

This version of the article has been accepted for publication, after peer review (when applicable) and is subject to Springer Nature's AM terms of use, but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: <https://doi.org/10.1007/s00033-022-01913-7>

Postprint of: Eremeev V., Scerrato D., Konopińska-Zmysłowska V., Ellipticity in couple-stress elasticity, ZEITSCHRIFT FÜR ANGEWANDTE MATHEMATIK UND PHYSIK Vol. 74 (2022), 18

Ellipticity in couple-stress elasticity

Victor A. Eremeyev , Daria Scerrato  and Violetta Konopińska-Zmysłowska 

Abstract. We discuss ellipticity property within the linear couple-stress elasticity. In this theory, there exists a deformation energy density introduced as a function of strains and gradient of macrorotations, where the latter are expressed through displacements. So the couple-stress theory could be treated as a particular class of strain gradient elasticity. Within the micropolar elasticity, the model is called Cosserat pseudocontinuum or medium with constrained rotations. Applying the classic definitions of ordinary ellipticity and strong ellipticity to static equations of the couple-stress theory, we conclude that these equations are neither elliptic nor strongly elliptic. As a result, one should be aware of extending properties of full strain gradient models such as Toupin–Mindlin strain gradient elasticity to models with incomplete set of second derivatives.

Mathematics Subject Classification. Primary 74B20, 74A30; Secondary 74A35, 35Q74, 46E35, 35J30, 35J70.

Keywords. Couple-stress theory, Ellipticity, Strong ellipticity, Strain gradient elasticity, Micropolar elasticity, Cosserat continuum.

1. Introduction

Elliptic systems of partial differential equations (PDEs) constitute a wide class of the models of mathematical physics. In the literature, one can find various definitions of ellipticity which extend the property of the Poisson equation. Among the latter, it is worth mentioning strong ellipticity, ellipticity known also as ordinary or Petrovskiy's ellipticity, Douglis–Nirenberg ellipticity and some others, see, for example, [1, 22, 46]. Ellipticity brings us such properties as a regularity (smoothness) of solutions and well-posedness of the corresponding boundary value problems. On the other hand, violation of ellipticity may lead to proper description of such physical phenomena as strain localization, material instabilities, shear bands formation, softening, wave propagation, see, for example, [7, 9, 10, 12, 27, 30, 31, 37, 42, 43].

The aim of this note is to discuss ellipticity within the linear couple-stress theory [28, 36, 44]. The latter could be treated as a Cosserat continuum with constrained rotations, so-called pseudocontinuum Cosserat [19, 21, 38] or a particular class of strain gradient models [5, 6, 11, 34, 35, 45]. Within nonlinear Cosserat continuum strong ellipticity was studied in [2, 13], see also [30], whereas relation of ellipticity loss on material stability within nonlinear strain gradient elasticity was analysed in [14]. Let us note that nowadays the couple-stress theory is widely used for modelling of composite materials and structures at small scales, see, for example, [3, 23, 26, 29, 39, 41] and the reference therein.

The paper is organized as follows. First, in Sect. 2 we recall the definitions of ellipticity and strong ellipticity for linear systems of PDEs. In Sect. 3, we introduce the equilibrium equations of the couple-stress theory including the modified one [48]. Section 4 is devoted to the analysis of ellipticity of these equations. Finally, we compare the results with ellipticity properties of linear micropolar and Toupin–Mindlin strain gradient elasticity.

Note that in the following we use the index-free tensor calculus as in [16, 40, 47].

2. Mathematical preliminaries

Following [1,22,46], let us consider a system of linear differential equations of order m

$$\sum_{|\alpha| \leq m} A_{ij}^{(\alpha)} D^\alpha u_j = f_i, \quad i, j = 1, 2, \dots, n, \quad (2.1)$$

where $\mathbb{A}^{(\alpha)} \equiv \{A_{ij}^{(\alpha)}\}$ are $n \times n$ matrices, $\alpha = (\alpha_1, \dots, \alpha_l)$ is a multiindex, α_k are natural numbers, $|\alpha| = \alpha_1 + \dots + \alpha_l$, $\mathbf{u} = (u_1, \dots, u_n)$ is a vector of unknown functions, $u_k = u_k(x_1, \dots, x_l)$, x_i are Cartesian coordinates, $k = 1, \dots, n$, and $\mathbf{f} = (f_1, \dots, f_n)$ is a vector of given functions, for example, of external loads. Differential operator D^α is defined by the formulae

$$D^\alpha = \partial_1^{\alpha_1} \dots \partial_l^{\alpha_l}, \quad \partial_p = \frac{\partial}{\partial x_p}, \quad p = 1, \dots, l.$$

In addition, hereinafter Einstein's summation rule over repeated indices is applied.

System (2.1) defines the differential operator \mathcal{A} given by

$$\mathcal{A} = \sum_{|\alpha| \leq m} \mathbb{A}^{(\alpha)} D^\alpha. \quad (2.2)$$

Motivated by strain gradient elasticity applications in the following, we consider $n = 3$, $l = 3$ and $m = 4$. For simplicity, we also assume that $\mathbb{A}^{(\alpha)}$, $|\alpha| = m$, does not depend on x_i .

Following [1,22,46], we call system (2.1) or operator \mathcal{A} *elliptic* or *Petrovskiy's elliptic* or *ordinary elliptic* if

$$\det \mathbb{A}_0(\mathbf{k}) \neq 0, \quad \forall \mathbf{k} \neq \mathbf{0}, \quad \mathbf{k} = (k_1, \dots, k_l), \quad (2.3)$$

where $\mathbb{A}_0(\mathbf{k})$ is the principal symbol given by the formula

$$\mathbb{A}_0(\mathbf{k}) = \sum_{|\alpha|=m} \mathbb{A}^{(\alpha)} \mathbf{k}^\alpha, \quad \mathbf{k}^\alpha = k_1^{\alpha_1} \dots k_l^{\alpha_l}. \quad (2.4)$$

Note that operator \mathcal{A} could be represented symbolically through the polynomial $\mathbb{A}(\mathbf{k}) = \sum_{|\alpha| \leq m} \mathbb{A}^{(\alpha)} \mathbf{k}^\alpha$ called symbol of \mathcal{A} . Symbolically $\mathbb{A}(\mathbf{k})$ could be obtained by formal replacement $\partial_p \rightarrow k_p$. So $\mathbb{A}_0(\mathbf{k})$ is a homogeneous polynomial of degree m in \mathbf{k} .

We call (2.1) or \mathcal{A} *strongly elliptic* if there is an inequality

$$(\mathbb{A}_0(\mathbf{k})\mathbf{a}, \mathbf{a}) \equiv \sum_{|\alpha|=m} A_{ij}^{(\alpha)} \mathbf{k}^\alpha a_i a_j \geq C \|\mathbf{k}\|^m \|\mathbf{a}\|^2 \quad (2.5)$$

for any vector \mathbf{k} and any vector $\mathbf{a} = (a_1, \dots, a_n)$, where C is a positive constant independent on \mathbf{k} and \mathbf{a} and

$$\|\mathbf{k}\|^2 = k_1^2 + \dots + k_l^2, \quad \|\mathbf{a}\|^2 = a_1^2 + \dots + a_n^2.$$

Obviously, strong ellipticity is more restrictive than ordinary ellipticity. Indeed, Eq. (2.5) means that matrix $\mathbb{A}_0(\mathbf{k})$ is positive definite, whereas (2.3) requires that $\mathbb{A}_0(\mathbf{k})$ does not have zero eigenvalues for any \mathbf{k} . So, strong ellipticity implies ellipticity. Let us note that the positive definiteness could be replaced by negative definiteness requirement since in this case one can get the positive definiteness by multiplication of (2.1) by -1 , see [1,46] for more details.

3. Couple-stress theory

In what follows, we restrict ourselves to small deformations. So let $\mathbf{u} = \mathbf{u}(\mathbf{x})$ be a vector of displacements and \mathbf{x} be a position vector. We introduce the linear strain tensor $\boldsymbol{\varepsilon}$ and rotation vector $\boldsymbol{\phi}$ as follows [36]

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad \boldsymbol{\phi} = \frac{1}{2} \nabla \times \mathbf{u}, \quad (3.1)$$

where ∇ is the 3D nabla-operator, the superscript T means transpose and \times denotes the cross product.

For a hyper-elastic solid there exists a deformation energy density W introduced as a function of strains and gradient of rotations [36]

$$W = W(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad \boldsymbol{\kappa} = \nabla \boldsymbol{\phi}. \quad (3.2)$$

So we have two stress measures, i.e. the classic symmetric stress tensor $\boldsymbol{\sigma}$ and the non-symmetric couple-stress tensor $\boldsymbol{\mu}$ both defined as follows

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\mu} = \frac{\partial W}{\partial \boldsymbol{\kappa}}.$$

Note that as $\boldsymbol{\phi}$ is introduced through Eq. (3.1)₂, there is no difference between $\boldsymbol{\varepsilon}$ and the strain tensor $\mathbf{e} = \nabla \mathbf{u} + \boldsymbol{\phi} \times \mathbf{I}$ usually used in the linear micropolar elasticity [16,19,21]. Hereinafter \mathbf{I} is the 3D unit tensor.

For an isotropic material, W takes the form [36]

$$W = \frac{1}{2} \lambda \text{tr}^2 \boldsymbol{\varepsilon} + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + 2\eta \boldsymbol{\kappa} : \boldsymbol{\kappa} + 2\zeta \boldsymbol{\kappa} : \boldsymbol{\kappa}^T, \quad (3.3)$$

where λ , μ , η and ζ are elastic moduli, and $:$ stands for the double dot product [16,19,21]. As a result, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ become

$$\boldsymbol{\sigma} = \lambda \mathbf{I} \text{tr} \boldsymbol{\varepsilon} + 2\mu \boldsymbol{\varepsilon}, \quad \boldsymbol{\mu} = 4\eta \boldsymbol{\kappa} + 4\zeta \boldsymbol{\kappa}^T. \quad (3.4)$$

Positive definiteness of W leads to inequalities [36]

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \eta > 0, \quad -\eta < \zeta < \eta. \quad (3.5)$$

Equilibrium equations have the following form [36]

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \eta \Delta \nabla \times (\nabla \times \mathbf{u}) + \mathbf{f} = \mathbf{0}. \quad (3.6)$$

where $\Delta = \nabla \cdot \nabla$ is the Laplace operator and \mathbf{f} is a vector of external loads. Note that ζ is not included into (3.6), it appears in static boundary conditions. Equation (3.6) is a particular form of general equations given also in [34,35,44].

Yang et al. [48] proposed to consider another symmetric strain measure $\boldsymbol{\chi}$ given by

$$\boldsymbol{\chi} = \frac{1}{2} (\nabla \boldsymbol{\phi} + \nabla \boldsymbol{\phi}^T)$$

with the constitutive relation in the form

$$W_m = \frac{1}{2} \lambda \text{tr}^2 \boldsymbol{\varepsilon} + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + 2\eta \boldsymbol{\chi} : \boldsymbol{\chi} \quad (3.7)$$

and symmetric couple-stress tensor $\mathbf{m} = 4\eta \boldsymbol{\chi}$. Equation (3.7) also results in (3.6).

4. Ellipticity

Calculation of the principal symbol of Eq. (3.6) is straightforward. It is determined by the highest order term of (3.6), that is, $\eta\Delta\nabla \times (\nabla \times \mathbf{u})$. Using formal substitution $\nabla \rightarrow \mathbf{k}$, we come to the formula

$$\mathbb{A}_0(\mathbf{k}) = \eta(\mathbf{k} \cdot \mathbf{k})\mathbf{k} \times (\mathbf{k} \times \mathbf{I}). \quad (4.1)$$

The determinant of $\mathbb{A}_0(\mathbf{k})$ is equal to zero: $\det \mathbb{A}_0(\mathbf{k}) = 0$. Indeed, we have

$$\det \mathbb{A}_0(\mathbf{k}) = \det [\eta(\mathbf{k} \cdot \mathbf{k})\mathbf{k} \times (\mathbf{k} \times \mathbf{I})] = \eta^3(\mathbf{k} \cdot \mathbf{k})^3 \det [\mathbf{k} \times (\mathbf{k} \times \mathbf{I})] = \eta^3(\mathbf{k} \cdot \mathbf{k})^3 \det [\mathbf{k} \times \mathbf{I} \times \mathbf{k}].$$

Using the identity

$$\mathbf{i} \times \mathbf{I} \times \mathbf{i} = \mathbf{i} \otimes \mathbf{i} - \mathbf{I},$$

valid for any unit vector \mathbf{i} , see, for example, [16, p. 104], we came to

$$\det \mathbb{A}_0(\mathbf{k}) = \eta^3(\mathbf{k} \cdot \mathbf{k})^6 \det [\mathbf{i} \otimes \mathbf{i} - \mathbf{I}] = 0, \quad \mathbf{i} = \mathbf{k}/\|\mathbf{k}\|,$$

as $\mathbf{i} \otimes \mathbf{i} - \mathbf{I}$ is a singular tensor. Hereinafter \otimes is the dyadic product.

Thus, despite positive definiteness of W equilibrium equations does not constitute an elliptic system. Since ordinary ellipticity is a necessary condition of the strong ellipticity, the latter is also violated. This statement could be also proved using (2.5). Indeed, we have that

$$\mathbf{a} \cdot \mathbb{A}_0(\mathbf{k}) \cdot \mathbf{a} = \eta(\mathbf{k} \cdot \mathbf{k}) [(\mathbf{a} \cdot \mathbf{k})^2 - (\mathbf{k} \cdot \mathbf{k})(\mathbf{a} \cdot \mathbf{a})],$$

which is zero if \mathbf{a} is collinear to \mathbf{k} .

5. Comparison with strain gradient and micropolar elasticity

5.1. Toupin–Mindlin strain gradient elasticity

Within the Toupin–Mindlin strain gradient elasticity, the deformation energy W_{TM} depends on strains $\boldsymbol{\varepsilon}$ and on its gradient (or on $\boldsymbol{\varepsilon}$ and $\nabla\nabla\mathbf{u}$) [34,35]. For an isotropic solid, W_{TM} takes the form

$$W_{\text{TM}} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\nabla\boldsymbol{\varepsilon} : \mathbf{D} : \nabla\boldsymbol{\varepsilon}, \quad (5.1)$$

where “:” and “::” are the double and triple dot products, respectively, and fourth- and sixth-order tensors of elastic moduli \mathbf{C} and \mathbf{D} have components

$$\begin{aligned} \mathbb{C}_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\ \mathbb{D}_{ijmkl} &= \frac{a_1}{2}(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{kl}\delta_{in}\delta_{jm}) + 2a_2\delta_{ij}\delta_{kl}\delta_{mn} \\ &\quad + \frac{a_3}{2}(\delta_{jk}\delta_{im}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{jl}\delta_{im}\delta_{kn}) \\ &\quad + a_4(\delta_{il}\delta_{jk}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn}) \\ &\quad + \frac{a_5}{2}(\delta_{jk}\delta_{in}\delta_{lm} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{jl}\delta_{km}\delta_{in} + \delta_{il}\delta_{km}\delta_{jn}), \end{aligned} \quad (5.2)$$

where δ_{ij} is Kronecker’s symbol, and a_1, a_2, a_3, a_4 and a_5 are five elastic moduli of higher order [34].

In this case, the equilibrium equations take the form

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times (\nabla \times \mathbf{u}) - \gamma_1\Delta\nabla\nabla \cdot \mathbf{u} + \gamma_2\Delta\nabla \times (\nabla \times \mathbf{u}) + \mathbf{f} = \mathbf{0}, \quad (5.4)$$

where $\gamma_1 = (\lambda + 2\mu)\ell_1^2$, $\gamma_2 = \mu\ell_2^2$, and ℓ_1, ℓ_2 are characteristic lengths given by

$$\ell_1 = \sqrt{\frac{a_1 + a_2 + a_3 + a_4 + a_5}{\lambda + 2\mu}}, \quad \ell_2 = \sqrt{\frac{a_3 + 2a_4 + a_5}{2\mu}}, \quad (5.5)$$

see [32, 34, 35] for more details.

Here the principal symbol takes the value

$$\mathbb{A}_{\text{TM}}(\mathbf{k}) = -\gamma_1(\mathbf{k} \cdot \mathbf{k})\mathbf{k} \otimes \mathbf{k} + \gamma_2(\mathbf{k} \cdot \mathbf{k})\mathbf{k} \times (\mathbf{k} \times \mathbf{I}). \quad (5.6)$$

It could be transformed into the more simple form

$$\mathbb{A}_{\text{TM}}(\mathbf{k}) = -(\mathbf{k} \cdot \mathbf{k})^2 [\gamma_1 \mathbf{i} \otimes \mathbf{i} + \gamma_2(\mathbf{i}_2 \otimes \mathbf{i}_2 + \mathbf{i}_3 \otimes \mathbf{i}_3)], \quad (5.7)$$

where \mathbf{i}_2 and \mathbf{i}_3 are two unit vectors orthogonal to $\mathbf{i} \equiv \mathbf{k}/\|\mathbf{k}\|$.

Since

$$\det \mathbb{A}_{\text{TM}}(\mathbf{k}) = -(\mathbf{k} \cdot \mathbf{k})^6 \gamma_1 \gamma_2,$$

the ordinary ellipticity is fulfilled if and only if $\gamma_1 \gamma_2 \neq 0$ or the two following inequalities are fulfilled

$$a_1 + a_2 + a_3 + a_4 + a_5 \neq 0, \quad a_3 + 2a_4 + a_5 \neq 0. \quad (5.8)$$

In other words, ordinary ellipticity requires that both characteristic lengths are nonzero.

Similarly, we get that

$$-\mathbf{a} \cdot \mathbb{A}_{\text{TM}}(\mathbf{k}) \cdot \mathbf{a} = (\mathbf{k} \cdot \mathbf{k}) [\gamma_1(\mathbf{a} \cdot \mathbf{k})^2 + \gamma_2((\mathbf{k} \cdot \mathbf{k})(\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{k})^2)]$$

So strong ellipticity condition (2.5) is satisfied if and only if $\gamma_1 > 0$ and $\gamma_2 > 0$ that is equivalent to [18]

$$a_1 + a_2 + a_3 + a_4 + a_5 > 0, \quad a_3 + 2a_4 + a_5 > 0. \quad (5.9)$$

Comparing equations (3.6) and (5.4), one can easily observe that (5.4) transforms into (3.6) if one assume that $\gamma_2 = \eta$ while $\gamma_1 = 0$. So the lack of ellipticity of (3.6) follows from the violation of the positivity of one characteristic length, that is, $\ell_1 = 0$, for the couple-stress theory.

Similar situation with ellipticity can be observed in another incomplete strain gradient model called dilatational strain gradient elasticity [15, 33]. Here a deformation energy is given by

$$W_{\text{dsg}} = \frac{\lambda}{2} \text{tr}^2 \boldsymbol{\varepsilon} + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \frac{\alpha}{2} \mathbf{k} \cdot \mathbf{k}, \quad (5.10)$$

where λ and μ are Lamé moduli and α is an additional dilatational elastic modulus. Corresponding equilibrium equation is given by

$$\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \nabla \cdot \mathbf{u} - \alpha \nabla \Delta (\nabla \cdot \mathbf{u}) + \mathbf{f} = \mathbf{0}. \quad (5.11)$$

Its principal symbol is given by the formula

$$\mathbb{A}_{\text{dsg}}(\mathbf{k}) = -\alpha(\mathbf{k} \cdot \mathbf{k})\mathbf{k}\mathbf{k}.$$

Obviously, $\mathbb{A}_{\text{dsg}}(\mathbf{k})$ is singular, and so, (5.11) is not elliptic. Comparing (5.11) with (5.4), one can see that here $\ell_2 = 0$.

5.2. Cosserat continuum

Within the linear micropolar elasticity translations \mathbf{u} and rotations $\boldsymbol{\varphi}$ are kinematically independent and $\boldsymbol{\varphi} \neq \boldsymbol{\phi}$, in general. A deformation energy W_{me} is a function of two strain measures defined as follows [19, 21]

$$\mathbf{e} = \nabla \mathbf{u} + \mathbf{I} \times \boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}. \quad (5.12)$$

For an isotropic material, W_{me} is given by

$$W_{\text{me}} = \frac{1}{2} \lambda \text{tr}^2 \mathbf{e} + (\mu + \kappa) \mathbf{e} : \mathbf{e} + \mu \mathbf{e} : \mathbf{e}^T + \frac{1}{2} \eta_1 \text{tr}^2 \boldsymbol{\kappa} + \frac{1}{2} \eta_2 \boldsymbol{\kappa} : \boldsymbol{\kappa} + \frac{1}{2} \eta_3 \boldsymbol{\kappa} : \boldsymbol{\kappa}^T, \quad (5.13)$$

where λ , μ , κ , η_1 , η_2 and η_3 are elastic moduli. Equilibrium equations take the form

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + (\mu + \kappa)\Delta\mathbf{u} + \kappa\nabla \times \boldsymbol{\varphi} + \mathbf{f} = \mathbf{0}, \quad (5.14)$$

$$(\eta_1 + \eta_2)\nabla\nabla \cdot \boldsymbol{\varphi} + \eta_2\Delta\boldsymbol{\varphi} + \kappa(\nabla \times \mathbf{u} - 2\boldsymbol{\varphi}) + \mathbf{c} = \mathbf{0}, \quad (5.15)$$

where \mathbf{f} and \mathbf{c} are vectors of volumetric forces and couples, respectively.

The principal symbol of system (5.14) and (5.15) has the form

$$\mathbb{A}_{\text{me}}(\mathbf{k}) = \begin{pmatrix} \mathbb{A}_{\text{me}}^{(1)}(\mathbf{k}) & \mathbf{0} \\ \mathbf{0} & \mathbb{A}_{\text{me}}^{(2)}(\mathbf{k}) \end{pmatrix},$$

$$\mathbb{A}_{\text{me}}^{(1)}(\mathbf{k}) = (\lambda + \mu)\mathbf{k} \otimes \mathbf{k} + (\mu + \kappa)(\mathbf{k} \cdot \mathbf{k})\mathbf{I}, \quad \mathbb{A}_{\text{me}}^{(2)}(\mathbf{k}) = (\eta_1 + \eta_3)\mathbf{k} \otimes \mathbf{k} + \eta_2(\mathbf{k} \cdot \mathbf{k})\mathbf{I}. \quad (5.16)$$

So ellipticity condition reduces to the inequality

$$\det \mathbb{A}_{\text{me}}^{(1)}(\mathbf{k}) \det \mathbb{A}_{\text{me}}^{(2)}(\mathbf{k}) \neq 0$$

that results in the following inequalities for the elastic moduli

$$\lambda + 2\mu + \kappa \neq 0, \quad \mu + \kappa \neq 0, \quad \eta_1 + \eta_2 + \eta_3 \neq 0, \quad \eta_3 \neq 0.$$

The strong ellipticity conditions have the form [2, 13, 19]

$$\lambda + 2\mu + \kappa > 0, \quad \mu + \kappa > 0, \quad \eta_1 + \eta_2 + \eta_3 > 0, \quad \eta_2 > 0. \quad (5.17)$$

Comparing W_{me} and W , one can see that they are coincide to each other if $\kappa = 0$, $\eta_1 = 0$, $\eta_2 = \eta$, $\eta_3 = \zeta$. Even in this case, (5.14) and (5.15) are strongly elliptic if and only if

$$\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta_2 > 0,$$

whereas (3.6) is not elliptic. So despite the obvious similarity between micropolar elasticity and couple-stress theory, the corresponding ellipticity conditions are different.

6. Conclusions and discussion

Within the linear couple-stress theory, we discussed the ellipticity and strong ellipticity conditions. It was shown that both conditions are violated. So equilibrium equation does not constitute nor strongly elliptic neither elliptic system of PDEs. This relates to a certain degeneration of a deformation energy within the Toupin–Mindlin strain gradient elasticity. Nevertheless, in order to get advantages of ellipticity one can consider more general conditions of ellipticity or transform somehow the problem under consideration. For example, let us consider the following transformation of (3.6). Let us represent displacement using the Helmholtz decomposition

$$\mathbf{u} = \nabla\psi - \nabla \times \boldsymbol{\Psi}, \quad \nabla \cdot \boldsymbol{\Psi} = 0, \quad (6.1)$$

where ψ and $\boldsymbol{\Psi}$ are potential of dilatation and rotation, respectively. Similarly we represent \mathbf{f} as a sum

$$\mathbf{f} = \nabla g - \nabla \times \mathbf{F}, \quad \nabla \cdot \mathbf{F} = 0. \quad (6.2)$$

Upon substituting (6.1) and (6.2) into (3.6), we obtain

$$\begin{aligned} & \mu\Delta\nabla\psi + (\lambda + \mu)\nabla\nabla \cdot \nabla\psi + \nabla g - \mu\Delta\nabla \times \boldsymbol{\Psi} - \eta\Delta\nabla \times (\nabla \times (\nabla \times \boldsymbol{\Psi})) - \nabla \times \mathbf{F} \\ & = \nabla [(\lambda + 2\mu)\Delta\psi + g] + \nabla \times [-\mu\Delta\boldsymbol{\Psi} + \eta\Delta^2\boldsymbol{\Psi} - \mathbf{F}] = \mathbf{0}. \end{aligned}$$

Thus, the problem under consideration can be reduced to the two equations

$$(\lambda + 2\mu)\Delta\psi + g = 0, \quad \eta\Delta^2\boldsymbol{\Psi} - \mu\Delta\boldsymbol{\Psi} - \mathbf{F} = \mathbf{0};$$

both of them are elliptic if and only if $\lambda + 2\mu \neq 0$ and $\eta \neq 0$, and strongly elliptic if $\lambda + 2\mu > 0$ and $\eta > 0$. Let us note that these inequalities constitute a part of ellipticity conditions for a basic simple material (without gradients of strain) and a general strain gradient elastic material.

So one should be aware to transmit ellipticity properties from a general model to a reduced one. For example, similarities between micropolar elasticity and couple-stress theory are obvious. Nevertheless, the ellipticity conditions are different since for micropolar elasticity equilibrium equations constitute a system of six scalar differential equations of second order, whereas within the couple-stress theory we have three differential equations of fourth order. Couple-stress theory could be also derived as a Cosserat continuum with constraint $\phi = \varphi$ [19, 38] that brings another mathematical form of equilibrium conditions, see also [4] for Lagrange multiplier technique to strain gradient and media with microstructure. On the other hand, even reduced model may become elliptic after certain transformations as was shown above.

In the literature, one can also find another gradient incomplete models [11], which are also result in non-elliptic but hypoelliptic differential operators [17]. Let us also note that for nonlinear models relations between ellipticity properties of different models could be less straightforward. For example, similar to micropolar elasticity and couple-stress theory, for a micromorphic medium and strain gradient material the ellipticity conditions are different, in general, but these models can demonstrate some similarities in behaviour, see [20]. The provided comparison could be also extended to more complex models of continua such as discussed in [8, 24, 25].

Acknowledgements

The first author acknowledges the support within the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant agreement EffectFact No 101008140 and support from the Deutsche Forschungsgemeinschaft DFG within the framework of Grant MU 1752/64-1 "Ein neuer Zugang zur Beschreibung struktureller Änderungen in Mikropolarmedien."

References

- [1] Agranovich, M.: Elliptic boundary problems. In: Agranovich, M., Egorov, Y., Shubin, M. (eds.) *Partial Differential Equations IX: Elliptic Boundary Problems*. Encyclopaedia of Mathematical Sciences, vol. 79, pp. 1–144. Springer, Berlin (1997)
- [2] Altenbach, H., Eremeyev, V.A., Lebedev, L.P., Rendón, L.A.: Acceleration waves and ellipticity in thermoelastic micropolar media. *Arch. Appl. Mech.* **80**(3), 217–227 (2010)
- [3] Berkache, K., Deogekar, S., Goda, I., Picu, R.C., Ganghoffer, J.F.: Identification of equivalent couple-stress continuum models for planar random fibrous media. *Continuum Mech. Thermodyn.* **31**(4), 1035–1050 (2019)
- [4] Bersani, A.M., dell'Isola, F., Seppecher, P.: *Lagrange Multipliers in Infinite Dimensional Spaces, Examples of Application*, pp. 1425–1432. Springer, Berlin (2020)
- [5] Bertram, A.: *Compendium on Gradient Materials*. Springer, Cham (2023)
- [6] Bertram, A., Forest, S. (eds.): *Mechanics of Strain Gradient Materials*. Springer, Cham (2020)
- [7] Bigoni, D., Gougiotis, P.A.: Folding and faulting of an elastic continuum. *Proc. R. Soc. A: Math. Phys. Eng. Sci.* **472**(2187), 20160,018 (2016)

- [8] Ciallella, A., Giorgio, I., Eugster, S.R., Rizzi, N.L., dell'Isola, F.: Generalized beam model for the analysis of wave propagation with a symmetric pattern of deformation in planar pantographic sheets. *Wave Motion* **113**, 102,986 (2022)
- [9] De Borst, R.: Simulation of strain localization: a reappraisal of the Cosserat continuum. *Eng. Comput.* **8**(4), 317–332 (1991)
- [10] De Borst, R., Muhlhaus, H.B.: Finite deformation analysis of inelastic materials with micro-structure. In: *Finite Inelastic Deformations-Theory and Applications*, pp. 313–322. Springer (1992)
- [11] dell'Isola, F., Steigmann, D.J.: *Discrete and Continuum Models for Complex Metamaterials*. Cambridge University Press, Cambridge (2020)
- [12] Dietsche, A., Steinmann, P., Willam, K.: Micropolar elastoplasticity and its role in localization. *Int. J. Plast.* **9**(7), 813–831 (1993)
- [13] Eremeyev, V.A.: Acceleration waves in micropolar elastic media. *Dokl. Phys.* **50**(4), 204–206 (2005)
- [14] Eremeyev, V.A.: Strong ellipticity conditions and infinitesimal stability within nonlinear strain gradient elasticity. *Mech. Res. Commun.* **117**, 103,782 (2021)
- [15] Eremeyev, V.A., Cazzani, A., dell'Isola, F.: On nonlinear dilatational strain gradient elasticity. *Continuum Mech. Thermodyn.* **33**(4), 1429–1463 (2021)
- [16] Eremeyev, V.A., Cloud, M.J., Lebedev, L.P.: *Applications of Tensor Analysis in Continuum Mechanics*. World Scientific, New Jersey (2018)
- [17] Eremeyev, V.A., dell'Isola, F., Boutin, C., Steigmann, D.: Linear pantographic sheets: existence and uniqueness of weak solutions. *J. Elast.* **132**(2), 175–196 (2018)
- [18] Eremeyev, V.A., Lazar, M.: Strong ellipticity within the Toupin–Mindlin first strain gradient elasticity theory. *Mech. Res. Commun.* p. 103944 (2022)
- [19] Eremeyev, V.A., Lebedev, L.P., Altenbach, H.: *Foundations of Micropolar Mechanics*. Springer-Briefs in Applied Sciences and Technologies. Springer, Heidelberg (2013)
- [20] Eremeyev, V.A., Reccia, E.: Nonlinear strain gradient and micromorphic one-dimensional elastic continua: comparison through strong ellipticity conditions. *Mech. Res. Commun.* p. 103909 (2022)
- [21] Eringen, A.C.: *Microcontinuum Field Theory. I. Foundations and Solids*. Springer, New York (1999)
- [22] Fichera, G.: *Linear Elliptic Differential Systems and Eigenvalue Problems*. Lecture Notes in Mathematics, vol. 8. Springer, Berlin (1965)
- [23] Ganghoffer, J.F., Goda, I., Novotny, A.A., Rahouadj, R., Sokolowski, J.: Homogenized couple stress model of optimal auxetic microstructures computed by topology optimization. *ZAMM-J. Appl. Math. Mech./Z. Angew. Math. Mech.* **98**(5), 696–717 (2018)
- [24] Giorgio, I., De Angelo, M., Turco, E., Misra, A.: A Biot–Cosserat two-dimensional elastic nonlinear model for a micromorphic medium. *Continuum Mech. Thermodyn.* **32**(5), 1357–1369 (2020)
- [25] Giorgio, I., dell'Isola, F., Misra, A.: Chirality in 2D Cosserat media related to stretch-micro-rotation coupling with links to granular micromechanics. *Int. J. Solids Struct.* **202**, 28–38 (2020)
- [26] Golmakani, M.E., Malikan, M., Pour, S.G., Eremeyev, V.A.: Bending analysis of functionally graded nanoplates based on a higher-order shear deformation theory using dynamic relaxation method. *Continuum Mech. Thermodyn.* pp. 1–20 (2021)
- [27] Gourgiotis, P.A., Bigoni, D.: Stress channelling in extreme couple-stress materials part I: strong ellipticity, wave propagation, ellipticity, and discontinuity relations. *J. Mech. Phys. Solids* **88**, 150–168 (2016)
- [28] Koiter, W.T.: Couple-stresses in the theory of elasticity. Pt I-II. *Proc. Koninkl. Neterland. Akad. Wetensch. B* **67**, 17–44 (1964)
- [29] Kong, S.: A review on the size-dependent models of micro-beam and micro-plate based on the modified couple stress theory. *Arch. Comput. Methods Eng.* **29**, 1–31 (2021)
- [30] Lakes, R.S.: Stability of Cosserat solids: size effects, ellipticity and waves. *J. Mech. Mater. Struct.* **13**(1), 83–91 (2018)
- [31] Lakes, R.S.: Softening of Cosserat sensitivity in a foam: warp effects. *Int. J. Mech. Sci.* **192**, 106,125 (2021)
- [32] Lazar, M., Po, G.: On Mindlin's isotropic strain gradient elasticity: green tensors, regularization, and operator-split. *J. Micromech. Mol. Phys.* **3**(03n04), 1840,008 (2018)
- [33] Lurie, S.A., Kalamkarov, A.L., Solyaev, Y.O., Volkov, A.V.: Dilatation gradient elasticity theory. *Eur. J. Mech.-A/Solids* **88**, 104,258 (2021)
- [34] Mindlin, R.D.: Micro-structure in linear elasticity. *Arch. Ration. Mech. Anal.* **16**(1), 51–78 (1964)
- [35] Mindlin, R.D., Eshel, N.N.: On first strain-gradient theories in linear elasticity. *Int. J. Solids Struct.* **4**(1), 109–124 (1968)
- [36] Mindlin, R.D., Tiersten, H.F.: Effects of couple-stresses in linear elasticity. *Arch. Ration. Mech. Anal.* **11**(1), 415–448 (1962)
- [37] Muhlhaus, H.B., Aifantis, E.C.: The influence of microstructure-induced gradients on the localization of deformation in viscoplastic materials. *Acta Mech.* **89**(1), 217–231 (1991)
- [38] Nowacki, W.: *Teoria Sprężystości*. PWN, Warszawa (1970)



- [39] Park, S.K., Gao, X.L.: Bernoulli–Euler beam model based on a modified couple stress theory. *J. Micromech. Microeng.* **16**(11), 2355 (2006)
- [40] Simmonds, J.G.: *A Brief on Tensor Analysis*, 2nd edn. Springer, New Yourk (1994)
- [41] Skrzat, A., Eremeyev, V.A.: On the effective properties of foams in the framework of the couple stress theory. *Continuum Mech. Thermodyn.* **32**(6), 1779–1801 (2020)
- [42] Soldatos, K.P.: On loss of ellipticity in second-gradient hyper-elasticity of fibre-reinforced materials. *Int. J. Non-Linear Mech.* **47**(2), 117–127 (2012)
- [43] Steinmann, P., Willam, K.: Localization within the framework of micropolar elasto-plasticity. In: *Advances in continuum mechanics*, pp. 296–313. Springer (1991)
- [44] Toupin, R.A.: Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **11**(1), 385–414 (1962)
- [45] Toupin, R.A.: Theories of elasticity with couple-stress. *Arch. Ration. Mech. Anal.* **17**(2), 85–112 (1964)
- [46] Vishik, M.I.: On strongly elliptic systems of differential equations. *Sb.: Math.* **29**(71), 615–657 (1951). (in Russian)
- [47] Wilson, E.B.: *Vector Analysis, Founded upon the Lectures of G. W. Gibbs*. Yale University Press, New Haven (1901)
- [48] Yang, F.A.C.M., Chong, A.C.M., Lam, D.C.C., Tong, P.: Couple stress based strain gradient theory for elasticity. *Int. J. Solids Struct.* **39**(10), 2731–2743 (2002)

Victor A. Eremeyev
 DICAAR
 University of Cagliari, Cagliari
 via Marengo 2
 09123 Cagliari
 Italy

Victor A. Eremeyev and Violetta Konopińska-Zmysłowska
 WILiŚ, Gdańsk University of Technology
 ul. Gabriela Narutowicza 11/12
 80-233 Gdańsk
 Poland
 e-mail: eremeyev.victor@gmail.com

Violetta Konopińska-Zmysłowska
 e-mail: violetta.konopinska@pg.edu.pl

Daria Scerrato
 Dipartimento di Scienze di Base ed Applicate per l'Ingegneria (SBAI)
 University of Rome La Sapienza
 00161 Rome
 Italy
 e-mail: daria.scerrato@uniroma1.it

