Postprint of : Belyakov N., Nosko O., Analytical solution of non-stationary heat conduction problem for two sliding layers with time-dependent friction conditions, International Journal of Heat and Mass Transfer, Vol. 98 (2016), pp. 624-630, DOI: 10.1016/j.ijheatmasstransfer.2016.03.054

Analytical solution of non-stationary heat conduction problem for two sliding layers with time-dependent friction conditions

N. Belyakov

Hilti Corporation

O. Nosko*

Department of Machine Design, KTH Royal Institute of Technology

Abstract

In this article we conduct an overview of various types of thermal contact conditions at the sliding interface. We formulate a problem of non-stationary heat conduction in two sliding layers with generalized thermal contact conditions allowing for dependence of the heat-generation coefficient and contact heat transfer coefficient on time. We then derive an analytical solution of the problem by constructing a special coordinate integral transform. In contrast to the commonly used transforms, e.g. Laplace or Fourier transforms, the one proposed is applicable to a product of two functions dependent on time. The solution is validated by a series of test problems with parameters corresponding to those of real tribosystems. Analysis shows an essential influence of both time-dependent heat generation and contact heat transfer coefficients on the partition of the friction heat between the layers. The solution can be used for simulating temperature fields in sliding components with account of this influence.

Keywords: Non-stationary heat conduction, Sliding layers, Imperfect thermal contact, Integral transform
PACS: 02.30.Uu, 44.05.+e, 46.55.+d, 81.40.Pq

1. Introduction

Thermal friction problem is considered to be one of the central problems in tribology due to the fact that thermal effects manifest in various forms and often can not be ignored. Accurate simulation of frictional processes in a tribosystem requires the knowledge of the temperature fields in its sliding components.

*Corresponding author
Email addresses: nikolay.s.belyakov@gmail.com (N. Belyakov), nosko@dr.com (O. Nosko)

These temperature fields can be determined by numerical methods. Despite the fact that such methods have become common practice, the analytical approach is preferable in many cases. Analytical expressions for the temperature fields enable parametric analysis, investigation of asymptotic behavior and special cases, or testing of numerical algorithms.

Thermal friction problem is usually formulated in the form of an initial-boundary-value problem of non-stationary heat conduction in two coupled bodies with a heat source at their interface. The contact conditions are specified at the interface to describe a certain relation between the spatial derivatives of the temperatures T_1 and T_2 of the bodies, i.e. the heat fluxes into the bodies, and the specific power q of heat generation. There are several basic types of the contact conditions, which are reviewed in [1].

Blok [2] partitioned the friction heat between the sliding bodies by introducing the heat-partition coefficient α_f, so that the contact conditions have the form

$$\lambda_1 \frac{\partial T_1}{\partial \vec{n}}\bigg|_S = \alpha_f q, \qquad -\lambda_2 \frac{\partial T_2}{\partial \vec{n}}\bigg|_S = (1-\alpha_f) q, \tag{1}$$

where S is the interface region, $\vec{n}$ is the unit normal vector at S directed from the first to the second body, λ_1 and λ_2 are the thermal conductivity coefficients of the bodies.

Ling [3] formulated the conditions of the perfect thermal contact which imply the energy balance and temperature continuity in the microscopic regions of contact of roughness asperities. The perfect thermal contact conditions are also often specified at the macroscopic interface, that is

$$\lambda_1 \frac{\partial T_1}{\partial \vec{n}}\bigg|_S - \lambda_2 \frac{\partial T_2}{\partial \vec{n}}\bigg|_S = q, \qquad T_1\big|_S = T_2\big|_S. \tag{2}$$

Podstrigach [4] considered a thermal interaction of two bodies through a thin intermediate layer. He proposed the conditions of imperfect thermal contact between the bodies which describe the heat conduction in the intermediate layer with the contact heat transfer coefficient γ. In the presence of a heat source at the interface, these conditions take the form

$$\begin{aligned} \lambda_1 \frac{\partial T_1}{\partial \vec{n}}\bigg|_S &= \frac{q}{2} - \gamma\,(T_1 - T_2)\big|_S, \\ -\lambda_2 \frac{\partial T_2}{\partial \vec{n}}\bigg|_S &= \frac{q}{2} + \gamma\,(T_1 - T_2)\big|_S. \end{aligned} \tag{3}$$

Independently, Barber [5, 6] and Protasov [7] introduced another type of imperfect thermal contact conditions, which can be presented in our notations as

$$\begin{aligned} \lambda_1 \frac{\partial T_1}{\partial \vec{n}}\bigg|_S &= \alpha q - \gamma\,(T_1 - T_2)\big|_S, \\ -\lambda_2 \frac{\partial T_2}{\partial \vec{n}}\bigg|_S &= (1-\alpha) q + \gamma\,(T_1 - T_2)\big|_S, \end{aligned} \tag{4}$$

where α is the heat-generation coefficient [7]. It is noteworthy that Barber's reasoning [5] was based on heat conduction theory. He assumed that the heat flux, passing in either of the sliding bodies, consists of two components: the first one is due to the frictional heating, while the second is caused by the temperature difference of the bodies. The coefficient α is determined through the microscopic thermal resistances of the rough surfaces of the bodies. Protasov [7], on the other side, investigated the friction heat generation considering adhesion-deformational interactions of roughness asperities and based his conclusion on the principles of thermodynamics. He introduced α as the fraction of the friction energy which is generated at the surface (adhesive mechanism) and in the subsurface layer (deformational mechanism) of the first sliding body.

There is a principal difference between α_f and α: the former means the partition of the friction heat, whereas the latter specifies the partition of the heat-generation power (q) between the sliding bodies. When using any of the conditions (2), (3), or (4), α_f is a priori unknown.

It should be mentioned that the equations (4) are a generalization of the contact conditions considered above, so that (4) would degenerate into (1) at $\gamma = 0$, into (2) at $\gamma \to \infty$, and into (3) at $\alpha = 1/2$.

If the friction conditions, such as the sliding velocity and contact pressure, vary with time, this would result in a change of q. They have also effects on the coefficients α and γ. It is known from literature (see, for instance, [8]) that γ generally increases with the contact pressure. According to the theoretical study [9], both α and γ depend on the sliding velocity. Thus, the quantities q, α, γ should be considered as variables dependent on time t. By this means, the contact conditions (4) are transformed into

$$\begin{aligned} \lambda_1 \frac{\partial T_1}{\partial \vec{n}}\bigg|_S &= \alpha(t) q(t) - \gamma(t)\left(T_1 - T_2\right)\big|_S, \\ -\lambda_2 \frac{\partial T_2}{\partial \vec{n}}\bigg|_S &= \left(1 - \alpha(t)\right) q(t) + \gamma(t)\left(T_1 - T_2\right)\big|_S. \end{aligned} \tag{5}$$

A number of analytical studies on temperatures in sliding components have been conducted using the Laplace transform, Fourier series, or other techniques. The existing mathematical techniques allow to derive analytical solutions of one-dimensional heat conduction problems for sliding bodies represented in the form of semispaces or layers. Classical solutions for the semispaces interacting due to the contact conditions (1) or (2) can be found in [10]. Temperature expressions for the elements of the pairs semispace–semispace, semispace–layer and layer–layer were derived for the contact conditions (3) [11, 12, 13, 14], and the contact conditions (4) [15, 16, 17]. At the same time, a literature review reveals that analytical solutions of heat conduction problems with the time-dependent conditions (5) are unknown.

The aim of this study is to provide an analytical solution of the initial-boundary-value problem of non-stationary heat conduction in two layers coupled through the contact conditions (5). For this purpose, an original integral transform is developed to map the differential operator under the given specific

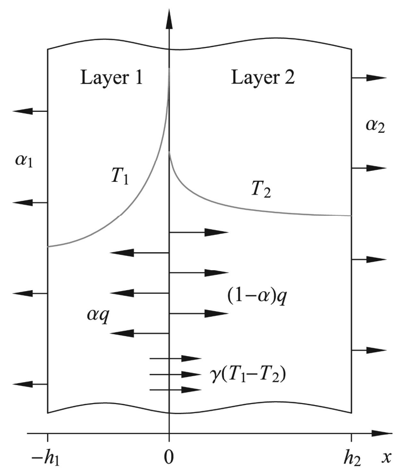

Figure 1: Problem schematic

contact conditions.

2. Problem Statement

We consider non-stationary heat conduction in two layers with thicknesses h_1 and h_2 which move relative to each other, so that there is friction between them. The friction leads to a heat release with a specific power given as a continuous function $q(t) \geq 0$. We assume that the thermal contact between the layers follows the conditions (5). At the free surfaces of the layers we specify convective heat transfer with coefficients α_1 and α_2. At $t = 0$ the temperatures of the layers are equal to some ambient temperature $T_0 \neq 0$. Under the given assumptions, the temperatures $T_1(x,t)$ and $T_2(x,t)$ in the layers change with time t along the direction x perpendicular to the sliding interface. Fig. 1 shows a schematic of the problem. The thermal conductivity and diffusivity coefficients of the layers are denoted by λ_1, λ_2 and a_1, a_2, respectively.

The dimensionless formulation of the problem described incorporates the heat conduction equations

$$\begin{aligned} \frac{\partial \Theta_1(\xi,\tau)}{\partial \tau} &= \frac{\partial^2 \Theta_1(\xi,\tau)}{\partial \xi^2}, \quad -1 < \xi < 0, \quad \tau > 0, \\ \frac{\partial \Theta_2(\xi,\tau)}{\partial \tau} &= \chi \frac{\partial^2 \Theta_2(\xi,\tau)}{\partial \xi^2}, \quad 0 < \xi < H, \quad \tau > 0 \end{aligned} \tag{6}$$

with zero initial conditions

$$\Theta_1(\xi,\tau)\big|_{\tau=0} = 0 = \Theta_2(\xi,\tau)\big|_{\tau=0}, \tag{7}$$

the contact conditions

$$
\begin{aligned}
\left.\frac{\partial \Theta_1(\xi,\tau)}{\partial \xi}\right|_{\xi=0} &= \alpha(\tau)Q(\tau) - B(\tau)\left(\Theta_1(\xi,\tau) - \Theta_2(\xi,\tau)\right)\big|_{\xi=0},\\
-\left.\frac{1}{\Lambda}\frac{\partial \Theta_2(\xi,\tau)}{\partial \xi}\right|_{\xi=0} &= (1-\alpha(\tau))Q(\tau) + B(\tau)\left(\Theta_1(\xi,\tau) - \Theta_2(\xi,\tau)\right)\big|_{\xi=0},
\end{aligned}
\tag{8}
$$

and the boundary conditions

$$
\begin{aligned}
\left.\frac{\partial \Theta_1(\xi,\tau)}{\partial \xi}\right|_{\xi=-1} &= \mathrm{Bi}\,\Theta_1(\xi,\tau)\big|_{\xi=-1},\\
-\left.\frac{1}{\Lambda}\frac{\partial \Theta_2(\xi,\tau)}{\partial \xi}\right|_{\xi=H} &= \Upsilon\,\mathrm{Bi}\,\Theta_2(\xi,\tau)\big|_{\xi=H}.
\end{aligned}
\tag{9}
$$

In the equations above we use the following dimensionless variables

$$
\xi = \frac{x}{h_1}, \quad \tau = \frac{a_1 t}{h_1^2}, \quad \Theta_1 = \frac{T_1 - T_0}{T_0}, \quad \Theta_2 = \frac{T_2 - T_0}{T_0}, \quad Q = \frac{q h_1}{\lambda_1 T_0}, \quad B = \frac{\gamma h_1}{\lambda_1},
$$

and parameters

$$
H = \frac{h_2}{h_1}, \quad \chi = \frac{a_2}{a_1}, \quad \Lambda = \frac{\lambda_1}{\lambda_2}, \quad \mathrm{Bi} = \frac{\alpha_1 h_1}{\lambda_1}, \quad \Upsilon = \frac{\alpha_2}{\alpha_1}.
$$

3. Analytical Solution

The equations (6)–(9) represent an initial-boundary-value heat conduction problem in a double domain with the special contact conditions (8). The main peculiarity of the problem lies in the time-dependent dimensionless contact heat transfer coefficient B. The general methods, which are commonly used for solving such initial-boundary-value problems (e.g. operational calculus or Fourier series approach), are not applicable in this case due to the occurrence of the product of two time-dependent functions. However, an analytical solution can be obtained in the form of a convolution integral by applying the method of generalized integral transforms for multiple domains [18].

3.1. Construction of the Integral Transform

In this section we construct a generalized integral transform [19, 20, 21] with respect to the spatial variable ξ which should eliminate the differentiation defined by the linear second-order differential operator

$$
\mathcal{D}[u(\xi,\tau)] = \begin{cases} \dfrac{\partial^2 u_1(\xi,\tau)}{\partial \xi^2}, & -1 < \xi < 0,\\[2mm] \chi\dfrac{\partial^2 u_2(\xi,\tau)}{\partial \xi^2}, & 0 < \xi < H, \end{cases}
\tag{10}
$$

where the arbitrary function

$$u(\xi,\tau) = \begin{cases} u_1(\xi,\tau), & -1 < \xi < 0, \\ u_2(\xi,\tau), & 0 < \xi < H \end{cases}$$

belongs to $C^{(2)}(-1,H)$ with respect to ξ. For the operator (10) we formulate the Sturm–Liouville problem with the differential equation

$$\mathcal{D}[u(\xi,\tau)] + p^2 u(\xi,\tau) = 0 \tag{11}$$

containing a parameter $p \in \mathbb{R}$.

Let Γ denote a set of functions from the space $C^{(2)}(-1,H)$ matching the equation (11) together with the contact conditions (8) and the boundary conditions (9) formulated for the function $u(\xi,\tau)$. Furthermore, we denote by Γ_h a similar set of functions satisfying the homogeneous contact conditions

$$\left.\frac{\partial u_1(\xi,\tau)}{\partial \xi}\right|_{\xi=0} = \frac{1}{\Lambda}\left.\frac{\partial u_2(\xi,\tau)}{\partial \xi}\right|_{\xi=0} = B(\tau)\left(u_2(\xi,\tau) - u_1(\xi,\tau)\right)|_{\xi=0}, \tag{12}$$

which coincide with (8) at $Q = 0$.

If we use the dot product

$$\langle u(\xi,\tau), v(\xi,\tau)\rangle = \int_{-1}^{H} u(\xi,\tau)v(\xi,\tau)\rho(\xi)\,d\xi \tag{13}$$

in the Hilbert space $L^2[-1,H]$, we can ask the operator (10) to be self-adjoint with respect to this dot product, which can be achieved by the special choice of the weight function

$$\rho(\xi) = \begin{cases} \rho_1 = 1, & -1 < \xi < 0, \\ \rho_2 = (\Lambda\chi)^{-1}, & 0 < \xi < H, \end{cases}$$

leading to the following corollaries [21].

Corollary 1. *For the Sturm-Liouville problem with the linear differential operator* (10) *for functions from the space* Γ_h^n*, the condition of the self-conjugacy implies existence of an infinite sequence of real eigenvalues*

$$0 < p_1^2 < p_2^2 < \cdots < p_k^2 < \ldots$$

and corresponding non-trivial solutions $K(\xi,p_k)$ *which form a complete orthogonal function system on the interval* $-1 \le \xi \le H$ *with respect to the weight function* $\rho(\xi)$.

Corollary 2. *Any function* $u(\xi,\tau) \in L^2$ *can be expanded into a Fourier series along the function system* $K(\xi,p_k)$ *and represented in the form of the infinite sum*

$$u(\xi,\tau) = \sum_{k=0}^{\infty} \frac{K(\xi,p_k)}{\|K(\xi,p_k)\|^2}\langle u(\xi,\tau), K(\xi,p_k)\rangle,$$

meaning that it converges to the function $u(\xi,\tau)$ *along the norm of the space* L^2*, and* $\|f(\xi)\| = \sqrt{\langle f(\xi), f(\xi)\rangle}$ *with the dot product defined by* (13).

By this means, based on the corollaries, the sought temperature function

$$\Theta(\xi,\tau) = \begin{cases} \Theta_1(\xi,\tau), & -1 < \xi < 0, \\ \Theta_2(\xi,\tau), & 0 < \xi < H \end{cases}$$

can be expanded into a Fourier series along the function system $K(\xi, p_k)$. This allows constructing the direct integral transform as

$$U(p,\tau) = \mathcal{T}[\Theta(\xi,\tau)] \equiv \langle \Theta(\xi,\tau), K(\xi,p)\rangle = \int_{-1}^{H} \Theta(\xi,\tau) K(\xi,p) \rho(\xi)\, d\xi, \qquad (14)$$

denoting by $\mathcal{T}$ the operator of the integral transform. The inverse transform gives the equality

$$\Theta(\xi,\tau) = \mathcal{T}^{-1}[U(p,\tau)] \equiv \sum_{k=0}^{\infty} \frac{K(\xi,p_k)}{\|K(\xi,p_k)\|^2} U(p_k,\tau) \qquad (15)$$

considered in $L^2[-1,H]$.

3.2. Construction of the Kernel

According to (14), (15), the temperature functions $\Theta_1(\xi,\tau)$ and $\Theta_2(\xi,\tau)$ differ only by their kernels $K_1(\xi,p)$ and $K_2(\xi,p)$ on the domains $[-1,0]$ and $[0,H]$, respectively. These kernels can be defined by the following theorem.

Theorem 1. *Assume $\Theta(\xi,\tau) \in \Gamma$. Let the kernel $K(\xi,p) \in L^2[-1,H]$ of the integral transform* (14), (15) *satisfy the Sturm–Liouville problem* (11) *with the homogeneous contact conditions* (12) *and the boundary conditions defined as*

$$\begin{aligned} &\left.\frac{\partial K_1(\xi,p)}{\partial \xi}\right|_{\xi=-1} - \mathrm{Bi}\, K_1(\xi,p)\big|_{\xi=-1} = 0, \\ &\left.\frac{1}{\Lambda}\frac{\partial K_2(\xi,p)}{\partial \xi}\right|_{\xi=H} + \Upsilon\, \mathrm{Bi}\, K_2(\xi,p)\big|_{\xi=H} = 0. \end{aligned} \qquad (16)$$

Then the image of the differential operator (10) *satisfies the property*

$$\mathcal{T}\big[\mathcal{D}[\Theta(\xi,\tau)]\big] = -p^2 \mathcal{T}[\Theta(\xi,\tau)] + Q(\tau)\Xi(p,\tau), \qquad (17)$$

where

$$\Xi(p,\tau) = \alpha(\tau) K_1(\xi,p)\big|_{\xi=0} + (1-\alpha(\tau)) K_2(\xi,p)\big|_{\xi=0}.$$

Proof. We begin our proof with applying the direct integral transform (14) to the differential operator (10) and integrating two times by parts, which results in

$$\mathcal{T}\big[\mathcal{D}[\Theta(\xi,\tau)]\big] = \int\limits_{-1}^{0} \frac{\partial^2\Theta_1(\xi,\tau)}{\partial\xi^2} K_1(\xi,p)\rho_1 \, d\xi + \int\limits_{0}^{H} \chi \frac{\partial^2\Theta_2(\xi,\tau)}{\partial\xi^2} K_2(\xi,p)\rho_2 \, d\xi$$

$$= \int\limits_{-1}^{0} \frac{\partial^2 K_1(\xi,p)}{\partial\xi^2} \Theta_1(\xi,\tau)\rho_1 \, d\xi + \int\limits_{0}^{H} \chi \frac{\partial^2 K_2(\xi,p)}{\partial\xi^2} \Theta_2(\xi,\tau)\rho_2 \, d\xi + \mathcal{N}_1(p,\tau) + \mathcal{N}_2(p,\tau). \tag{18}$$

Here $\mathcal{N}_1(p,\tau)$ and $\mathcal{N}_2(p,\tau)$ are unknown terms which shall be defined further from the boundary and contact conditions.

The equality (18) should not include the integral terms. To eliminate them, we assume without loss of generality that

$$\frac{\partial^2 K_1(\xi,p)}{\partial\xi^2} = -p^2 K_1(\xi,p), \qquad \chi \frac{\partial^2 K_2(\xi,p)}{\partial\xi^2} = -p^2 K_2(\xi,p),$$

which yields

$$\int\limits_{-1}^{0} \frac{\partial^2 K_1(\xi,p)}{\partial\xi^2} \Theta_1(\xi,\tau)\rho_1 \, d\xi + \int\limits_{0}^{H} \chi \frac{\partial^2 K_2(\xi,p)}{\partial\xi^2} \Theta_2(\xi,\tau)\rho_2 \, d\xi = -p^2 \mathcal{T}[\Theta(\xi,\tau)]$$

and hence eliminates integrals.

We are still left with the task of determining $\mathcal{N}_1(p,\tau)$ and $\mathcal{N}_2(p,\tau)$. From the condition $\Theta(\xi,\tau) \in \Gamma$, for $\xi = -1$ we have

$$\left(\frac{\partial\Theta_1(\xi,\tau)}{\partial\xi} K_1(\xi,p) - \Theta_1(\xi,\tau) \frac{\partial K_1(\xi,p)}{\partial\xi} \right) \Bigg|_{\xi=-1} = \Theta_1(\xi,\tau) \left(\mathrm{Bi}\, K_1(\xi,p) - \frac{\partial K_1(\xi,p)}{\partial\xi} \right) \Bigg|_{\xi=-1}$$

which should be equal to zero under the assumption (16). Thereby, we eliminate the unknown function $\Theta_1(\xi,\tau)$. The same approach for $\xi = H$ allows elimination of $\Theta_2(\xi,\tau)$.

Furthermore, for $\xi = 0$ we obtain

$$\left(\frac{\partial\Theta_1(\xi,\tau)}{\partial\xi} K_1(\xi,p) - \Theta_1(\xi,\tau) \frac{\partial K_1(\xi,p)}{\partial\xi} \right) \Bigg|_{\xi=0} - \frac{1}{\Lambda} \left(\frac{\partial\Theta_2(\xi,\tau)}{\partial\xi} K_2(\xi,p) - \Theta_2(\xi,\tau) \frac{\partial K_2(\xi,p)}{\partial\xi} \right) \Bigg|_{\xi=0}$$

$$= \Theta_1(\xi,\tau) \left(\frac{\partial K_1(\xi,p)}{\partial\xi} - B(\tau)(K_2(\xi,p) - K_1(\xi,p)) \right) \Bigg|_{\xi=0}$$

$$+ \Theta_2(\xi,\tau) \left(\frac{1}{\Lambda} \frac{\partial K_2(\xi,p)}{\partial\xi} - B(\tau)(K_2(\xi,p) - K_1(\xi,p)) \right) \Bigg|_{\xi=0} + Q(\tau)\Xi(p,\tau).$$

Substitution of the obtained expressions into (18) leads to the satisfaction of the property (17), which finishes the proof. □

Thereby, according to the Theorem 1, the kernel of the integral transform (14), (15) is defined as a solution of the initial differential equations (6) and has the form

$$K_1(\xi,p) = M_1(p)\sin(p\xi) + N_1(p)\cos(p\xi)\,,$$
$$K_2(\xi,p) = M_2(p)\sin\left(\frac{p\xi}{\sqrt{\chi}}\right) + N_2(p)\cos\left(\frac{p\xi}{\sqrt{\chi}}\right),$$

where the coefficients $M_k(p)$ and $N_k(p)$, $k=1,2$, do not depend on ξ.

We apply the boundary conditions (16) to decrease the number of the coefficients by two, i.e.

$$K_1(\xi,p) = M_1(p)\left(\sin(p\xi) + \frac{m_1(p)}{n_1(p)}\cos(p\xi)\right),$$
$$K_2(\xi,p) = M_2(p)\left(\sin\left(\frac{p\xi}{\sqrt{\chi}}\right) + \frac{m_2(p)}{n_2(p)}\cos\left(\frac{p\xi}{\sqrt{\chi}}\right)\right)$$

with

$$m_1(p) = \mathrm{Bi}\sin p + p\cos p, \quad n_1(p) = \mathrm{Bi}\cos p - p\sin p,$$
$$m_2(p) = \frac{p}{\sqrt{\chi}}\cos\left(\frac{pH}{\sqrt{\chi}}\right) + \Lambda\Upsilon\,\mathrm{Bi}\sin\left(\frac{pH}{\sqrt{\chi}}\right),$$
$$n_2(p) = \frac{p}{\sqrt{\chi}}\sin\left(\frac{pH}{\sqrt{\chi}}\right) - \Lambda\Upsilon\,\mathrm{Bi}\cos\left(\frac{pH}{\sqrt{\chi}}\right).$$

The missing coefficients $M_1(p)$ and $M_2(p)$ can be found by use of the following equalities derived from the contact conditions (12):

$$\left.\frac{\partial K_1(\xi,p)}{\partial\xi}\right|_{\xi=0} = \frac{1}{\Lambda}\left.\frac{\partial K_2(\xi,p)}{\partial\xi}\right|_{\xi=0} = B(\tau)\big(K_2(\xi,p) - K_1(\xi,p)\big)\big|_{\xi=0}, \tag{19}$$

which reads, when transformed and written in the matrix form,

$$\begin{pmatrix} \Lambda\sqrt{\chi} & -1 \\ p + B(\tau)\dfrac{m_1(p)}{n_1(p)} & -B(\tau)\dfrac{m_2(p)}{n_2(p)} \end{pmatrix}\begin{pmatrix} M_1(p) \\ M_2(p)\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation has a non-trivial solution only in case of zero determinant, that is,

$$p + B(\tau)\left(\frac{m_1(p)}{n_1(p)} - \Lambda\sqrt{\chi}\frac{m_2(p)}{n_2(p)}\right) = 0. \tag{20}$$

The characteristic equation (20) has an infinite number of real solutions p_k, $k=1,2,\dots$, which are situated symmetrically with respect to the origin.

From (19) we first find

$$M_2(p) = \Lambda\sqrt{\chi}M_1(p)$$

and then determine the last missing coefficient, for instance, $M_1(p)$. From the condition

$$1 = \int\limits_{-1}^{H} [K(\xi,p)]^2\, \rho(\xi)\, d\xi = \int\limits_{-1}^{0} [K_1(\xi,p)]^2 \rho_1\, d\xi + \int\limits_{0}^{H} [K_2(\xi,p)]^2 \rho_2\, d\xi$$

we finally derive

$$\begin{aligned} M_1(p) = \frac{1}{4p}\Bigg[& 2p - \sin(2p) - 4(\sin(p))^2 \frac{m_1(p)}{n_1(p)} + \big(2p + \sin(2p)\big) \left(\frac{m_1(p)}{n_1(p)}\right)^2 \\ & + 2p\Lambda H - \Lambda\sqrt{\chi} \sin\left(\frac{2pH}{\sqrt{\chi}}\right) + 4\Lambda\sqrt{\chi} \left(\sin\left(\frac{pH}{\sqrt{\chi}}\right)\right)^2 \frac{m_2(p)}{n_2(p)} \\ & + \Lambda\left(2pH + \sqrt{\chi}\sin\left(\frac{2pH}{\sqrt{\chi}}\right)\right)\left(\frac{m_2(p)}{n_2(p)}\right)^2 \Bigg]. \end{aligned}$$

The obtained function system $K(\xi, p_k)$ forms an orthonormal system on the closed interval $[-1, H]$.

3.3. Application of the Integral Transform

Since the kernel $K(\xi, p)$ incorporates the time-dependent coefficient B, we face a certain difficulty to apply the integral transform (14), (15) to the problem (6)–(9) as we cannot swap the operator $\mathcal{T}$ of the integral transform and the time derivative:

$$\mathcal{T}\left[\frac{\partial \Theta(\xi,\tau)}{\partial \tau}\right] \neq \frac{\partial}{\partial \tau}\Big(\mathcal{T}[\Theta(\xi,\tau)]\Big).$$

To overcome this difficulty we use the kernel splitting procedure proposed in [22] for multiple domains. Let us introduce the notation

$$\Omega_1(p,\tau) = \int\limits_{-1}^{0} \Theta_1(\xi,\tau)\,[\sin(p\xi) + i\cos(p\xi)]\; d\xi,$$

$$\Omega_2(p,\tau) = \frac{1}{\chi}\int\limits_{0}^{H} \Theta_2(\xi,\tau)\left[\sin\left(\frac{p\xi}{\sqrt{\chi}}\right) + i\cos\left(\frac{p\xi}{\sqrt{\chi}}\right)\right]\; d\xi,$$

where i is the imaginary unit, and

$$\omega_1(p) = M_1(p)\left(1 - i\frac{m_1(p)}{n_1(p)}\right),$$

$$\omega_2(p) = \lambda\sqrt{\chi}M_1(p)\left(1 - i\frac{m_2(p)}{n_2(p)}\right),$$

which allows representing the image $U(p,\tau)$ in the form

$$U(p,\tau) = \mathrm{Re}\Big\{\omega_1(p)\Omega_1(p,\tau) + \omega_2(p)\Omega_2(p,\tau)\Big\}. \tag{21}$$

We can propose, yet without proof,

Lemma 1. *For the image* (21) *the following equations hold:*

$$
\begin{aligned}
&\mathcal{T}\left[\frac{\partial\Theta(\xi,\tau)}{\partial\tau}\right] = \mathrm{Re}\left\{\omega_1(p)\frac{d\Omega_1(p,\tau)}{d\tau} + \omega_2(p)\frac{d\Omega_2(p,\tau)}{d\tau}\right\},\\
&\mathcal{T}\left[\mathcal{D}[\Theta(\xi,\tau)]\right] = -p^2\,\mathrm{Re}\{\omega_1(p)\Omega_1(p,\tau) + \omega_2(p)\Omega_2(p,\tau)\} + Q(\tau)\Xi(p,\tau).
\end{aligned}
\tag{22}
$$

By applying the direct integral transform (14) to the problem (6)–(9) and taking (21), (22) into account, we formulate the initial-value problem

$$
\begin{aligned}
&\mathrm{Re}\left\{\sum_{k=1}^{2}\left(\omega_k(p)\frac{d\Omega_k(p,\tau)}{d\tau} + p^2\omega_k(p)\Omega_k(p,\tau)\right)\right\} = Q(\tau)\Xi(p,\tau),\\
&\Omega_k(p,\tau)\big|_{\tau=0} = 0,\ \ k=1,2,
\end{aligned}
\tag{23}
$$

considering p as a parameter and hence using full derivatives. While the functions $Q(\tau)$ and $\Xi(p,\tau)$ are real, and based on the properties of complex numbers, the problem (23) can be represented in the form

$$
\begin{aligned}
&\frac{d\Omega_k(p,\tau)}{d\tau} + p^2\Omega_k(p,\tau) = \frac{Q(\tau)\Xi(p,\tau)}{|\omega_1(p)|^2 + |\omega_2(p)|^2}\bar{\omega}_k(p),\\
&\Omega_k(p,\tau)\big|_{\tau=0} = 0,\ \ k=1,2,
\end{aligned}
\tag{24}
$$

where by $\bar{\omega}$ we denote the complex-conjugate function to ω.

The solution of (24) is found by using common methods for solving linear differential equations:

$$
\Omega_k(p,\tau) = \int\limits_0^\tau \frac{\bar{\omega}_k(p)}{|\omega_1(p)|^2 + |\omega_2(p)|^2} Q(t)\Xi(p,t)e^{-p^2(\tau-t)}\,dt,\ \ k=1,2.
$$

The substitution of the expression above into (21) and explicit indication of the dependence of p on the time variable yield

$$
\begin{aligned}
U(p(\tau),\tau) = \int\limits_0^\tau &\frac{\mathrm{Re}\left\{\bar{\omega}_1(p(t))\omega_1(p(\tau)) + \bar{\omega}_2(p(t))\omega_2(p(\tau))\right\}}{|\omega_1(p(t))|^2 + |\omega_2(p(t))|^2}\\
&\times Q(t)\Xi(p(t),t)e^{-(p(t))^2(\tau-t)}\,dt,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathrm{Re}\left\{\bar{\omega}_1(p(t))\omega_1(p(\tau))\right\} = M_1(p(t))M_1(p(\tau))\left(1 + \frac{m_1(p(t))m_1(p(\tau))}{n_1(p(t))n_1(p(\tau))}\right),\\
&\mathrm{Re}\left\{\bar{\omega}_2(p(t))\omega_2(p(\tau))\right\} = \Lambda^2\chi M_1(p(t))M_1(p(\tau))\left(1 + \frac{m_2(p(t))m_2(p(\tau))}{n_2(p(t))n_2(p(\tau))}\right),\\
&|\omega_1(p(t))|^2 = (M_1(p(t)))^2\left[1 + \left(\frac{m_1(p(t))}{n_1(p(t))}\right)^2\right],\\
&|\omega_2(p(t))|^2 = \Lambda^2\chi\,(M_1(p(t)))^2\left[1 + \left(\frac{m_2(p(t))}{n_2(p(t))}\right)^2\right].
\end{aligned}
$$

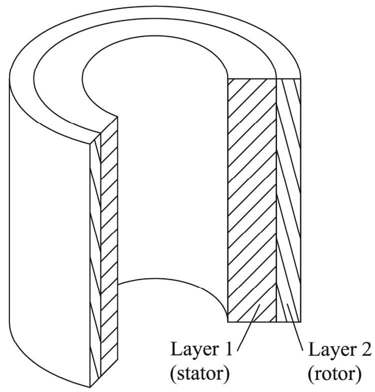

Figure 2: Sliding of hollow cylinders

We note here that $p(t)$ and $p(\tau)$ are calculated at the time moments t and τ, respectively.

Finally, by applying the inverse integral transform (15), we derive the sought solution in the form

$$\begin{bmatrix} \Theta_1(\xi,\tau) \\ \Theta_2(\xi,\tau) \end{bmatrix} = \sum_{k=1}^{\infty} \begin{bmatrix} K_1(\xi, p_k(\tau)) \\ K_2(\xi, p_k(\tau)) \end{bmatrix} U(p_k(\tau), \tau), \tag{25}$$

where $p_k(\tau)$ are positive roots of the equation (20).

4. Validation of the Solution

An analysis of the solution (25) shows that it coincides with the known analytical solutions [10, 11, 12, 13, 15, 16] in particular cases. As an example, we consider a problem of sliding of hollow cylinders as depicted in Fig. 2. The stator cylinder is aluminum, while the rotor cylinder is steel, i.e. $\Lambda = 9.5$, $\chi = 0.17$. We assume that the cylinders are long and thin-walled, which allows representing the given problem as a problem of sliding of two layers. The values of the parameters are chosen as follows: $H = 0.5$, $\alpha = 0.5$, $B = 0.0082$, $Q = 0.038$, $\mathrm{Bi} = 0.0011$, $\Upsilon = 3.1$. Figures 3 and 4 illustrate the evolution of the non-stationary contact temperatures ($\xi = 0$) and the distributions of the stationary temperatures in the layers ($\tau \to \infty$), respectively. The solution of Eq. (25) and the solution [11] for the contact conditions (3) are provided.

To solve the problem (6)–(9) numerically, an algorithm based on the finite-difference method with an implicit scheme is applied. A comparison of the numerical solution and that obtained by (25) for different combinations of the parameters confirms the validity of (25). For instance, we consider a problem

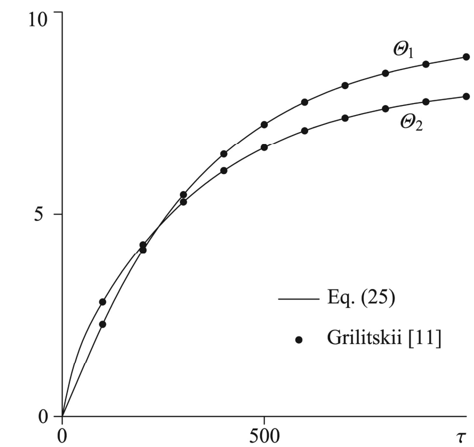

Figure 3: Non-stationary contact temperatures

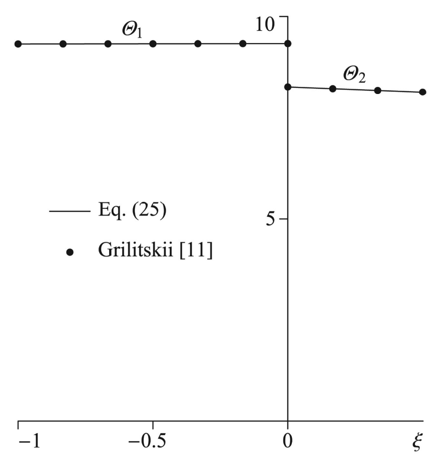

Figure 4: Distributions of the stationary temperatures

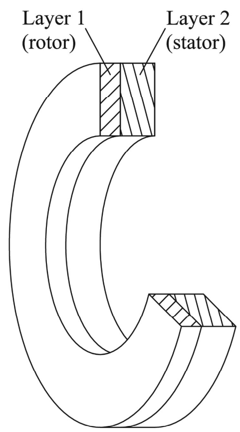

Figure 5: Sliding of discs

of sliding of two thin discs, as shown in Fig. 5. The rotor disc is polymer, whereas the stator disc is steel, i.e. $\Lambda = 0.032$, $\chi = 13$. The stator disc is 1.5 times thicker than the rotor disc, leading to $H = 1.5$. The sliding duration is $\tau_0 = 0.038$. The dimensionless specific power Q of heat generation decreases linearly from the initial value $Q_0 = 780$ to zero:

$$Q(\tau) = Q_0 \left(1 - \frac{\tau}{\tau_0}\right). \tag{26}$$

The heat-generation coefficient α and contact heat transfer coefficient B change linearly with time:

$$\alpha(\tau) = \alpha_0 \left(1 + k_\alpha \frac{\tau}{\tau_0}\right), \tag{27}$$

$$B(\tau) = B_0 \left(1 + k_B \frac{\tau}{\tau_0}\right), \tag{28}$$

where $\alpha_0 = 0.26$ and $B_0 = 5.9$ are the initial values, while $k_\alpha = 0.1$ and $k_B = -0.4$ are the rates of change of α and B, respectively. The convective heat transfer parameters are set as $\mathrm{Bi} = 0.35$ and $\Upsilon = 0.67$. This problem describes the heating of the friction discs during braking when α increases by 10% and B decreases by 40%. The evolution of the contact temperatures ($\xi = 0$) is presented in Fig. 6. Fig. 7 shows the temperature distributions at the end of braking ($\tau = \tau_0$).

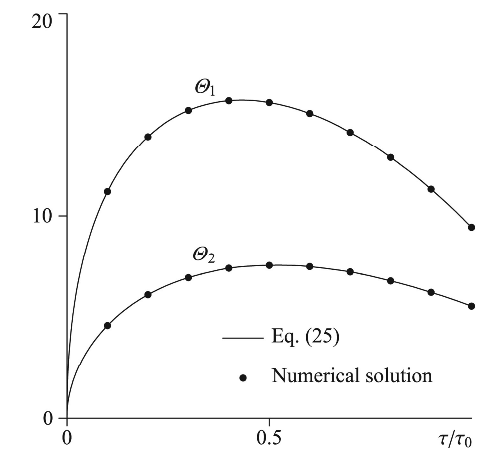

Figure 6: Evolution of the contact temperatures

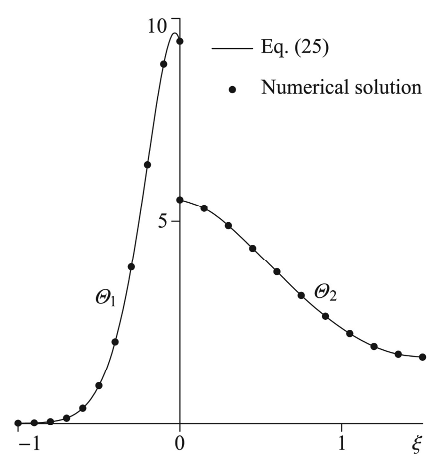

Figure 7: Temperature distributions at the end of braking

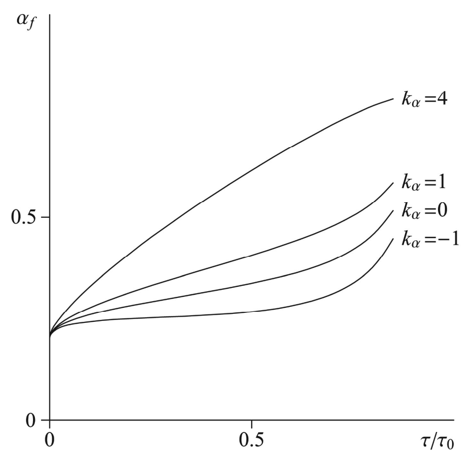

Figure 8: Heat partition dependent on the heat-generation coefficient α

5. Analysis of the Heat Partition

The dependence on time of the coefficients α and B inevitably affects the heat partition

$$\alpha_f(\tau) = \alpha(\tau) - \frac{B(\tau)}{Q(\tau)} \left(\Theta_1(\xi,\tau) - \Theta_2(\xi,\tau)\right)\big|_{\xi=0}.$$

We consider a friction pair similar to that shown in Fig. 5. The discs are assumed to have the same sizes and properties, i.e. $H = \Lambda = \chi = \Upsilon = 1$. During a single braking, Q decreases due to (26) with $Q_0 = 1$ and $\tau_0 = 0.1$. The convective heat transfer at the free surfaces of the discs is neglected, meaning that $\mathrm{Bi} = 0$. The friction surfaces of the discs have different roughness characteristics, which results in an asymmetric heat generation, so that α changes due to (27) with $\alpha_0 = 0.2$. The coefficient B changes according to (28) with $B_0 = 1$. By using (25) we conduct simulations of α_f at various values k_α and k_B. Fig. 8 shows the results for $k_B = 0$ and k_α varying from -1 (α decreases to zero) to 4 (α increases to 1). Similarly, Fig. 9 presents the results for $k_\alpha = 0$ and k_B varying from -1 (B decreases to zero) to 4 (B increases by 5 times). These simulations exhibit an essential influence of the time-dependent α and B on the heat partition, which emphasizes the practical value of the solution (25) obtained in this study.

6. Conclusions

The initial-boundary-value problem (6)–(9) of non-stationary heat conduction in two sliding layers with generalized thermal contact conditions at their interface is formulated allowing for variation of the heat-generation coefficient

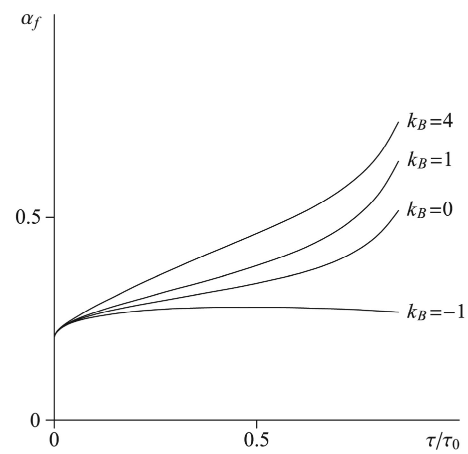

Figure 9: Heat partition dependent on the contact heat transfer coefficient B

and contact heat transfer coefficient with time. The integral transform (14), (15) is developed to eliminate the differentiation with respect to the spatial variable in the heat conduction equation and to take the contact conditions inside the double domain with the time-dependent contact heat transfer coefficient into account. On the basis of the integral transform, the analytical solution (25) of the problem is derived. The validity of the solution is confirmed by comparisons with the known analytical expressions as well as numerical results. The heat partition is shown to be sensitive to time-dependent heat-generation coefficient and contact heat transfer coefficient. The solution is applicable for simulation of temperature fields in sliding components of tribosystems.

7. References

[1] A.A. Yevtushenko, M. Kuciej, One-dimensional thermal problem of friction during braking: The history of development and actual state, International Journal of Heat and Mass Transfer 55 (2012) 4148–4153.

[2] H. Blok, Theoretical study of temperature rise at surfaces of actual contact under oiliness lubricating conditions, General Discussion on Lubrication, Institution of Mechanical Engineers, London, 1937, v. 2, pp. 222–235.

[3] F.F. Ling, A quasi-iterative method for computing interface temperature distributions, Zeitschrift für angewandte Mathematik und Physik 10 (5) (1959) 461–474.

[4] Ya.S. Podstrigach, The temperature field in a system of rigid bodies coupled by thin interface, Journal of Engineering Physics 6 (10) (1963) 129–136 (in Russian).

[5] J.R. Barber, The conduction of heat from sliding solids, International Journal of Heat and Mass Transfer 13 (5) (1970) 857–869.

[6] G.A. Berry, J.R. Barber, The division of frictional heat — A guide to the nature of sliding contact, Journal of Tribology 106 (1984) 405–415.

[7] B.V. Protasov, Energy proportions in a tribosystem and prognostication of its service durability, Saratov University Publisher, Saratov, 1979, pp. 32–73 (in Russian).

[8] M.A. Lambert, L.S. Fletcher, Review of models for thermal contact conductance of metals, Journal of Thermophysics and Heat Transfer 11 (2) (1997) 129–140.

[9] P. Chantrenne, M. Raynaud, A microscopic thermal model for dry sliding contact, International Journal of Heat and Mass Transfer 40 (5) (1997) 1083–1094.

[10] H.S. Carslaw, J.C. Jaeger, Conduction of heat in solids, second ed., Oxford University Press, London, 1959, pp.75–77, pp.87–89.

[11] D.V. Grilitskii, Thermoelastic contact problems in tribology, Institute of Topics and Methods of Teaching, Kyiv, 1996, pp. 70–92 (in Ukrainian).

[12] P.P. Krasnyuk, Quasistatic contact interaction of two layers with heat generation caused by friction, Material Science 35 (2) (1999) 180–192.

[13] A.A. Yevtushenko, M. Kuciej, Influence of the convective cooling and the thermal resistance on the temperature of the pad/disc tribosystem, International Communications in Heat and Mass Transfer 37 (4) (2010) 337–342.

[14] A.A. Yevtushenko, M. Kuciej, The thermal problem of friction for two plane-parallel strips, Numerical Heat Transfer Part A 61 (2012) 764–785.

[15] N.S. Belyakov, A.P. Nosko, Mathematical simulation of thermal friction processes under conditions of nonideal contact, High Temperature 47 (1) (2009) 123–130.

[16] N.S. Belyakov, A.P. Nosko, Thermoelastic problem of friction of plane-parallel layers with allowance for nonstationarity of thermal processes, Journal of Friction and Wear 31 (5) (2010) 317–325.

[17] O. Nosko, Partition of friction heat between sliding semi spaces due to adhesion-deformational heat generation, International Journal of Heat and Mass Transfer 64 (2013) 1189–1195.

[18] N.S. Belyakov, Singular integral transforms for heat transfer problems in semi-infinite solids with coatings, J. Heat Mass Transfer 3 (46) (2010) 355–346.

[19] R.M. Cotta, Integral Transforms in Computational Heat and Fluid Flow, CRC Press, Boca Raton, 1993.

[20] I.K. Volkov, A.N. Kanatnikov, Integral transforms and operational calculus, Izdatelstvo MGTU im. N. E. Baumana, Moscow, 2002 (in Russian).

[21] M.Ya. Antimirov, A.A. Kolyshkin, R. Vaillancourt, Applied integral transforms, AMS, 1993.

[22] A.V. Attetkov, N.S. Belyakov, I.K. Volkov, Temperature field of the infinite solid body incorporating cylindrical channel with multi-layer coating, under non-stationary heat exchange conditions, Vestnik MSTU n.a. N.E. Bauman 3 (2006) 37–50 (in Russian).