


# Distortion in the group of circle homeomorphisms

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*Abstract.* Let  $G$  be the group  $\text{PAff}_+(\mathbb{R}/\mathbb{Z})$  of piecewise affine circle homeomorphisms or the group  $\text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$  of smooth circle diffeomorphisms. A constructive proof that all irrational rotations are distorted in  $G$  is given.

**Key words:** homeomorphisms, distortion, rotation

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## 1. Introduction

Let  $G$  be a group with some finite generating set  $\mathcal{G}$ . We define the metric  $d_{\mathcal{G}}$  on  $G$  by taking  $d_{\mathcal{G}}(g_1, g_2)$  to be the infimum over all  $k \geq 0$  such that there exist  $f_1, \dots, f_k \in \mathcal{G}$  and  $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$  satisfying  $g_2 = f_1^{\epsilon_1} \cdots f_k^{\epsilon_k} g_1$ .

Now let  $H$  be an arbitrary group. An element  $f \in H$  is called *distorted* in  $H$  if there exists a finitely generated subgroup  $G \subset H$  containing  $f$  such that

$$\lim_{n \rightarrow \infty} \frac{d_{\mathcal{G}}(f^n, \text{id})}{n} = 0$$

for some (and hence every) generating set  $\mathcal{G}$ . Since the limit always exists, it is enough to verify it for some subsequence. The notion of distortion comes from geometric group theory and was introduced by Gromov in [7].

The problem of the existence of distorted elements in some groups of homeomorphisms has been intensively studied for many years (see [2, 3–6, 8, 10, 11]). Substantial progress has been achieved for groups of diffeomorphisms of manifolds. In particular, Avila [1] proved that rotations with irrational rotation number are distorted in the group of smooth diffeomorphisms of the circle. In this note we give a constructive proof that all irrational rotations are distorted both in the group of piecewise affine circle homeomorphisms,

$\text{PAff}_+(\mathbb{R}/\mathbb{Z})$ , and in the group of smooth circle diffeomorphisms,  $\text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$ . The result gives an answer to Question 11 in [9] (see also Question 2.5 in [5]). So far it has not even been known whether there exist distorted elements in  $\text{PAff}_+(\mathbb{R}/\mathbb{Z})$ . Now from [8] it follows that each distorted element is conjugate to a rotation.

From now on let  $G$  be either  $\text{PAff}_+(\mathbb{R}/\mathbb{Z})$  or  $\text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$ . We say that  $g \in G$  is *trivial on some set* if there exists a non-empty open set  $I \subset [0, 1)$  such that  $g(x) = x$  for  $x \in I$ . The set of all homeomorphisms in  $G$  which are trivial on some set will be denoted by  $G_{\text{triv}}$ . By  $T$  we denote the set of all rotations, and let  $T_\alpha$  be the rotation with rotation number  $\alpha$ .

This paper is devoted to the proof of the following theorem.

**THEOREM.** *All irrational rotations are distorted in  $G$ .*

2. *Proofs*

We first formulate two lemmas and deduce the theorem. The proofs of the lemmas will be given at the end of the paper.

**LEMMA 1.** *For any irrational rotation  $T_\alpha$  and  $g \in G_{\text{triv}} \cup T$  there exist a finite generating set  $\mathcal{G}_g \subset G$  and a constant  $C > 0$  such that*

$$d_{\mathcal{G}_g}(T_\alpha^n g T_\alpha^{-n}, \text{id}) \leq C \log n \quad \text{for all } n \geq 1.$$

**LEMMA 2.** *In  $G$  there exist  $g_1, \dots, g_l \in G_{\text{triv}} \cup T$  and  $k, k_1, \dots, k_l \in \mathbb{Z}$  with  $k \neq k_1 + \dots + k_l$ , such that for each sufficiently small  $\beta > 0$  the element  $x = T_\beta$  satisfies*

$$x^{k_1} g_1 x^{k_2} g_2 \dots x^{k_l} g_l = x^k. \tag{1}$$

*Proof of the theorem.* Fix an irrational rotation  $T_\alpha$ . From Lemma 2 it follows that in  $G$  there exists an equation of the form (1) such that  $x = T_\beta$ , for all sufficiently small  $\beta$ , is its solution. Let  $\mathcal{G} = \mathcal{G}_{g_1} \cup \dots \cup \mathcal{G}_{g_l}$ , where  $\mathcal{G}_{g_i}, i = 1, \dots, l$ , are finite generating sets derived from Lemma 1 for  $T_\alpha$ . We may rewrite equation (1) in the form

$$x^{k_1} g_1 x^{-k_1} x^{k_2+k_1} g_2 x^{-k_2-k_1} \dots x^{k_1+\dots+k_l} g_l x^{-k_1-\dots-k_l} = x^{k-k_1-\dots-k_l}. \tag{2}$$

Let  $\beta_0$  be a positive constant such that  $x = T_\beta$  for  $\beta \in (0, \beta_0)$  satisfies (2). Set  $m := k - k_1 - \dots - k_l$ , and let  $(n_i)$  be an increasing sequence of integers such that  $n_i \alpha \in (0, \beta_0) \pmod{1}$ . From Lemma 1 it follows that

$$d_{\mathcal{G}}(T_\alpha^{n_i(k_1+\dots+k_j)} g_j T_\alpha^{-n_i(k_1+\dots+k_j)}, \text{id}) \leq C_j \log n_i \quad \text{for all } i \geq 1 \text{ and } j = 1, \dots, l.$$

Since  $x = T_{n_i \alpha}$  satisfies (2), we obtain

$$d_{\mathcal{G}}(T_\alpha^{n_i m}, \text{id}) \leq \sum_{j=1}^l C_j \log n_i := C \log n_i \quad \text{for all } i \geq 1.$$

Hence

$$\lim_{n \rightarrow \infty} \frac{d_{\mathcal{G}}(T_\alpha^n, \text{id})}{n} = \lim_{i \rightarrow \infty} \frac{d_{\mathcal{G}}(T_\alpha^{n_i m}, \text{id})}{n_i m} \leq \frac{C}{m} \lim_{i \rightarrow \infty} \frac{\log n_i}{n_i} = 0$$

and the proof is complete. □

*Proof of Lemma 1.* The proof relies on the observation that for a given interval  $I \subset (0, 1)$  there exists a finite generating set  $\mathcal{G} \subset G$  such that for any  $n \geq 1$  there exists a homeomorphism  $h_n$  with  $d_{\mathcal{G}}(h_n, \text{id}) \leq C \log n$  for some constant  $C > 0$  independent of  $n$ , and  $h_n(x) = T_{\alpha}^n(x)$  for  $x \notin I$ . Without loss of generality we may assume that  $I = (a, 1)$ . Let  $m \geq 1$  be an integer such that  $a + 2/m < 1$ . Let  $h \in G$  be any homeomorphism such that  $h(x) = x/2$  for  $x \in [0, a + 2/m)$ , and let  $r(x) = x + 1/m$ .

We shall define  $h_n$  by induction. Set  $h_0 = \text{id}$ . If  $n$  is odd we put  $h_n = T_{\alpha} h_{n-1}$ . If  $n$  is even, we take  $s_n := h_{n/2} h$  and observe that  $s_n((0, a)) = (n\alpha/2, a/2 + n\alpha/2)$ . Let  $k \in \{1, \dots, m\}$  be such that  $n\alpha/2 + k/m \in [0, 1/m) \pmod{1}$ . Then  $r^k s_n((0, a)) \subset (0, a/2 + 1/m)$ . Therefore

$$h^{-1} r^k h_{n/2} h(x) = 2(x/2 + n\alpha/2 + k/m) = x + n\alpha + 2k/m = T_{\alpha}^n(x) + 2k/m \tag{3}$$

for  $x \in (0, a)$ . Put  $h_n := r^{-2k} h^{-1} r^k h_{n/2} h$ , and let  $\mathcal{G} := \{T_{\alpha}, h, r\}$ . Note that

$$d_{\mathcal{G}}(h_n, \text{id}) \leq 3m + 3 + d_{\mathcal{G}}(h_{\lfloor n/2 \rfloor}, \text{id}).$$

Thus we obtain  $d_{\mathcal{G}}(h_n, \text{id}) \leq C \log n$ . Finally, observe that for any  $g \in G_{\text{triv}}$  such that  $g(x) = \text{id}$  on  $I$  we have

$$T_{\alpha}^n g T_{\alpha}^{-n} = h_n g h_n^{-1}. \tag{4}$$

Indeed, from (3) and the definition of  $h_n$  and  $r$  it follows that  $h_n(x) = T_{\alpha}^n(x)$  for  $x \in (0, a)$ , and

$$h_n((0, a)) = T_{\alpha}^n((0, a)) = (n\alpha, a + n\alpha). \tag{5}$$

Therefore, we have

$$h_n^{-1}(x) = T_{\alpha}^{-n}(x) \in (0, a) \quad \text{for } x \in (n\alpha, a + n\alpha).$$

Since  $g(x) = x$  for  $x \in (a, 1)$  and  $g$  is a homeomorphism, we have  $g((0, a)) = (0, a)$ .

To justify equality (4), first fix  $x \in (n\alpha, a + n\alpha)$ . Then we have

$$h_n^{-1}(x) = T_{\alpha}^{-n}(x) \in (0, a)$$

and

$$(g h_n^{-1})(x) = (g T_{\alpha}^{-n})(x) \in (0, a).$$

Consequently, we obtain

$$h_n g h_n^{-1}(x) = T_{\alpha}^n g T_{\alpha}^{-n}(x) \quad \text{for } x \in (n\alpha, a + n\alpha),$$

by the fact that  $h_n(x) = T_{\alpha}^n(x)$  for  $x \in (0, a)$ . On the other hand, if  $x \notin (n\alpha, a + n\alpha)$ , from (5) and the fact that  $T_{\alpha}^n$  and  $h_n$  are homeomorphisms, we obtain

$$T_{\alpha}^{-n}(x) \in (a, 1] \quad \text{and} \quad h_n^{-1}(x) \in (a, 1].$$

Since  $g(x) = x$  for  $x \in (a, 1]$ , we have

$$(T_{\alpha}^n g T_{\alpha}^{-n})(x) = (T_{\alpha}^n T_{\alpha}^{-n})(x) = x$$

and

$$(h_n g h_n^{-1})(x) = (h_n h_n^{-1})(x) = x.$$

Thus equality (4) holds true.

Finally, we obtain

$$d_{\mathcal{G}}(T_{\alpha}^n g T_{\alpha}^{-n}, \text{id}) \leq C \log n.$$

In the case where  $g$  is a rotation the conclusion of the lemma is obvious. □

*Proof of Lemma 2.* Let  $\beta \in (0, 10^{-3})$ , and let  $f_1 \in G_{\text{triv}}$  be arbitrary such that

$$f_1(x) = 0.4 + 2(x - 0.4) \text{ for } x \in [0.4, 0.6] \quad \text{and} \quad f_1(x) = x \text{ for } x \in [0.9, 1.1].$$

Set

$$H_1 = T_{2\beta}^{-1} f_1 T_{2\beta} f_1^{-1}.$$

It is obvious that

$$H_1(x) = x + 2\beta \text{ for } x \in [0.41, 0.79] \quad \text{and} \quad H_1(x) = x \text{ for } x \in [0.91, 1.09].$$

Define

$$H_2 = T_{1/2} H_1^{-1} T_{1/2} H_1,$$

and observe that

$$H_2(x) = x - 2\beta \quad \text{for } x \in [0.95, 1].$$

Simple computation gives

$$T_{1/2} H_2 T_{1/2} H_2 = \text{id}.$$

Set

$$H_3 = T_{2\beta} H_2.$$

Then we have

$$H_3(x) = x \quad \text{for } x \in [0.95, 1]$$

and

$$T_{2\beta+1/2} H_3 T_{-2\beta-1/2} H_3 = T_{4\beta}. \tag{6}$$

Take an arbitrary  $f_2 \in G_{\text{triv}}$  satisfying

$$f_2(x) = 2x \quad \text{for } x \in [0, 0.49],$$

and define

$$H_4 = f_2^{-1} H_3 f_2.$$

It is easy to see that

$$H_4(x) = \begin{cases} H_3(2x)/2 & \text{for } x \in [0, 1/2), \\ x & \text{for } x \in [1/2, 1). \end{cases}$$

Let

$$H_5 = T_{1/2}H_4T_{1/2}H_4. \tag{7}$$

Observe that the graph of  $H_5$  is built from two scaled copies of  $H_3$ , that is,

$$H_5(x) = \begin{cases} H_3(2x)/2 & \text{for } x \in [0, 1/2), \\ H_3(2x - 1)/2 + 1/2 & \text{for } x \in [1/2, 1). \end{cases}$$

Therefore, by (6) and (7), we finally obtain

$$T_{\beta+1/4}H_5T_{-\beta-1/4}H_5 = T_{2\beta}. \tag{8}$$

Indeed, this is easy to see if we realize that (8) is simply equation (6) rewritten in the new coordinates  $(x/2, y/2)$ . Subsequently plugging  $H_5, H_4, H_3, H_2$  and  $H_1$  into formula (8), we have

$$\begin{aligned} & T_{\beta}T_{1/4}T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2}f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2f_1^{-1}f_2T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2} \\ & \cdot f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2f_1^{-1}f_2T_{\beta}^{-1}T_{-1/4}T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2}f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2 \\ & \cdot f_1^{-1}f_2T_{1/2}f_2^{-1}T_{\beta}^2T_{1/2}f_1T_{\beta}^{-2}f_1^{-1}T_{\beta}^2T_{1/2}T_{\beta}^{-2}f_1T_{\beta}^2f_1^{-1}f_2 = T_{\beta}^2. \end{aligned}$$

Since  $\beta \in (0, 10^{-3})$  was arbitrary, we obtain that each  $T_{\beta}$  sufficiently small satisfies equation (1) with the functions  $g_1, \dots, g_l \in \{f_1, f_2, f_1^{-1}, f_2^{-1}, T_{1/2}, T_{-1/2}, T_{1/4}, T_{-1/4}\} \subset G_{\text{triv}} \cup T$  and  $k_1, \dots, k_l \in \mathbb{Z}$ . Obviously, some of the  $k_i$  are equal to 0 ( $k_2$ , for instance) but  $k_1 + \dots + k_l = 8$ . Since  $k = 2$ , the proof of the lemma is complete.  $\square$

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