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VALUATION OF EMBEDDED OPTIONS IN NON-MARKETABLE CALLABLE BONDS: A NEW NUMERICAL APPROACH

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Abstract. The issue of how to price options embedded in callable bonds has attracted a lot of interest over the years. The usual bond valuation methods rely on yield curves, risk premium, and other parameters to estimate interest rates used in discounted cash flow calculations. The option to retire the bond is, however, neglected in the standard pricing models, causing a systematic overvaluation of callable bonds. In the event of a decline in interest rates, investors are exposed to the risk of a lower return on investment than indicated by the yield to maturity. We propose a novel approach to valuing the risk that the issuer will use the right to buy back the bond at a specific call price. While prior models are focused on valuing marketable callable bonds, we deliver a unique approach to valuing bonds with an embedded European option (or a multiple option) that are traded solely through private transactions. These can typically be characterized by the lack of historical records on transaction prices. The modular character of calculation we propose allows us to take into account additional information, such as probable behaviour of the issuer, available opportunities for achieving alternative earnings or different estimates in terms of interest rate development.

Keywords: non-marketable callable bonds, options embedded in bonds, embedded options valuation, option pricing, bond pricing, loss of interest income, interest rate volatility, probability of call.

JEL Classification: G12, G13.

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Introduction

In recent decades, corporate financing has experienced considerable changes. Many innovations on how enterprises can raise finance have emerged, complementing the traditional finance models based on equity, debt, and convertibles (Onuferová et al., 2020; Bukalska, 2020; Kliestik et al., 2020; Zinecker et al., 2021a, 2021b; Meluzín et al., 2021; Priem, 2021; Schinckus et al., 2021; Valaskova et al., 2021a, 2021b; Setianto et al., 2022). Callable bonds represent one type of financial innovation associated with the bond market. This type of security, also known as redeemable bonds, or bonds with an embedded option, allows the issuing company to retire the bond prior to its maturity under certain conditions (Xie, 2009; Blume & Keim, 1988; Boyce & Kalotay, 1979). Xue (2011) and Ben-Ameur et al. (2007) note that these options cannot be traded separately in the open market, which differentiates these from call (and put) options on stocks. The coupon rates for a callable bond are higher, making this financial tool more expensive for the issuer and thus more attractive for investors. According to Xie (2009), a rationally behaving company will call a bond if the coupon payments exceed corporate earnings due to a lower return from the market.

Banko and Zhou (2010) emphasize substantial changes the U.S. callable bond market has experienced since the 1980s. First, the share of public callable bonds has dropped from 80% to about 30%. Second, the yield on the ten-year Treasury has decreased dramatically because of the low interest rate environment following the 2008-09 financial crisis. Third, the callable bond market has been dominated by below investment-grade securities that are traded solely through private transactions. Hence, the academic research is increasingly focused on optimal financial decision-making related to non-marketable callable bonds.

Faced with the increasing importance of below investment-grade callable bonds, a growing body of academic literature examines the issue how to value a callable bond (Goldberg et al., 2021; François & Pardo, 2015; Lim et al., 2012; Jarrow et al., 2010; Banko & Zhou, 2010; Xie, 2009; Ben-Ameur et al., 2007; D'Halluin et al., 2001; Duffie & Singleton, 1999; Büttler & Waldvogel, 1996; Büttler, 1995; Ho et al., 1992; Hull & White, 1990, 1993; Katolay et al., 1993; Brennan & Schwartz, 1977). The seminal work by Dai and Singleton (2000) can be recalled here to point out that the vast majority of prior studies have dealt with non-callable bonds, although "the majority of dollar-denominated corporate bonds are callable" (Jarrow et al., 2010). Unlike plain coupon bonds, a callable bond has an embedded option making maturity and cash flows uncertain, which implies that this security needs to be approached differently (Ho et al., 1992). According to Jarrow et al. (2010), the difficulties in pricing callable bonds represent the main reason for the lack of empirical research as there exists no exact valuation method "suitable for a large-scale empirical analysis". Xie (2009) suggests that early studies on bond valuation share a common shortcoming assuming the market interest rate as a constant (e.g., Merton, 1974; Black & Cox, 1976; Geske, 1977). Further works attempt to address the interest rate effect on the value of bonds (e.g., Hull & White, 1990, 1993). Recently, many reduced-form approaches for valuing a callable corporate bond have been introduced to respond to the limitations of the traditionally applied structural approach for valuing callable bonds assuming "a stochastic process for firm value" and determining "the optimal call policy by minimizing the present value of liabilities" (Jarrow, 2010; Duffie & Singleton, 1999). The first limitation consists in determining the optimal call policy as this requires information on the company's value process and the company's dynamic liability structure, which remain mostly hidden from market participants. Second, the impact of market frictions having an essential impact on the optimal call policy can be incorporated only with difficulties. The last limitation can be seen in the fact that the valuation is a challenge from the computational point of view (Jarrow et al., 2010).

Although options embedded in non-marketable callable bonds are an integral part of investment portfolios and investors need to monitor them constantly and report their performance, there is, to our best knowledge, a lack of methods which are suitable for pricing this type of security. In this article, we are focused on the financial valuation of the risk that a bond will be exercised at the request of the issuer. We contribute to the literature two-fold. First, we deliver a unique valuation approach on callable bonds without a secondary market, i.e., no historical records on transaction prices are available. Second, there has been neither theoretical nor empirical work so far analysing the specific features of non-marketable callable bonds based on the following assumptions: 1) Transaction cost on the side of the issuer making the value of the call option different from the investors' and issuers' perspective; 2) Existence of multiple terms, when the call option might be exercised; 3) The probability that the call option will be exercised is determined by the random walk theory. In this paper, we assume the European-type bonds with embedded option the typical feature of which is that these have only one possible call date.

The remainder of the paper is structured as follows. In the first part, the determinants of callable bonds are analysed to shed light on the problem of their pricing. Next, the paper presents the methodological framework including the model design in particular. Third, a numerical example obtained from our method is presented. Finally, we provide a discussion of the research results and propose a new agenda for the upcoming research.

1. Literature review

When examining the determinants of callable bonds, four main hypotheses have been articulated in academic research: 1) The interest rate hypothesis; 2) The information asymmetry hypothesis; 3) The risk-shifting hypothesis; and 4) The underinvestment hypothesis (Banko & Zhou, 2010).

The interest rate hypothesis is assumed to be the most fundamental reason behind the decision to issue a callable bond. Xue (2011) argues that the issuer will redeem the bond before its original maturity (at the call date) if the market interest rate has decreased, causing an increase in the price of the bond. Under such conditions, the issuer may choose a cheaper source to refinance its debt (bond) and calls the bonds it originally issued. As this argument shows, bond issuers will choose the callable bond if avoiding the risk of interest rate lowering is in the forefront of their interest. Blume and Keim (1988) note that the financial strategy based on repeatedly calling and reissuing new callable bonds is like "marking to market" changes in interest rate. Xie (2009) interprets the interest rate hypothesis from the bond issuer´s perspective somewhat differently. He outlines the situation in which the bond issuer will invest the raised capital in a variety of assets. The overall return from all possible market investments has to exceed the bond coupons the investor has to pay. The optimal financial decision at any given time depends on how much return can be earned in the market. Xie (2009) concludes that the bonds should not be called unless the overall investment return rate remains at a very low level for a certain period of time. Thus, the issuer monitors throughout the whole life of the bond the market investment return and decides if it is efficient to use the call option immediately. Banko and Zhou (2010) analysed the U.S. callable bond market using data from the period between 1980 and 2003 and show that interest rate hedging plays a role in issuing investment grade bonds and when interest rates are high. Issuers of below-investment grade bonds remain rather unaffected by changes in interest rate levels in their financial decision-making.

In addition to the interest rate hypothesis, the economic theory attempts to explain callable bond issues based on agency conflicts between shareholders and bondholders. These include three theoretical explanations, the information asymmetry hypothesis, the risk-shifting hypothesis, and the underinvestment hypothesis (Banko & Zhou, 2010).

The underinvestment problem will occur in "a firm with risky debt outstanding" where managers act in its shareholders' interest (Myers, 1977). Their decision-making will vary from decision rules in not leveraged firms or firms issuing risk-free debt (Kramoliš & Dobeš, 2020; Chang & Wu, 2021). As a result, risky debt issuers might "pass up" positive net present value projects because managers assume that bondholders will benefit more than shareholders will. Consequently, the firms with high default risk will suffer from the lack of positive NPV projects (Panova, 2020; Kaczmarek et al., 2021; Valaskova et al., 2021a, 2021b; Karas & Režňáková, 2021). Bodie and Taggart (1978) and Barnea et al. (1980) argue that the underinvestment problem can be resolved with callable bonds. Following the underinvestment problem, Banko and Zhou (2010) suggest that callable bonds are predominantly issued by firms with more growth opportunities (or positive NPV investments) and by firms facing higher default risk.

The risk-shifting hypothesis also introduced by Barnea et al. (1980) argues that issuing callable bonds represents a mechanism to mitigate or even eliminate the shareholders' incentive from increasing default risk at the expense of the bondholders. When shareholders conduct activities increasing the default risk, the value of both the bond and call provision falls. Vice versa, if the default risk lowers the value of the embedded call provision, the price of the bond increases. Banko and Zhou (2010) imply that firms with higher default risk are more likely to issue callable bonds because of "stronger incentives to engage in risk-shifting activities". Furthermore, issuers of bonds with an embedded option "have more flexible asset structures and higher free cash flows" that might be flexibly diverted from conventional assets to riskier investment opportunities.

Barnea et al. (1980) and Robbins and Schatzberg (1988) document that call provisions might support issuers in resolving the information asymmetry problem. This occurs when confounding announcements in regard to intra-firm uncertainty are spread and when it is difficult to reveal or signal positive information towards external investors. Therefore, the information asymmetry hypothesis implies that callable bond issuers are non-public and non-transparent firms and thus subject to a higher default risk. Banko and Zhou (2010) conclude that call provisions might represent an effective tool mitigating the information asymmetry and lowering the cost of capital when good news are published.

The pricing of bonds with an embedded option has been attracting a lot of research interest recently (Goldberg et al., 2021; François & Pardo, 2015; Lim et al., 2012; Jarrow et al., 2010; Banko & Zhou, 2010). The researchers agree that the task is challenging due to discontinuities in the "bond value" or "its derivative at call and/or notice dates" (for details, see D'Halluin et al., 2001). According to Ho et al. (1992), the callable bond can be interpreted as an "interest-rate-contingent claim". The value of the call option is determined by "the complex interrelationships among a number of variables" (Ho et al., 1992). Traditionally, the value of any bond can be expressed as the present value of its future cash flows (e.g., Katolay et al., 1993). This is, however, not so easily determined in the real world. The reasons are twofold: the interest rate uncertainty and investment risks to which the bondholder is exposed. This means that, in addition to internal factors, external market factors have a substantial impact on the call value. These include the volatility of interest rates and the shape or slope of the yield curve (Ho et al., 1992). The cash flows are difficult to predict, because the exercise of options embedded in a bond is related tightly to the interest rate development. This means that the investor needs to accept the possibility that the cash flow will be altered by the issuer decision to call the bond.

For a given term structure, the more uncertainty is reflected in the volatility of interest rates, the more likely it is that interest rates will fall to a level low enough to make the bond calling effective from the issuer perspective and the more valuable will be the call. Furthermore, the constellations in which the short-term rates exceed long-term rates also support the value of the call. This implies that "the value of the call option is a decreasing function of the slope of the term structure" (Ho et al., 1992). Several methods were developed to calculate these relationships and thus quantify "a fair value for the call" (Ho et al., 1992).

Ho et al. (1992) introduced the following definition of valuing a callable bond (P_{cb}): This is the difference between the value of the underlying noncallable bond (P_{ncb}) and the value of the call option (P_c). The investor will sell the bond, purchase the noncallable bond, and short sell the option if the price of the callable bond exceeds $P_{ncb} - P_c$. The possibility that the investor will achieve the arbitrate profit, P_{cb} cannot exceed the difference between $P_{ncb} - P_c$. Subsequently, the only equilibrium condition for the prices of the three securities can be derived as follows (Ho et al., 1992):

$$P_{cb} = P_{ncb} - P_c. \tag{1}$$

This also means (Katolay et al., 1993):

$$P_{ncb} = P_{cb} + P_c. \tag{2}$$

Early approaches to determine a callable bond price were based on the isolation of the implied value of an underlying option-free bond by adding an estimate of the embedded option's value to the bond's market price. The option-pricing theory was used here to estimate the value of the call option (Katolay et al., 1993). Ben-Ameur et al. (2007) summarize the major streams of academic research in the field and point out the main contributions provided so far by: 1) Brennan and Schwartz (1977) who "used a finite-difference approach in time-homogenous diffusion models"; 2) Hull and White (1990), who proposed "trinomial trees in the context of generalized versions with time-dependent parameters for the state process"; 3) Büttler (1995), who proved that "finite-difference methods under the Vasicek (1977) model resulted in a poor numerical accuracy" because of "the presence of slowly decaying oscillations in the solution after each coupon/call date", and 4) Büttler and Waldvogel (1996) who suggested an alternative approach under the Vasicek (1977) and Cox-Ingersoll-Ross (CIR) (Cox et al., 1985a, 1985b) models based on the "explicit form of the Green's function in these models".

Furthermore, Katolay et al. (1993) designed "a binominal interest rate tree" to simulate that future interest rates follow a random development. The bond value can be determined by discounting the cash flows while using "the volatility-dependent one-period forward rate". These are the result of the tree implementation facing, however, substantial challenges. D'Halluin et al. (2001) argue that the traditionally used finite difference approach to value callable bonds requiring notice cannot be used because of poor accuracy of calculations caused by "discontinuities and difficulties in handling boundary conditions". Because of a limited applicability, other approaches such as the numerical partial differential equation (PDE) method emerged. D'Halluin et al. (2001) propose that a fully numerical PDE approach can be exploited to price callable bonds with notice in a more accurate manner, particularly in cases where the calculation of the analytic Green´s function is impossible. This typically happens when "time-dependent parameters are used to match the initial term structure". Lim et al. (2012) propose a method based on "the eigenfunction expansion of the pricing operator". Under the assumption of a set of call and put dates, the pricing function for callable and putable bonds can be defined as "the value function of a stochastic game with stopping times". The authors argue that for the commonly used "short rate diffusion models" (CIR, Vasicek), the approach is significantly faster compared to the methods described in the literature so far.

Ben-Ameur et al. (2007) propose another modification for the pricing of callable bonds based on a dynamic programming (DP) method. Similarly to Büttler and Waldvogel (1996) and D'Halluin et al. (2001), the authors use the Vasicek (1977) and Cox-Ingersoll-Ross (CIR) (Cox et al., 1985a, 1985b) specification of the interest rate dynamics. The calculation results suggest that the method is efficient and robust and can be also used in the case of more general models calibrated to reflect the term structure of interest rates.

In contrast to the alternative approaches described in the academic literature so far we are focused on valuing options embedded in non-marketable callable bonds. The valuation procedure consists of three separate steps, which incorporate specific data available on the bond issuer, the bond holder, and estimates in terms of interest rate developments. Moreover, the approach presented in this paper also provides a methodology for valuing bonds with a multiple embedded European option.

2. Research design

The assumption of subjectivity plays an essential role in our numerical model. Contrary to the approaches described in the theoretical review, we expect that the absolute value of embedded options differs from the issuer- and investor-related perspective. The differences in transaction cost assessment are substantial in this regard. Furthermore, we assume that the planned cash flow will remain unchanged in terms of its amount and distribution over time, both on the side of the issuer and the bondholder. At the time of the valuation, the borrower would not have borrowed a different amount with a different maturity date than it had done in the case of the original issue. Finally, we assume that there has been no change in the issuer's credit rating from the time of issue to the time of valuation. Changes in the interest rate available to an individual borrower are affected only by a change in market conditions, not by a change in the borrower's credit rating. Hence, the borrower responds only to these changes when deciding to call the bond. In case of non-compliance with the above assumptions, the model requires some adjustments compared to the initial proposal. These are discussed in the concluding parts of this article.

The problem solving process can be accomplished by the steps as follows: 1) The first question is under what conditions the bond will be called? We look for an answer to this question from the position of the issuer because the issuer decides to call the option (to prematurely redeem the security). Decision-making is supported by the cost-benefit analysis (CBA), which compares scenarios of calling or not calling the option at a certain point in time. If we abstract from specific conditions on the issuer's side, the decision depends on current market interest rates available to the issuer at the time of calling the embedded option. It means that for the issuer the interest rate is the only stimulus to call the bond; 2) Second, the bond holder has to assess the cost that is incurred if the embedded option is exercised. This question can be answered by comparing the investment opportunities of the callable bond holder at the time of calling the embedded option with the situation of a bond holder without an option embedded in the bond. An alternative for the investor is to hold the subject bond (but without the embedded option) until maturity. The solution lies in the quantification of the loss of interest income, the amount of which depends on the current market interest rate at the time of calling the embedded option; 3) Third, the probability whether interest rates will decline to such a level that the issuer will be stimulated to exercise the option has to be calculated. Regarding this, models describing the evolution of interest rates have to be applied, e.g. the Vasicek (1977), Cox-Ingersoll-Ross (CIR) (Cox et al., 1985), and Marsh and Rosenfeld (1983) models. For the sake of simplification, we assume that changes in interest rates in time correspond with the Gaussian random walk, i.e., a concept frequently used in financial modelling (see, e.g., the Black-Scholes model/Black-Scholes-Merton (BSM) model, 1973). This concept views the evolution of a quantity over time as a sequence of random steps. The individual steps are assumed to be independent of each other and consist of identically distributed random variables. Each individual step corresponds to the inverse of a cumulative normal distribution with mean zero. The proposed pricing model, however, allows also for other assumptions about the behaviour of interest rates. For each level of discount rate at which the embedded option can be applied, there is a probability that the real market interest rate at the time of calling the embedded option reaches the stated interest rate or that it drops under this level.

To answer the questions posed above, the negative value of a callable bond option can be defined as the sum of losses on the bond holder's side (according to point 2) which the investor will have to bear in contrast to a situation where the investor is a holder of a non-callable bond. Subsequently, these losses will be weighed by the values of the probability that a situation like this occurs (according to point 3). In line with the general approach to financial instrument valuation based on discounted cash flows (see, e.g., Myers, 1977), we express these potential losses at the present value. The discount rate at the call date of the embedded option on which the amount of bond holder's loss is dependent and which is associated with a certain level of probability that the actual discount rate reaches its value, is a continuous quantity. For this reason, we estimate the negative value of the embedded option for the bond holder as an integral of the product of loss and probability of loss in the number of discount rates that meet the call condition described in point 1.

Figure 1 presents the overall methodological framework for the valuation of options embedded in non-marketable callable bonds.

While valuing the non-marketable callable bonds, the valuer will stumble on the lack of input variables and eventually will face the need to include some specific information. This is why our general requirement for the valuation of options embedded in non-marketable callable bonds is a sufficiently flexible structure of the valued model that enables reaction to individual parameters. In our model, the role of the cost/benefit analysis (CBA) described in point (1) is crucial as it makes it possible to take into consideration the specific conditions on the issuer's side (e.g. financial distress, changes in cash flow planning, changes in investment planning, specific transaction cost, etc.) since all of them can influence its decision to call the option. In the same way, it is possible to adjust the loss according to the point (2) borne by the investor when calling the embedded option, reflecting the achievability of alternative investments or costs (of the transaction, for example).

The model proposed in this article makes it possible to model the expected fluctuation of interest rates and the resulting probability of calling the embedded option depending on available data. As mentioned above, the literature has described a number of approaches to modelling the behaviour of interest rates (for details, see, e.g., Vasicek, 1977; Cox-Ingersoll-Ross, 1985a, 1985b). As shown in Figure 1, the autonomy in addressing partial issues,

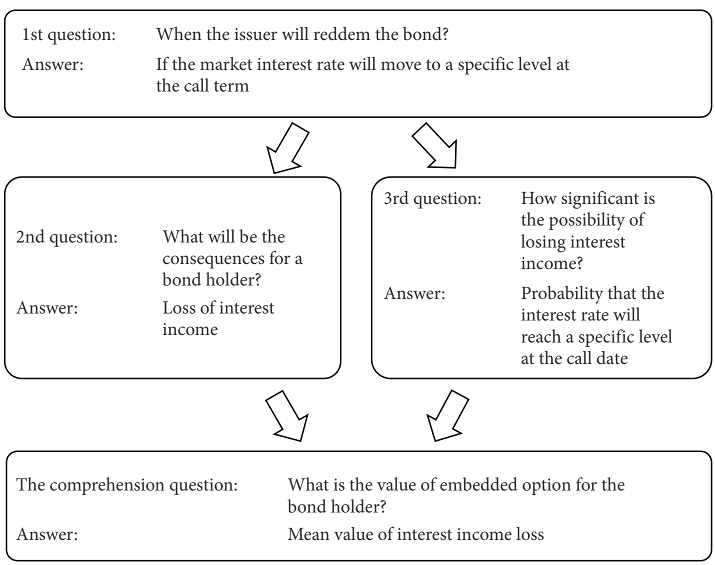

Figure 1. Methodological framework (source: own processing)

taking into consideration the possibilities and conditions of valuation, enables the valuer to respond to specific conditions of the case (e.g., the availability of the database) and to optimise the relationship between the valuation accuracy and the costs incurred.

3. The model proposal

We presume the issuance terms and conditions allow that the bonds can be called back by the issuer on specified call dates before its maturity. Let us call these dates (a_1) to (a_n). As of these dates, the emission may be redeemed early by the issuer before its maturity. This option will be exercised if the current costs of such a solution, including the cost of new issue, are lower than the present cost of the debt, i.e., if the following inequality is fulfilled:

$$Vol * \sum_{t=1}^{n} \left(C_{old} - C_{new}\right)\left(1+r_t\right)^{-\left(tfc+\left(t-1\right)*l\right)/365} - E \geq 0, \tag{3}$$

where: (Vol) – value of the original issue in monetary units); (C_{old}) – coupon rate of the original issue in per cent); (C_{new}) – coupon rate of the new issue in per cent); (n) – number of periods to the maturity date at which point the coupon must be paid; (t) – number of time periods); (r_t) – the current interest rate at the end of the period (t) reflecting the individual risk of the issuer at the valuation date; (tfc) – the time interval to the first coupon payment in days); (l) – the time interval (t) in days; (E) – the issuer's costs related to the early redemption of the current bond issue and cost of the new issue).

At the same time, we presume that the issuer does not strive to change the amount of debt (this being determined by the value of the original issue) or the structure of the debt time frame. If this was the case, the variable (E) would be reduced (note: as part of the cost/benefit analysis, the call of the value issue would reduce the (E) value by a part of the issuer's cost associated with the intended change in the debt structure).

If we presume a new bond issue with a coupon where the issue value fully covers the original issue, the nominal value of such bond will be identical with the original issue (at the par value) and it will provide a coupon in the same frequency and at the same dates as the original issue coupon. The coupon amount at the valuation date should correspond with such a rate (r_t) which corresponds with the remaining bond duration (r_{Dur}) at the day when the option contract is used. Depending on the bond coupon frequency, it is possible to derive the amount of the coupon payment achievable by the borrower at the moment of valuation as follows:

$$C_{new\_actual} = \left(1+r_{Dur}\right)^{1/f} - 1, \tag{4}$$

where: (r_{Dur}) – the current achievable yield of the issuer covering the time interval of the remaining duration at the call date; (f) – coupon payment frequency (on an annual basis).

This value (r_{Dur}) which, when replaced by (3), will fulfill the condition of equality, will be called the border discount rate (r_{Border}) to exercise a call option (related to the original bond issue). If the current rate ($r_{t=Dur}$) at the time of the potential call date is lower than (r_{Border}), the bonds will be called.

For (m) of future issue callability dates, the values (r_{Border_1}) to (r_{Border_m}) will be determined according to relationships (1) and (2) for each of them. Individual-determined border rates are associated with various time periods with respect to the remaining bond at the call date.

Based on historical data on the variability of discount rates of bonds with the same maturity as the individual specified border rates (r_{Border_1}) to (r_{Border_m}), we will identify the variability of these discount rates over a selected period of time:

$$cr_{Border\_(p)} = \frac{r_{Border_{(p)}}}{r_{Border_{(p-1)}}} - 1, \tag{5}$$

where ($r_{Border\_(p)}$) – border discount rate at the end of the period (p) of length (l).

In the next step, we will determine the variability of these changes $var(cr_{Border\_(p)})$, which we will use as an estimate of the variability of the corresponding rates (r_{Dur}) in time.

The value of the bond call option then depends on: (a) the amount of saving from the lower coupon caused by the decrease in discount rates at the call date while fulfilling inequality (1), (b) the probability of such discount rate reduction, and (c) in the case of subsequent callability also the probability that the bond will not be called during the previous call dates.

Assuming that the changes in discount rates between periods have a normal distribution with the mean value at zero and variance $var(cr_{Border\_(p)})$, with the changes in discount rates mutually independent between individual periods, we can assume the value of the call option associated with the first call date as the result of the first two members (a) and (b). The present value of saving is as follows:

$$save_1\left(cr_1\right) = Vol * \sum_{t=1}^{rest_1}\left(C_{old} - (1 + r_{Dur1} * \left(1 + cr_1\right))^{\frac{1}{f}} + 1\right) * \left(1 + r_t\right)^{-\left(tfc + \left(t-1\right)*l\right)/365}. \tag{6}$$

The amount of savings depends on the difference between the amount of the current coupon (C_{old}) and the coupon achievable for the issuer at the time of valuation ($C_{new\text{-}actual}$), which is calculated at the rate r_{Dur1}. However, this rate may change by cr_1 depending on the future development of the discount rate. The value ($rest_1$) corresponds to the number of coupon payments of the original issue from the first call date to maturity. Future differences in coupon payments are then discounted to the present value by discount rates corresponding to the moment of individual coupon payment maturity. With the future change of current value of (r_{tDur1}) by the value (cr_1) = x_1, we will arrive at the value of saving $Save(x_1)$.

The probability of change in the relevant discount rate (r_{Dur1}) decisive for the determination of the new coupon value by the value (cr_1) < x_1 (i.e., the decline is greater than the value x_1 in %) can be determined as follows:

$$p_1\left(cr_1 < x_1\right) = \int_{-\infty}^{x_1} f\left(cr_1\right) dcr_1, \tag{7}$$

where: ($f(cr_1)$) – probability density of the normal distribution of changes in the discount rate, while maturity corresponds to (r_{Dur1}) with the mean value zero (μ = 0) and the standard deviation (σ); (σ) is calculated based on formula (5). Therefore, we apply the following formula:

$$\sigma^2 = \frac{L}{l} var\left(cr_1\right), \tag{8}$$

where: (L) – time until the first call date; (l) – time interval in (5), which was used to calculate the discount rate variability corresponding to (r_{Dur1}).

The first call date value can then be determined as the mean value of individual amounts where the discount rate (r_{Duri1}) is so low that it provokes the bond call and the probability that this discount rate until the call date drops so much (or remains low) that the call will not be realised.

$$Call_1 = \int_{-1}^{x_1} save_1\left(cr_1\right) * p\left(cr_1\right) dcr_1. \tag{9}$$

The lower limit of integration (–1) refers to the assumption that the discount rate is positive and its decrease is impossible below the value (r_{Dur1}) = 0. The assumption can be mitigated and the lower limit of integration may be minus infinity. The upper limit of integration corresponds to such change (cr_1) of the discount rate (r_{Dur1}) where the value ($save_1(cr_1)$) from the relationship (6) is zero after substituting all other variables.

In the case of the possibility of calling the bond at the next call date (e.g., on an issue date anniversary), we proceed analogously by determining the call value at the first call date. In the case of point (a), we construct the value ($save_2(cr_2)$) corresponding to the saving on coupon payments. This results from the current value of the discount rate (r_{Dur2}), the potential future changes of this rate (cr_2) and the remaining number of coupon payments ($rest_2$) which can be changed. The probability (b) of the call at the second call is constructed on the basis of the discount rate (cr_2) variability with maturity corresponding to the discount rate (r_{Dur2}) decisive for the new coupon at the second call date.

The holder cannot call the issue at the second call date, however, if the issue was called at the first call date. This probability is expressed in the relationship (7). Thus, the value of the call option for the first two call dates can be expressed as follows.

$$Val_{1+2} = PV\left[Call_1\right] + \left(1 - p_1\right) * PV\left[Call_2\right], \tag{10}$$

where PV[.] – present value.

The value of the call option for all call dates will be expressed as follows:

$$Val_{all} = \sum_{i=1}^{m} PV\left[Call_i\right] * \prod_{j=0}^{i-1}\left(1 - p_j\right), \tag{11}$$

where: m – remaining number of call dates; ($Call_i$) – the option value at the call date (i); (p_j) – probability that the call option will be exercised at date j.

The process of valuing the issuer's call option can be demonstrated in Figure 2.

4. Numerical example

From the perspective of the bond holder, we would like to value a callable bond on the valuation date (31/12/2020). This non-marketable security was issued on 02/07/2018, the amount of the issue was EUR 10 million, the nominal value of one bond was EUR 10,000 and the final maturity date is 02/07/2023. The coupon of 6.25% is paid semi-annually, always at 02/01 and 02/07 of the calendar year. The day count convention for calculating interest yield is 30E/360. The issuer may make the bonds due on 02/01/2020 and then on each subsequent issue anniversary (i.e. at 02/07/2020; 02/07/2021; 02/07/2022; 02/07/2023). The discount rates correspond to the issuer's rating and the maturity of individual amounts. This is followed by the bond valuation in line with the model, see Figure 2.

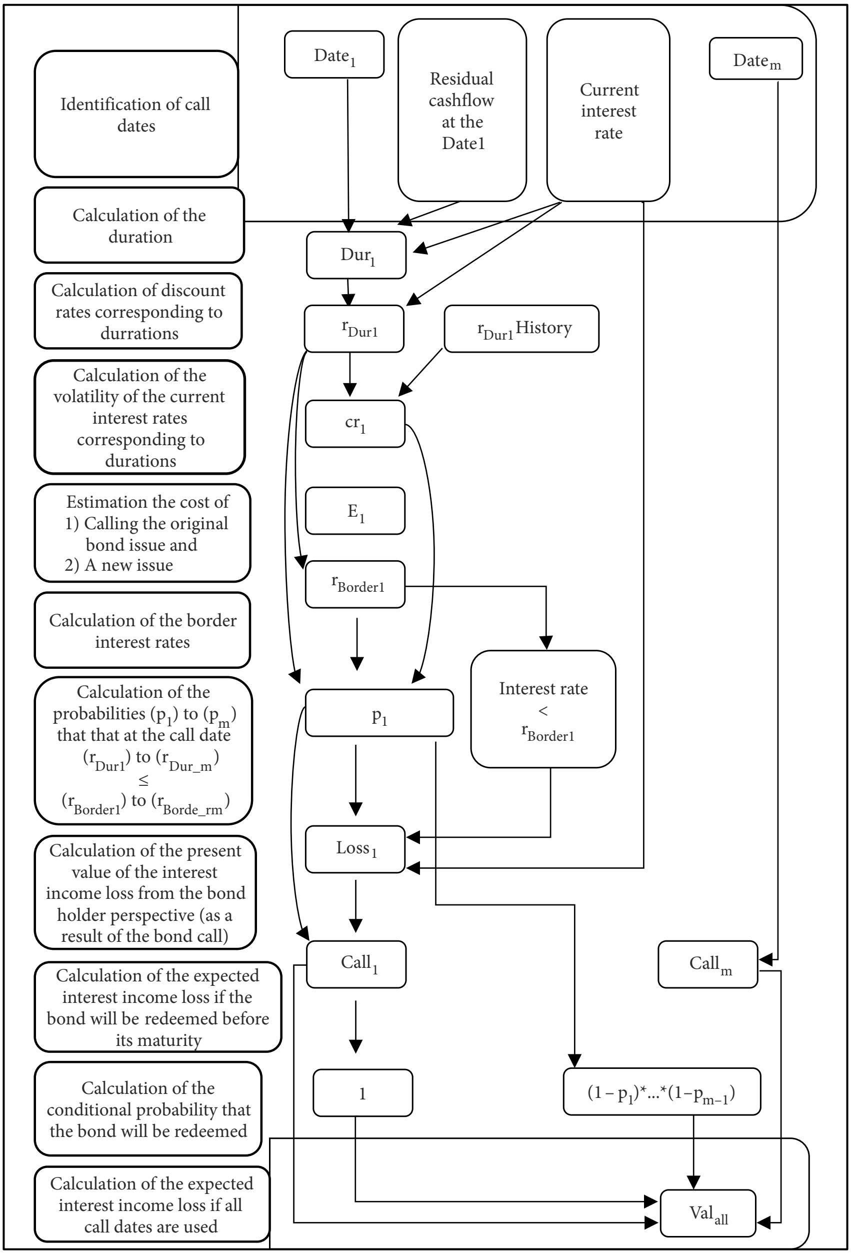

Figure 2. Model proposal

Figure 2 delivers the answers to the three issues highlighted in Figure 1. The first set of steps in the valuation procedure shown in Figure 2 addresses the question when and under what conditions the call option will be exercised? This involves describing sub-steps such as identification of call dates, calculation of border interest rates and defining of essential input variables (e.g. bond duration and bond cash flow). The second set of steps (calculation of the interest income loss) aims at quantifying the extent of the loss to the bondholder when the bond is called. The third set of steps and related variables answers the question of what the probability of the loss is (i.e. calculation of the probabilities, the conditional probability; variables include e.g. current interest rates and their volatility). The negative value of the call option from the bondholder's perspective is determined as the range of possible losses weighted by their probabilities.

Step 1. Identification of call dates. At 31/12/2020, the issuer has only two dates left at which it can call the issue: $Date_1$ – 02/07/2021 and $Date_2$ – 02/07/2022.

Step 2 and Step 3. Bond duration at the call date and the discount rate alculation. At the first call date ($Date_1$), the bond holder can expect the cash flow at the dates shown in Table 1. After assigning the relevant discount rates (at 31/12/2020) to the determined maturity, it is possible to determine the current cash flow value based on the corresponding yield curve and to determine the expected duration (Dur_1) at the call date ($Date_1$). The discount rate (r_{Dur1}) is calculated on the basis of the relevant bond yield curve.

We will apply the same procedure to determine the corresponding discount rate (r_{Dur2}) at call date ($Date_2$), for details see Table 2.

Table 1. Calculation of the discount rate r_{Dur1} at the call date ($Date_1$ – 02/07/2021)

Cash flow payment days	Cash Flow (in EUR)	Number of days from the call date ($Date_1$)	Corresponding discount rate (in per cent)	Present value of the cash flow (in EUR)
02/01/2022	312.50	184	4.48	305.67
02/07/2022	312.50	365	4.58	298.80
02/01/2023	312.50	549	4.67	291.76
02/07/2023	10312.50	730	4.76	9,397.47
			In total	10,293.70
Duration (Dur_1) at the $Date_1$ = 698 days			Corresponding discount rate r_{Dur1} = 4.74%	

Table 2. Calculation of the discount rate r_{Dur2} at the call date ($Date_2$ – 02/7/2022)

Cash flow payment days	Cash flow (in EUR)	Number of days from the call date ($Date_2$)	Corresponding discount rate (in per cent)	Present value of the cash flow (in EUR)
02/01/2023	312.50	184	4.48	305.67
02/07/2023	10312.50	365	4.58	9,860.48
			In total	10,166.15
Duration (Dur_2) at the ($Date_2$) = 360 days			Corresponding discount rate r_{Dur2} = 4.58%	

Step 4. Estimation of the cost of a new bond issue. Based on the analysis of comparable transactions and the original bond issue cost, the cost of calling the issue and re-issue is estimated at EUR 300,000.

Step 5. Calculation of border interest rates. We are looking for such (r_{Dur1}) and (r_{Dur2}) for which, after substituting into equations (2) and (1) the condition of achieving equality in relationship (1) is satisfied. These values will then be marked as ($r_{Border1}$) and ($r_{Border2}$). In the case of the first call date ($Date_1$) the calculation parameters are shown in Table 3, second column. In the case of call date ($Date_2$) the parameters are shown in Table 3, third column.

Table 3. Calculation of border interest rates (($r_{Border1}$) and ($r_{Border2}$)) that are congruent with the condition to call the bond issue at ($Date_1$) and ($Date_2$)

Parameter	Dates	
	$Date_1$ – 02/07/2021	$Date_2$ – 02/07/ 2022
Volume of the bond issue (Vol) (in EUR)	10, 000,000	10,000,000
Original coupon payment (C_{old}) (in per cent)	6.25 / 2	6.25 / 2
Number of coupon payments remaining until the call date (n)	4	2
Length of the period (l) (in days)	180	180
Discount rate (r_1) for the 1st coupon payment (following to $Date_1$ or $Date_2$), (in per cent)	4.48	4.48
Discount rate (r_2) for the 2nd coupon payment (following to $Date_1$ or $Date_2$), (in per cent)	4.58	4.58
Discount rate (r_3) for the 3rd coupon payment (following to $Date_1$), (in per cent)	4.67	
Discount rate(r_4) for the 4th coupon payment (following to $Date_1$), (in per cent)	4.76	
Time interval to the 1st coupon payment (tfc) following to $Date_{1/2}$ (in days)	2	2
Coupon payments frequency (f)	2	2
Cost of a new bond issue (E) (in EUR)	300,000	300,000
(r_{Dur1}) and (r_{Dur1}) corresponding to condition (1)	$r_{Border1}$ =	$r_{Border2}$ =
($r_{Border1}$) and ($r_{Border2}$), (in per cent)	4.72	3.17

Step 6. Calculation of the variability of the discount rate. We have the past discount rate values (r_{Dur1}) and (r_{Dur2}) related to the bond, always on the last day of the month. Based on the relationship (3) we calculate the monthly variability of discount rates decisive for determining the coupon for a new issue in order to define the probability that these rates will reach the border discount rates ($r_{Border1}$) and ($r_{Border2}$). We assume the length of the interval between individual values (l) at the level of 30 days. Changes in the corresponding values are the point of departure for the variability calculation, i.e., var($cr_{Border1}$) and var($cr_{Border2}$). If we consider the variability of changes in discount rates between individual time intervals (in

our case expressed in months) to be mutually independent, the variability of discount rates for (n) period can be determined as follows:

$$\sigma^2\left(X+Y\right)=\sigma^2\left(X\right)+\sigma^2\left(Y\right)+2Cov\left(X,Y\right). \quad (12)$$

And written also as:

$$n^{*}\ \mathrm{var}(cr_{Border1}). \quad (13)$$

In the case of the first call date ($Date_1$) it is 6 months. In the case of the second call date (Date22), it is 18 months, for details see Table 4

Table 4. Variability of changes in discount rates until the ($Date_1$) and ($Date_2$)

Call date	$Date_1$ – 02/07/2021	$Date_2$ – 02/07/2022
Variability of monthly changes in discount rates (3)	0.00978	0.01043
Number of months until the call date	6	18
Variability σ_1^2 and σ_2^2 of changes in rates $r_{Border1}$ and $r_{Border2}$ until the call dates ($Date_1$ and $Date_2$)	0.05872	0.18785

Step 7. Calculation of the probabilities that the current interest rates (r_{Dur1}) and (r_{Dur2}) ≤ ($r_{Border1}$) and ($r_{Border2}$) at the call dates ($Date_1$ and $Date_2$). The discount rates (r_{Dur1}) and (r_{Dur1}) that determine the decision to exercise the call were defined in Step 3 on the basis of the durations (Dur_1) and (Dur_2). In Step 4, we set the rate border values to exercise callability, that is, ($r_{Border1g}$) and ($r_{Border2}$). We can therefore deduce by what value the discount rates must change in order to provoke a call.

If we assume the mean value in such changes of discount rates to be zero and the distribution to be normal, then we can use the variability of changes in the corresponding discount rate and the number of time intervals to the call date determined in the previous step to determine probabilities (p_1) and (p_2) that at the call date the current discount rates (r_{Dur1}) and (r_{Dur2}) will be lower or equal to the border discount rates ($r_{Border1}$) and ($r_{Border2}$).

In case of ($Date_2$), the probability of call is conditioned by the probability that the call is not exercised in case of ($Date_1$), for details see Table 5.

Table 5. Probabilities that on $Date_1$ and $Date_2$ the bond will be redeemed

Call date	$Date_1$ – 02/07/2021	$Date_2$ – 02/07/2022
Current interest rates (r_{Dur1}) and (r_{Dur2}), (in per cent)	4.74	4.58
Border interest rates ($r_{Border1}$) and ($r_{Border1}$), (in per cent)	4.72	3.17
Border change of interest rates ($cr_{Border1}$) and ($cr_{Border2}$), (in per cent)	−0.52	−30.76
Variability σ_1^2 and σ_2^2 in changes of interest rates until the call date	0.05872	0.18785
Probabilities (p_1) and (p_2) (based on (5))	49.20	18.80
Probabilities that the bond will be redeemed	49.20	12.10

Step 8. Calculation of the loss from the bondholder perspective at the individual call dates. In the case of call date ($Date_1$) the creditor can lose on four coupon payments (at 02/01/2022, at 02/07/2022, at 02/01/2023 and at 02/07/2023). The mean value of the loss on one coupon payment, if the bond is called at ($Date_1$), can be quantified as follows (in relation to the entire issue) (14):

$$loss_{E1.1} = \int_{-1}^{cr_{Border1}} Vol * (C_{old} - \left(1 + r_{Dur1} * \left(1 + cr_1\right)\right)^{\frac{1}{f}} - 1) * Normdist\left(0; \sigma_1^2\right) dcr_1. \quad (14)$$

After substituting values from Table 3, we will obtain the value of the original coupon payment (EUR 312,500). From the corresponding calculations we will substitute the remaining variables in relationship (14).

$$loss_{E1.1} = \int_{-1}^{,-0,0052} \left[312{,}500 - 10{,}0000{,}000 * \left[\left(1 + 4.74\% * \left(1 + cr_1\right)\right)^{\frac{1}{2}} - 1 \right] \right] *$$

$$\left(2 * 0.05872 * \pi\right)^{-\frac{1}{2}} * e^{-cr_1^2/(2*0.05872)} dcr_1,$$

$$loss_{E1.1} = 60{,}905.5 \text{ EUR}.$$

The loss arising to creditors at call date ($Date_1$) relates to four coupon payments. Table 6 summarises them after conversion to the present value at the valuation date 31/12/2020.

Table 6. Calculation of the expected interest income loss on $Date_1$

	Expected loss in coupon payments (in EUR)	Remaining time to achieve the loss (in days)	Corresponding discount rate (in per cent)	Discount factor	Present value (in EUR)
1st coupon payment (02/01/2022)	60,905.5	367	4.59	0.9559	58,221.1
2nd coupon payment (02/07/2022)	60,905.5	548	4.67	0.9338	56,871.6
3rd coupon payment (02/01/2023)	60,905.5	732	4.76	0.9110	55,486.2
4.th coupon payment (02/07/2023)	60,905.5	913	4.84	0.8885	54,112.4
Present value of the loss at $Date_1$					224,691.2

In the case of the second call date ($Date_2$), there remain only two coupon payments which can change as a result of the call, i. e. at 02/01/2023 and 02/07/2023. The mean value of the loss on one coupon payment which results from the first call date ($Date_2$) and which corresponds to the entire issue is expressed in the following relationship (15):

$$loss_{E2.1} = \int_{-1}^{cr_{Border2}} Vol * (C_{old} - \left(1 + r_{Dur2} * \left(1 + cr_2\right)\right)^{\frac{1}{f}} - 1) * Normdist\left(0; \sigma_2^2\right) dcr_2. \quad (15)$$

After substituting values from Table 3 we will obtain the value of the original coupon payment, which was EUR 312,500. Further variables will be substituted from the corresponding tables.

$$loss_{E2.1} = \int_{-1}^{-0,3076} \left[312,500 - 10,0000,000 * \left[\left(1 + 4.58\% * \left(1 + cr_2\right)\right)^{\frac{1}{2}} - 1 \right] \right] *$$

$$\left(2 * 0.18785 * \pi\right)^{-\frac{1}{2}} * e^{-cr_1^2/(2*0.18785)} dcr_2,$$

$$loss_{E2.1} = 47,233.6 \text{ EUR.}$$

The loss arising to creditors when exercising the call date (Event$_2$) relates to two coupon payments. Table 7 summarises them after conversion to the present value at the valuation date 31/12/2020.

Table 7. Calculation of the expected interest income loss at Date$_2$

	Expected loss in coupon payments (in EUR)	Remaining time to achieve the loss (in days)	Corresponding discount rate (in per cent)	Discount factor	Present value (in EUR)
1st coupon (02/01/2023)	47,233.6	732	4.76	0.9110	43,030.8
2nd coupon (02/07/2023)	47,233.6	913	4.84	0.8885	41,965.4
Present value of the loss at Date$_2$					84,996.2

Step 9. Calculation of the expected interest income loss if all call dates are used (from the bond holder perspective). Applying Eq. (9), we will sum up the losses achieved at individual call dates which will be subsequently adapted to reflect the fact that the bond has already been called (at previous call dates). Thus, in our case, it is the probability of call in case of Date$_1$.

$$Val_{all} = PV\left[Call_1\right] + \left(1 - p_1\right) * PV\left[Call_2\right], \tag{16}$$

$$Val_{all} = 224,691.2 + \left(1 - 49.2\%\right) * 84,996.2 = 267,911 \text{ EUR.}$$

The loss of interest income per one bond amounts to 2.679% of the nominal value.

We emphasize here that this result should be interpreted from the perspective of the bondholder. It is the amount that the investor has to deduct from the expected value of the bond (assuming the bond is non-marketable) determined at the valuation date, if we apply the discount rates at that date for bonds that do not have a call option. This discount (the negative value of the call option) corresponds to the risk that the bond held by the investor will be called at a time of unfavorable conditions for, i.e., at a time of low market interest rates that will no longer allow the investor to earn the original yield.

The calculation result, however, cannot automatically be interpreted as the value of the call option from the perspective of the borrower. The latter is burdened with the transaction costs associated with the call of the original issue and the execution of the new issue. The significance of these costs may vary considerably from issuer to issuer. On the other hand, the value of the call option increases the borrower's ability to adjust its cash flows to changing internal needs before the maturity date of the bond. Both circumstances may take on different significance for different borrowers – contributing to different degrees to the reduction or increase in the value of the call option for the borrower.

5. Discussion

The proposed model focuses on the valuation of the embedded European option (or a multiple option) that is traded solely through private transactions, which sets it apart from previous approaches in the literature (see, e.g., Ho et al., 1992; Kalotay et al., 1993; D'Halluin et al., 2001; Jarrow et al., 2010). The construction of the model, its link to the yield curve, and risk premia imply that it can be applied without additional modifications to the valuation of call options even for those redeemable bonds that are traded in public markets. Due to the availability of market data in these cases and the already existing concepts concerning the relationship between the value of a call option, the value of a bond with a call option, and the value of a bond without a call option, for details, see, e.g. Ho et al. (1992) and Katolay et al. (1993), some steps in our proposed valuation procedure can be simplified, but need not be modified. Additionally, the model takes into account the issuer's transaction cost which plays a significant role in reality and allows the valuer to flexibly adapt to the available database scope. At the same time, the model does not explicitly require the adoption of assumptions about a certain behaviour of interest rates that usually require longer time series for the estimation of parameters (e. g., Cox-Ingersoll-Ross, 1985a, 1985b; Marsh & Rosenfeld, 1983). On the other hand, the model enables the acceptance of these assumptions. Unlike many alternative approaches (e.g. Kalotay, 1993), the model is not merely schematic; i.e. it works with the data if these are available.

In the valuation model proposal, we used the interest rate hypothesis (Banko & Zhou, 2010; Blume & Keim, 1988), which considers the motive for reducing the interest rate on the issuer's side as the fundamental reason behind the issuance of a callable bond if the interest rates happen to drop. This is also a limitation of our model because it fails to consider other motives of the issuer, such as a change in the financial plan (regardless of the interest rate level), which could also be an impulse for calling the bond. It means that the valuation model does not systematically address other motives, but, on the other hand, if they are known, they are fully or partially reflected in reducing the transaction costs (E). Thus, the valuation model reduces the systematic deviation in the bond valuation compared to the situation where the call option is not considered. However, it can be expected that despite our proposed model, in real economic practice the call costs of the call will always be undervalued from the investor's point of view due to the existence of other call that are present in the issuer's domain. The cost of reissue (E) will be effectively lower for the issuer because with the new issue they will be able to react to the need to adjust the cash flow originally prepared at the date of the original bond issue. This will increase the probability of the call and thus the negative value of the embedded option for the bond holder. In this respect, our model is not much different from the alternative models described in the literature, which are also based only on changes in the interest rate.

Conclusions

This paper deals with the issue of how to value options embedded in non-marketable callable bonds. As there is a lack of methods which are suitable for pricing this type of security, we attempt to close this knowledge gap by proposing a novel approach to valuing the risk that the issuer will use the right to buy back the bond at a specific call price. The model can help investors and financial analysts value callable bonds without the secondary market. Moreover, different levels of transaction cost from the investors' and issuers' perspective, multiple terms of the call option execution, and the assumption of the random walk theory regarding the probability that the call option will be exercised are embodied in the model.

The main limitation of the model lies in the fact that the only current interest rate used in them is the one that is relevant for the valued callable bond. Based on this interest rate (or rather its history), an interest rate model is built, which is a condition for the expectation of the call. In reality, there is always a structure of interest rates. The interest rate relevant for the borrower is influenced by the general level of interest rates and their changes, but also by the credit rating, i.e. the positioning of the relevant interest rate in the structure of interest rates. Once the credit rating of the borrower changes, the relevant interest rate changes to a different level within the interest rate structure. The model of interest rates derived from other than currently relevant levels of interest rates within the interest rate structure conditioned by the credit rating of the borrower can be misleading for deriving the probability that the call option will be exercised as a result of the interest rate development. The models with the embedded call option based on various interest rate models described in the literature do not address this issue. In our proposed model, it is possible to deal with this fact by assigning different interest rate levels to the borrower with a variability that corresponds to the new credit rating. In this way (regardless of the interest rate fluctuations), in case of an improvement in the credit rating, the probability of call option being exercised will grow immensely. A modification is necessary on the side of the bond holder who invested in a bond the credit rating of which has changed. This fact should be taken into account when looking for a comparable investment (or transaction). The concept of our proposed valuation model respects the objection of prior approaches that point out that the value of the call option has a connection with the slope of the yield curve slope. This fact is captured by comparing interest rates with different durations. Option valuation models proposed in the literature usually work with short-term changes in interest rate models.

Regarding another possible adjustment of the valuation model, we can mainly consider a method to capture the potential for a change in the issuer's credit rating. In the case of non-marketable bonds, public ratings from rating agencies are not common, however, market participants usually compile their own (internal) ratings according to internal or external methodologies and orient themselves accordingly. Thus, the expanded model would take into account the probability of a change in rating. In the event of an improvement of the credit rating, another element of asymmetry would be incorporated that influences the callability assessment on the creditor and the issuer side. In the present version of the valuation model, the asymmetry is represented in the form of re-issue cost on the side of the borrower. In this case, the value of the call option from the borrower's point of view is lower than the negative value of the call option from the creditor's point of view. If the credit rating improves, the level of interest rates within the interest rate structure is different for the borrower from that pertaining to the original bond and to which the original bond holder concentrates its attention considering the undertaken risk. In the perspective of the proposed model, in the case of options embedded in non-marketable callable bonds, it is not a regular option, the value of which is equal in its absolute value for the issuer and the holder.

Author contributions

Roman Skalický conceived the study and was responsible for the design of the study and the model proposal. Marek Zinecker was responsible for the theoretical framework and design and development of the numerical example. Adam Balcerzak and Michal Pietrzak were responsible for the mathematical analysis and interpretation of the research findings. Elżbieta Rogalska contributed to the theoretical section and discussion of the research results.

Disclosure statement

All authors declare that they have no competing financial, professional, or personal interests with other parties.

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