^* = \alpha_{\mathrm{f}} q_0;$$
$$-q_2^* = (1-\alpha_{\mathrm{f}}) q_0$$

Eq.(31) suggests that to a first approximation, neglecting the variations of q_1^* and q_2^*, one can accept that

$$\alpha_{\mathrm{f}} = \frac{e_{\tau 1}}{e_{\tau 1} + e_{\tau 2}} \qquad (33)$$

Many theoretical methods have been proposed aiming at determination of α_{f}. Table 1 overviews the heat partition models based solely on the thermophysical properties of the contacting bodies. Comparison of the expressions presented in Table 1 shows that Eq.(33) degrades to the model by Blok [43] and Jaeger [44] at $k_1\tau_1 = k_2\tau_2$ and model by Charron [45] at $\tau_1 = \tau_2$.

Table 1. Heat partition models based solely on the thermophysical properties

Model	Description	Heat partition coefficient
Blok [43], Jaeger [44]	Stationary heat conduction in the semispaces coupled in a bounded region (circle, square).	$\alpha_{\mathrm{f}} = \frac{K_1}{K_1 + K_2}$
Charron [45]	Non-stationary parabolic heat conduction in coupled semispaces.	$\alpha_{\mathrm{f}} = \frac{e_1}{e_1 + e_2} = \frac{K_1/\sqrt{k_1}}{K_1/\sqrt{k_1} + K_2/\sqrt{k_2}}$
Hasselgruber [46]	Non-stationary parabolic heat conduction in coupled semispaces with account of the efficient thermal layers. Here $\rho_{1,2}$ are the densities.	$\alpha_{\mathrm{f}} = \frac{K_1/\rho_1}{K_1/\rho_1 + K_2/\rho_2}$
This study, Eq.(33)	Hyperbolic heat conduction in coupled semispaces.	$\alpha_{\mathrm{f}} = \frac{e_{\tau 1}}{e_{\tau 1} + e_{\tau 2}} = \frac{K_1/\sqrt{k_1\tau_1}}{K_1/\sqrt{k_1\tau_1} + K_2/\sqrt{k_2\tau_2}}$

4.2. Initial temperature difference

In the absence of interfacial heat source, ϑ^* is found by simplifying Eq.(24) into

$$\frac{a^2\Theta}{\vartheta_0 b}\vartheta^*(\eta) = a\Theta + c\big(1 - a(1+\Theta)\big) - ac\eta - (1-a)\big(c + a\Theta(1-c)\big)\exp\{-a\eta\}$$
$$+\frac{c}{2\Theta}\int_0^\eta (a(1+\Theta) - 1 + a(\eta - \varsigma)$$
$$+ (1-a)(1-a\Theta)\exp\{-a(\eta-\varsigma)\})\exp\left\{-\frac{1+\Theta}{2\Theta}\varsigma\right\}\left((1+\Theta)I_0\left(\frac{1-\Theta}{2\Theta}\varsigma\right)\right. \quad (34)$$
$$\left. - (1-\Theta)I_1\left(\frac{1-\Theta}{2\Theta}\varsigma\right)\right)d\varsigma$$

while $Q_1^* = Q_2^*$, defined in the space of images by Eq.(19), reads

$$\frac{Q_1^*(\eta)}{\vartheta_0 b} = (1-c)\exp\{-a\eta\}$$
$$+\frac{1}{2}\int_0^\eta \exp\{-a(\eta-\varsigma)\}\left(\exp\left\{-\frac{\varsigma}{2}\right\}\left(I_0\left(\frac{\varsigma}{2}\right) + I_1\left(\frac{\varsigma}{2}\right)\right)\right. \quad (35)$$
$$\left. -\frac{c}{\Theta}\exp\left\{-\frac{\varsigma}{2\Theta}\right\}\left(I_0\left(\frac{\varsigma}{2\Theta}\right) + I_1\left(\frac{\varsigma}{2\Theta}\right)\right)\right)d\varsigma$$

Figure 6 shows the influence of Λ on Q_1^*. At $\Lambda \to 0$ the heat flux through the interface equals zero, i.e. $Q_1^* = 0$. For a fixed value of η, Q_1^* increases with increasing Λ and reaches its maximum at $\Lambda \to \infty$.

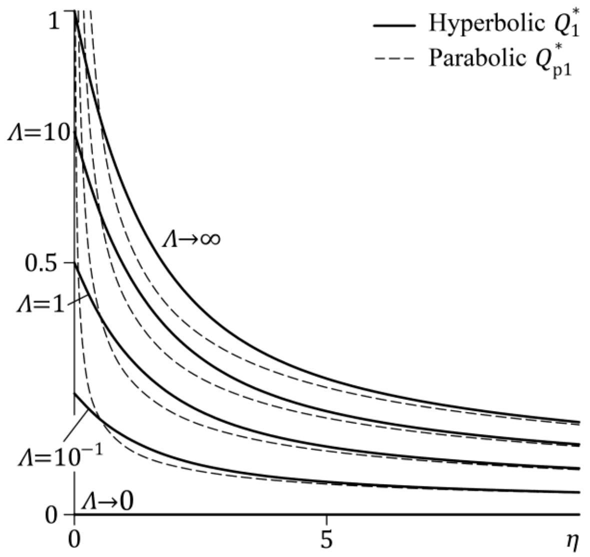

Fig.6. Influence of Λ on the contact heat flux Q_1^* at $\Theta = \vartheta_0 = 1$

Figure 7 shows the influence of Θ on the thermal behaviour. It is seen that an increase in Θ leads to a smaller value of ϑ^*. In the case of $\Theta = 1$, ϑ^* is constant in time, being identically equal to ϑ_p^*. As $\Theta \to \infty$, there is no heat flux through the interface, i.e. $Q_1^* = 0$, which is similar to the case of '$\Lambda \to 0$' in Fig.6, and $\vartheta^* = 0$.

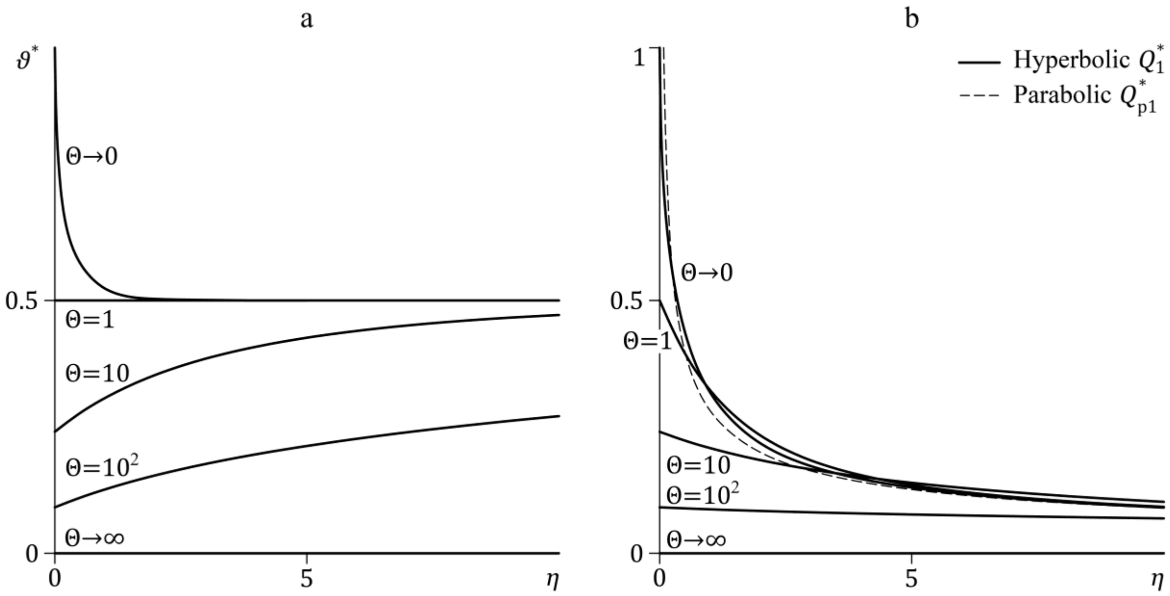

Fig.7. Influence of Θ on the contact temperature ϑ^* (a) and heat flux Q_1^* (b) at $\Lambda = \vartheta_0 = 1$

Analysis of Eq.(34) and Eq.(35) shows that ϑ^* and Q_1^* change at $\eta = 0$ in a step-wise manner:

$$\vartheta^*|_{\eta\to 0} = Q_1^*|_{\eta\to 0} = \frac{\vartheta_0 \Lambda}{\Lambda + \sqrt{\chi\Theta}} \tag{36}$$

The expression of Eq.(36) is illustrated in Fig.8. Apparently, its value increases with increasing Λ or decreasing Θ.

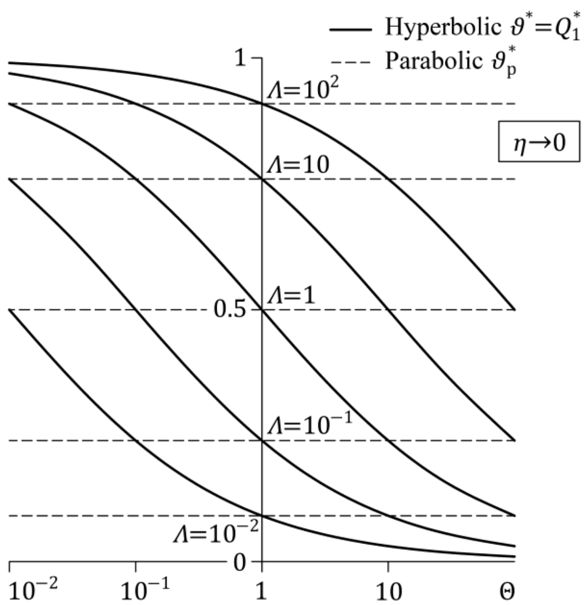

Fig.8. Influence of Λ and Θ on the jump of the contact temperature ϑ^* and heat flux Q_1^* at $\vartheta_0 = 1$

The dimensional representations of Eq.(36) can be written as

$$(T^* - T_{01})|_{t\to 0} = \frac{e_{\tau 2}}{e_{\tau 1} + e_{\tau 2}}(T_{02} - T_{01}) \tag{37}$$

and

$$q_1^*|_{t\to 0} = \frac{T_{02} - T_{01}}{e_{\tau 1}^{-1} + e_{\tau 2}^{-1}} \tag{38}$$

According to Eq.(26), it is true that (Carslaw and Jaeger [2], p.88)

$$q_{\mathrm{p}1}^*\Big|_{t\to 0} \cong \frac{T_{02} - T_{01}}{\sqrt{\pi}(e_1^{-1} + e_2^{-1})\sqrt{t}} \tag{39}$$

Comparison of Eq.(39) and Eq.(38) shows a qualitative difference between them: $q_{\mathrm{p}1}^*$ tends to infinity as $t \to 0$, whereas q_1^* takes a finite value. The infiniteness of $q_{\mathrm{p}1}^*$ suggests that the parabolic conduction equation cannot simulate the perfect thermal contact of bodies with initially different temperatures in a physical manner. The fundamental property of finiteness of heat flux can be, however, described by Eq.(38) based on the hyperbolic conduction equation. It should be mentioned that a similar finding was reported by Baumeister and Hamill [23].

5. Solution applications

The proposed solution of Eq.(24) and its particular expressions of Eqs.(26)–(39) can be useful in the field of mechanics, electromagnetism, optics, chemistry, etc., where an interaction of two bodies is studied involving an instantaneous interfacial source of heat. This section illustrates their application to simulation of microscopic problems of ultra-short laser pulse welding and sliding friction. Note that the following examples schematise spatial problems in the form of one-dimensional problems, and the simulation results should be therefore treated as qualitative ones.

5.1. Ultra-short laser pulse welding

Welding by repetitive ultra-short laser pulses is a modern technology that is used to form high-strength joints of materials, in particular, glasses (Cvecek et al. [47]). Consider a problem of heating of two dissimilar glass pieces 1 and 2 at their interface by a single ultra-short laser pulse. Fig.9 presents the relevant schematic. The thermal conductivities of the pieces differ substantially between each other and equal $K_1 = 10$ W/(m °C) and $K_2 = 1$ W/(m °C), while their thermal diffusivities equal $k_1 = 10^{-5}$ m^2/s and $k_2 = 10^{-6}$ m^2/s. The pieces are assumed to have the same thermal relaxation time $\tau = \tau_1 = \tau_2$. The initial temperature of the pieces equals $T_{01} = T_{02} = 20$ °C. The laser pulse is oriented perpendicular to the interface $x = 0$. Piece 1 is transparent to the laser wavelength and is not thermally affected when the laser pulse passes through it. The energy of the laser pulse transforms into heat in the interfacial region. The specific power of the laser pulse is 10^{11} W/m^2, while its duration t equals 0.1 ns.

Figure 9 shows the temperature distribution T in pieces 1 and 2 obtained by Eq.(24) depending on the value of τ. For $\tau = 0.01$ ns ($\tau \ll t$), T is slightly higher than T_p calculated by Eq.(26). In the case of $\tau = t = 0.1$ ns, T significantly exceeds T_p in the vicinity of the interface. One can clearly see the thermal waves with sharp fronts. If $\tau = 1$ ns ($\tau \gg t$), T is about 3 times higher than T_p in the vicinity of the interface. Note that the considered range of τ is typical for dielectric materials (Guillemet and Bardon [11]).

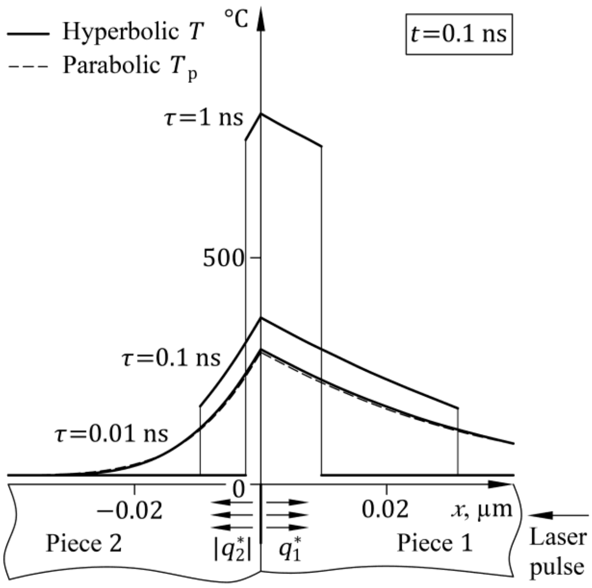

Fig.9. Influence of τ on the temperature T in glass pieces 1 and 2 heated by a laser pulse

5.2. Sliding friction

The present example belongs to the field of tribology. Consider a single interaction of roughness asperities 1 and 2 located at the sliding discs of an aircraft brake in the middle of the braking process (Chichinadze et al. [48]). The discs are made of a carbon composite material with thermal conductivity $K_{1,2} = 10$ W/(m °C) and thermal diffusivity $k_{1,2} = 10^{-5}$ m^2/s. The characteristic size of either of the asperities is 0.1 μm. The velocity of sliding between the asperities equals 10 m/s, which corresponds to the interaction duration of order $t = 10$ ns. The friction heat is generated in the interfacial region $x = 0$ with specific power q_0=10^{10} W/m^2. The initial temperatures of the asperities equal $T_{01} = 400$ °C and $T_{02} = 100$ °C.

Figure 10 shows the temperature distributions T and T_p in asperities 1 and 2 for different values of the thermal relaxation time $\tau = \tau_1 = \tau_2$. It is seen that at $\tau = 1$ ns ($\tau \ll t$), the difference between T and T_p is negligibly small. By contrast, at $\tau = t = 10$ ns, T is significantly higher than

T_p in the vicinity of the interface. Finally, for $\tau = 100$ ns ($\tau \gg t$), T is about 2 times higher than T_p in the vicinity of the interface. These results agree qualitatively with those reported by Nosko [34] for a hyperbolic conduction semispace subjected to the action of a surface heat source and an exponentially distributed volumetric heat source.

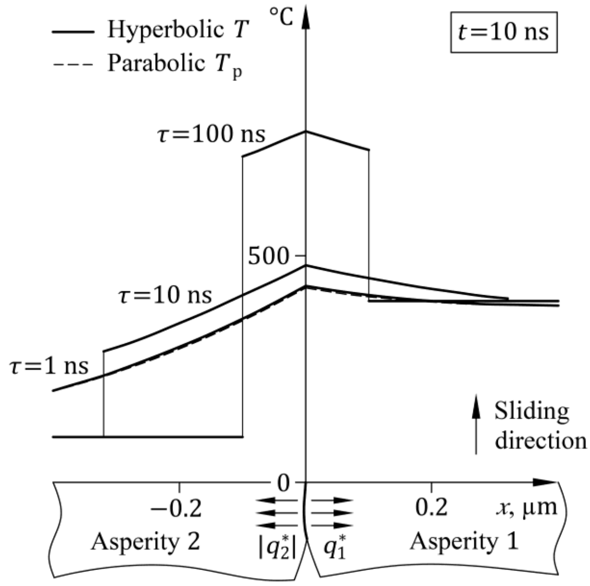

Fig.10. Influence of τ on the temperature T in sliding roughness asperities 1 and 2

In the calculations above, τ takes values in the range of 10^{-9} to 10^{-7} s which are substantially larger than those of order 10^{-12} to 10^{-10} s reported for dielectric materials (Guillemet and Bardon [11]). Several studies including Kaminski [12], Mitra et al. [13] and Roetzel et al. [14] claim that τ of non-homogeneous materials can be of order 10^{-1} to 10 s. Although this range is likely to be a substantial overestimate, the hypothesis that a non-homogeneous friction material can have τ measured in nanoseconds looks plausible.

Thereby, consideration of the wave nature of heat conduction is critically important when investigating microscopic problems of ultra-short laser pulse welding and high-speed tribology.

6. Conclusion

A hyperbolic heat conduction problem for two semispaces coupled by the perfect thermal contact conditions with an interfacial heat source was defined. Its solution was derived using the Laplace integral transform approach and represented in the form of Eq.(24). The contact temperature and heat fluxes were analysed for different ratios of thermal conductivities K_1 and K_2, thermal diffusivities k_1 and k_2, and thermal relaxation times τ_1 and τ_2 of the semispaces. Special attention was paid to the initial instance of time when the thermal behaviour differs qualitatively from that inherent in the similar parabolic conduction problem. The applicability of the solution and its particular expressions was illustrated on the example of microscopic problems of ultra-short laser pulse welding and sliding friction. The main findings can be summarised as follows:

1. Heat generation at the interface between the hyperbolic conduction semispaces results in a step-wise change in the contact temperature, described by Eq.(30). This contrasts with a continuous change in the contact temperature in the case of parabolic conduction.
2. The initial heat partition between the hyperbolic conduction semispaces occurs due to the ratio of $K_1/\sqrt{k_1\tau_1}$ and $K_2/\sqrt{k_2\tau_2}$, described by Eq.(31). The coefficients $K_{1,2}/\sqrt{k_{1,2}\tau_{1,2}}$ play the role of hyperbolic conduction thermal effusivities.
3. The thermal contact of the hyperbolic conduction semispaces with different initial temperatures is characterised by a finite heat flux through the interface, described by Eq.(38). This fundamental property of finiteness of heat flux cannot be described by the parabolic heat conduction theory.

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