

Distortion in the group of circle homeomorphisms

JULIUSZ BANECKI† and TOMASZ SZAREK[✉]‡‡

† *Institute of Mathematics Polish Academy of Sciences, Abrahamia 18,
81-967 Sopot, Poland*

(*e-mail: juliusz.banecki@autonomik.pl, tszarek@impan.pl*)

‡ *Faculty of Physics and Applied Mathematics, Gdańsk University of Technology,
ul. Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland*

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Abstract. Let G be the group $\text{PAff}_+(\mathbb{R}/\mathbb{Z})$ of piecewise affine circle homeomorphisms or the group $\text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$ of smooth circle diffeomorphisms. A constructive proof that all irrational rotations are distorted in G is given.

Key words: homeomorphisms, distortion, rotation

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1. Introduction

Let G be a group with some finite generating set \mathcal{G} . We define the metric $d_{\mathcal{G}}$ on G by taking $d_{\mathcal{G}}(g_1, g_2)$ to be the infimum over all $k \geq 0$ such that there exist $f_1, \dots, f_k \in \mathcal{G}$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$ satisfying $g_2 = f_1^{\epsilon_1} \cdots f_k^{\epsilon_k} g_1$.

Now let H be an arbitrary group. An element $f \in H$ is called *distorted* in H if there exists a finitely generated subgroup $G \subset H$ containing f such that

$$\lim_{n \rightarrow \infty} \frac{d_{\mathcal{G}}(f^n, \text{id})}{n} = 0$$

for some (and hence every) generating set \mathcal{G} . Since the limit always exists, it is enough to verify it for some subsequence. The notion of distortion comes from geometric group theory and was introduced by Gromov in [7].

The problem of the existence of distorted elements in some groups of homeomorphisms has been intensively studied for many years (see [2, 3–6, 8, 10, 11]). Substantial progress has been achieved for groups of diffeomorphisms of manifolds. In particular, Avila [1] proved that rotations with irrational rotation number are distorted in the group of smooth diffeomorphisms of the circle. In this note we give a constructive proof that all irrational rotations are distorted both in the group of piecewise affine circle homeomorphisms,

$\text{PAff}_+(\mathbb{R}/\mathbb{Z})$, and in the group of smooth circle diffeomorphisms, $\text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$. The result gives an answer to Question 11 in [9] (see also Question 2.5 in [5]). So far it has not even been known whether there exist distorted elements in $\text{PAff}_+(\mathbb{R}/\mathbb{Z})$. Now from [8] it follows that each distorted element is conjugate to a rotation.

From now on let G be either $\text{PAff}_+(\mathbb{R}/\mathbb{Z})$ or $\text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$. We say that $g \in G$ is *trivial on some set* if there exists a non-empty open set $I \subset [0, 1)$ such that $g(x) = x$ for $x \in I$. The set of all homeomorphisms in G which are trivial on some set will be denoted by G_{triv} . By T we denote the set of all rotations, and let T_α be the rotation with rotation number α .

This paper is devoted to the proof of the following theorem.

THEOREM. *All irrational rotations are distorted in G .*

2. *Proofs*

We first formulate two lemmas and deduce the theorem. The proofs of the lemmas will be given at the end of the paper.

LEMMA 1. *For any irrational rotation T_α and $g \in G_{\text{triv}} \cup T$ there exist a finite generating set $\mathcal{G}_g \subset G$ and a constant $C > 0$ such that*

$$d_{\mathcal{G}_g}(T_\alpha^n g T_\alpha^{-n}, \text{id}) \leq C \log n \quad \text{for all } n \geq 1.$$

LEMMA 2. *In G there exist $g_1, \dots, g_l \in G_{\text{triv}} \cup T$ and $k, k_1, \dots, k_l \in \mathbb{Z}$ with $k \neq k_1 + \dots + k_l$, such that for each sufficiently small $\beta > 0$ the element $x = T_\beta$ satisfies*

$$x^{k_1} g_1 x^{k_2} g_2 \dots x^{k_l} g_l = x^k. \tag{1}$$

Proof of the theorem. Fix an irrational rotation T_α . From Lemma 2 it follows that in G there exists an equation of the form (1) such that $x = T_\beta$, for all sufficiently small β , is its solution. Let $\mathcal{G} = \mathcal{G}_{g_1} \cup \dots \cup \mathcal{G}_{g_l}$, where $\mathcal{G}_{g_i}, i = 1, \dots, l$, are finite generating sets derived from Lemma 1 for T_α . We may rewrite equation (1) in the form

$$x^{k_1} g_1 x^{-k_1} x^{k_2+k_1} g_2 x^{-k_2-k_1} \dots x^{k_1+\dots+k_l} g_l x^{-k_1-\dots-k_l} = x^{k-k_1-\dots-k_l}. \tag{2}$$

Let β_0 be a positive constant such that $x = T_\beta$ for $\beta \in (0, \beta_0)$ satisfies (2). Set $m := k - k_1 - \dots - k_l$, and let (n_i) be an increasing sequence of integers such that $n_i \alpha \in (0, \beta_0) \pmod{1}$. From Lemma 1 it follows that

$$d_{\mathcal{G}}(T_\alpha^{n_i(k_1+\dots+k_j)} g_j T_\alpha^{-n_i(k_1+\dots+k_j)}, \text{id}) \leq C_j \log n_i \quad \text{for all } i \geq 1 \text{ and } j = 1, \dots, l.$$

Since $x = T_{n_i \alpha}$ satisfies (2), we obtain

$$d_{\mathcal{G}}(T_\alpha^{n_i m}, \text{id}) \leq \sum_{j=1}^l C_j \log n_i := C \log n_i \quad \text{for all } i \geq 1.$$

Hence

$$\lim_{n \rightarrow \infty} \frac{d_{\mathcal{G}}(T_\alpha^n, \text{id})}{n} = \lim_{i \rightarrow \infty} \frac{d_{\mathcal{G}}(T_\alpha^{n_i m}, \text{id})}{n_i m} \leq \frac{C}{m} \lim_{i \rightarrow \infty} \frac{\log n_i}{n_i} = 0$$

and the proof is complete. □

Proof of Lemma 1. The proof relies on the observation that for a given interval $I \subset (0, 1)$ there exists a finite generating set $\mathcal{G} \subset G$ such that for any $n \geq 1$ there exists a homeomorphism h_n with $d_{\mathcal{G}}(h_n, \text{id}) \leq C \log n$ for some constant $C > 0$ independent of n , and $h_n(x) = T_{\alpha}^n(x)$ for $x \notin I$. Without loss of generality we may assume that $I = (a, 1)$. Let $m \geq 1$ be an integer such that $a + 2/m < 1$. Let $h \in G$ be any homeomorphism such that $h(x) = x/2$ for $x \in [0, a + 2/m)$, and let $r(x) = x + 1/m$.

We shall define h_n by induction. Set $h_0 = \text{id}$. If n is odd we put $h_n = T_{\alpha} h_{n-1}$. If n is even, we take $s_n := h_{n/2} h$ and observe that $s_n((0, a)) = (n\alpha/2, a/2 + n\alpha/2)$. Let $k \in \{1, \dots, m\}$ be such that $n\alpha/2 + k/m \in [0, 1/m) \pmod{1}$. Then $r^k s_n((0, a)) \subset (0, a/2 + 1/m)$. Therefore

$$h^{-1} r^k h_{n/2} h(x) = 2(x/2 + n\alpha/2 + k/m) = x + n\alpha + 2k/m = T_{\alpha}^n(x) + 2k/m \tag{3}$$

for $x \in (0, a)$. Put $h_n := r^{-2k} h^{-1} r^k h_{n/2} h$, and let $\mathcal{G} := \{T_{\alpha}, h, r\}$. Note that

$$d_{\mathcal{G}}(h_n, \text{id}) \leq 3m + 3 + d_{\mathcal{G}}(h_{\lfloor n/2 \rfloor}, \text{id}).$$

Thus we obtain $d_{\mathcal{G}}(h_n, \text{id}) \leq C \log n$. Finally, observe that for any $g \in G_{\text{triv}}$ such that $g(x) = \text{id}$ on I we have

$$T_{\alpha}^n g T_{\alpha}^{-n} = h_n g h_n^{-1}. \tag{4}$$

Indeed, from (3) and the definition of h_n and r it follows that $h_n(x) = T_{\alpha}^n(x)$ for $x \in (0, a)$, and

$$h_n((0, a)) = T_{\alpha}^n((0, a)) = (n\alpha, a + n\alpha). \tag{5}$$

Therefore, we have

$$h_n^{-1}(x) = T_{\alpha}^{-n}(x) \in (0, a) \quad \text{for } x \in (n\alpha, a + n\alpha).$$

Since $g(x) = x$ for $x \in (a, 1)$ and g is a homeomorphism, we have $g((0, a)) = (0, a)$.

To justify equality (4), first fix $x \in (n\alpha, a + n\alpha)$. Then we have

$$h_n^{-1}(x) = T_{\alpha}^{-n}(x) \in (0, a)$$

and

$$(g h_n^{-1})(x) = (g T_{\alpha}^{-n})(x) \in (0, a).$$

Consequently, we obtain

$$h_n g h_n^{-1}(x) = T_{\alpha}^n g T_{\alpha}^{-n}(x) \quad \text{for } x \in (n\alpha, a + n\alpha),$$

by the fact that $h_n(x) = T_{\alpha}^n(x)$ for $x \in (0, a)$. On the other hand, if $x \notin (n\alpha, a + n\alpha)$, from (5) and the fact that T_{α}^n and h_n are homeomorphisms, we obtain

$$T_{\alpha}^{-n}(x) \in (a, 1] \quad \text{and} \quad h_n^{-1}(x) \in (a, 1].$$

Since $g(x) = x$ for $x \in (a, 1]$, we have

$$(T_{\alpha}^n g T_{\alpha}^{-n})(x) = (T_{\alpha}^n T_{\alpha}^{-n})(x) = x$$

and

$$(h_n g h_n^{-1})(x) = (h_n h_n^{-1})(x) = x.$$

Thus equality (4) holds true.

Finally, we obtain

$$d_{\mathcal{G}}(T_{\alpha}^n g T_{\alpha}^{-n}, \text{id}) \leq C \log n.$$

In the case where g is a rotation the conclusion of the lemma is obvious. □

Proof of Lemma 2. Let $\beta \in (0, 10^{-3})$, and let $f_1 \in G_{\text{triv}}$ be arbitrary such that

$$f_1(x) = 0.4 + 2(x - 0.4) \text{ for } x \in [0.4, 0.6] \quad \text{and} \quad f_1(x) = x \text{ for } x \in [0.9, 1.1].$$

Set

$$H_1 = T_{2\beta}^{-1} f_1 T_{2\beta} f_1^{-1}.$$

It is obvious that

$$H_1(x) = x + 2\beta \text{ for } x \in [0.41, 0.79] \quad \text{and} \quad H_1(x) = x \text{ for } x \in [0.91, 1.09].$$

Define

$$H_2 = T_{1/2} H_1^{-1} T_{1/2} H_1,$$

and observe that

$$H_2(x) = x - 2\beta \quad \text{for } x \in [0.95, 1].$$

Simple computation gives

$$T_{1/2} H_2 T_{1/2} H_2 = \text{id}.$$

Set

$$H_3 = T_{2\beta} H_2.$$

Then we have

$$H_3(x) = x \quad \text{for } x \in [0.95, 1]$$

and

$$T_{2\beta+1/2} H_3 T_{-2\beta-1/2} H_3 = T_{4\beta}. \tag{6}$$

Take an arbitrary $f_2 \in G_{\text{triv}}$ satisfying

$$f_2(x) = 2x \quad \text{for } x \in [0, 0.49],$$

and define

$$H_4 = f_2^{-1} H_3 f_2.$$

It is easy to see that

$$H_4(x) = \begin{cases} H_3(2x)/2 & \text{for } x \in [0, 1/2), \\ x & \text{for } x \in [1/2, 1). \end{cases}$$

Let

$$H_5 = T_{1/2}H_4T_{1/2}H_4. \tag{7}$$

Observe that the graph of H_5 is built from two scaled copies of H_3 , that is,

$$H_5(x) = \begin{cases} H_3(2x)/2 & \text{for } x \in [0, 1/2), \\ H_3(2x - 1)/2 + 1/2 & \text{for } x \in [1/2, 1). \end{cases}$$

Therefore, by (6) and (7), we finally obtain

$$T_{\beta+1/4}H_5T_{-\beta-1/4}H_5 = T_{2\beta}. \tag{8}$$

Indeed, this is easy to see if we realize that (8) is simply equation (6) rewritten in the new coordinates $(x/2, y/2)$. Subsequently plugging H_5, H_4, H_3, H_2 and H_1 into formula (8), we have

$$\begin{aligned} & T_{\beta}T_{1/4}T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2}f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2f_1^{-1}f_2T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2} \\ & \cdot f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2f_1^{-1}f_2T_{\beta}^{-1}T_{-1/4}T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2}f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2 \\ & \cdot f_1^{-1}f_2T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2}f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2f_1^{-1}f_2 = T_{\beta}^2. \end{aligned}$$

Since $\beta \in (0, 10^{-3})$ was arbitrary, we obtain that each T_{β} sufficiently small satisfies equation (1) with the functions $g_1, \dots, g_l \in \{f_1, f_2, f_1^{-1}, f_2^{-1}, T_{1/2}, T_{-1/2}, T_{1/4}, T_{-1/4}\} \subset G_{\text{triv}} \cup T$ and $k_1, \dots, k_l \in \mathbb{Z}$. Obviously, some of the k_i are equal to 0 (k_2 , for instance) but $k_1 + \dots + k_l = 8$. Since $k = 2$, the proof of the lemma is complete. \square

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