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A NUMERICAL STUDY ON THE DYNAMICS OF DENGUE DISEASE MODEL WITH FRACTIONAL PIECEWISE DERIVATIVE

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Abstract

The aim of this paper is to study the dynamics of Dengue disease model using a novel piecewise derivative approach in the sense of singular and non-singular kernels. The singular kernel operator is in the sense of Caputo, whereas the non-singular kernel operator is the Atangana–Baleanu Caputo operator. The existence and uniqueness of a solution with piecewise derivative are examined for the aforementioned problem. The suggested problem's approximate solution was found using the piecewise numerical iterative Newton polynomial approach. In terms of singular and non-singular kernels, a numerical scheme for piecewise derivatives has been established. The numerical simulation for the piecewise derivable problem under consideration is drawn using data for various fractional orders. This work makes the idea of piecewise derivatives and the dynamics of the crossover problem clearer.

Keywords: Dengue Disease; Piecewise Derivative; Caputo Operator; Atangana–Baleanu Operator.

1. INTRODUCTION

In the second part of the 20th century, medical research succeeded in terms of immunization, antibiotics, and improved living circumstances led to the expectation that infectious illnesses would be eradicated. As a result, in industrialized countries, efforts have been focused on diseases such as cancer. Infectious illnesses, however, continue to cause pain and death in underdeveloped nations around the turn of the century. Malaria, yellow fever, AIDS, Ebola, and other diseases will live on in the collective memory of mankind.

Among these diseases, Dengue fever, the most prevalent in Southeast Asia, is spreading across the globe, affecting nations with tropical and warm climates. It is spread to humans by the Aedes mosquito, and there are two types of dengue fever: basic dengue and Dengue Haemorrhagic Fever (DHF), which can progress to an extreme condition called Dengue Shock Syndrome (DSS). The fact that dengue is caused by four different serotypes classified as DEN1, DEN2, DEN3, and DEN4 which is a serious issue. A person who has been attacked by one of the four serotypes will never be infected by that serotype again (homologous immunity), but he will lose resistance to the other three serotypes in around 12 weeks (heterologous immunity), making him more susceptible to dengue haemorrhagic fever.

Dengue (Breakbone) fever is a mosquito-borne viral infection that has been rapidly spreading over the world. Dengue virus is the name given to the virus that causes dengue fever (DENV). A severe case of dengue causes significant sickness and death, although many cases of DENV generate relatively minor symptoms. Dengue fever has been linked to variety of symptoms. If a person has a high fever (40°C/104°F) and two of the symptoms/indications (severe headache, discomfort behind the eyes, muscle and joint aches, nausea, vomiting, swollen glands, and rash) during the febrile phase, dengue may be considered. For the time being, we must combat the illness by limiting vector transmission.[1,2] It is important to remember that this break bone fever is caused by a virus, with transmission occurring through the bite of female mosquitoes. In particular, Aedes albopictus and Aedes aegypti.[3,4] The broadcast has taken place when an infected human comes into contact with mosquitoes and becomes infected, the mosquito bites the sick person, infecting them and keeping them infected until death. In contrast to mosquitoes, infected people heal from their infections within a short period of time, and these healed people are unable to transfer the virus again to mosquitoes, allowing them to keep their immunity against transmission.[4–7]

Furthermore, environmental degradation, climate changes, filthy habitat, poverty, and uncontrolled urbanization are all favorable conditions for the spread of infectious diseases in general, and dengue fever in particular. The global frequency of dengue fever has risen considerably in recent decades. The illness has already spread to over 100 African and Latin American countries. The virus is wreaking havoc on Southeast Asia and the Western Pacific.

During dengue fever outbreaks, the infection rate among susceptibles is usually between 40% and 50%, but it can reach 80–90% under favorable geographic and environmental circumstances. Each year, around 500, 000 cases of dengue hemorrhagic fever necessitate hospitalization.

Dengue fever is a viral illness spread by mosquitoes of the genus Aedes. It is caused by the contact of susceptible people with any of the four serotypes. Aedes aegypti and Aedes albopictus are the two species of vectors that transmit dengue fever. The first is extremely anthropophilic, thriving in densely populated places and biting largely during the day, whilst the second is less anthropophilic and prefers to live in rural settings. As a result, dengue is important in two ways: (i) Even in the absence of deadly forms, the illness causes enormous economic and social costs due to its global dissemination and various serotypes (absenteeism, immobilization debilitation, medication). (ii) The danger of the disease evolving into a hemorrhagic form and dengue shock syndrome, both of which have large economic implications and can cause death.

Mathematical modeling has shown to be a useful approach for gaining a better understanding of certain diseases and developing treatment plans. The model's formulation and the feasibility of a simulation with parameter estimates enable sensitivity testing and conjuncture comparisons.[8] In case of dengue fever, the mathematical models we identified in the literature for dengue illness offer compartmental dynamics with Susceptible, Exposed, Infectious, and Removed compartments (immunized). SEIRS models[9] and[10] with only one virus or two viruses working concurrently[11] were examined in particular.

Fractional differential equations are naturally connected to memory storage in a variety of biological systems.[12,13] In the characterization of illnesses, memory effects played a key part.[14] If the memory effect was present in the previous episodes, it will have an impact on future illness descriptions. The history of illness description is shown by the distance of memory effects. As a result, fractional derivatives can be used to examine such memory effects on the contagious fatal illness.[15–18] In Ref. 19, the authors introduced a newly established fractional-order dengue mathematical model that is more reliable than the previous models. They studied the dynamical behavior and described the solution to the fractional dengue epidemic model (1).

$$\begin{aligned}
&{}_0^{\mathrm{ABC}}D_t^{\sigma}(\mathrm{S_H}(t)) \\
&\quad = \mathcal{B} - \frac{\vartheta_1 \mathrm{S_H}(t)\mathrm{I_H}(t) + \vartheta_2 \mathrm{S_H}(t)\mathrm{I_H}(t)}{\Lambda^H} - \nu_0 \mathrm{S_H}(t), \\
&{}_0^{\mathrm{ABC}}D_t^{\sigma}(\mathrm{I_H}(t)) \\
&\quad = -\frac{\vartheta_1 \mathrm{S_H}(t)\mathrm{I_H}(t) + \vartheta_2 \mathrm{S_H}(t)\mathrm{I_H}(t)}{\Lambda^H} \\
&\qquad - (\nu_0 + \nu_2 + \beta_1)\mathrm{I_H}(t), \\
&{}_0^{\mathrm{ABC}}D_t^{\sigma}(\mathrm{R_H}(t)) = \beta_1 \mathrm{I_H}(t) - \nu_0 \mathrm{R_H}(t), \\
&{}_0^{\mathrm{ABC}}D_t^{\sigma}(\mathcal{S_M}(t)) \\
&\quad = \gamma - \frac{\vartheta_3 \mathcal{S_M}(t)\mathrm{I_H}(t)}{\Lambda^H} - \nu_1 \mathcal{S_M}(t), \\
&{}_0^{\mathrm{ABC}}D_t^{\sigma}(\mathcal{I_M}(t)) \\
&\quad = \frac{\vartheta_3 \mathcal{S_M}(t)\mathcal{I_M}(t)}{\Lambda^H} - \nu_1 \mathcal{I_M}(t), \\
&\mathrm{S_H}(0) = \mathrm{S_{H0}} \geq 0, \quad \mathrm{I_H}(0) = \mathrm{I_{H0}} \geq 0, \\
&\mathrm{R_H}(0) = \mathrm{R_{H0}} \geq 0, \quad \mathcal{S_M}(0) = \mathcal{S}_{\mathcal{M}0} \geq 0, \\
&\mathcal{I_M}(0) = \mathcal{I}_{\mathcal{M}0} \geq 0, \quad 0 < \sigma \leq 1, \quad t \in [0, T].
\end{aligned} \tag{1}$$

Different operators, such as fractal derivative, non-integer order derivative with kernel of singularity and non-singularity, fractal-fractional operator, and other derivative operators, have been presented for the study of crossover issues.[20–24] Although the incorporation of randomness in the form of a stochastic equation results in more realistic results, the crossover dynamics remain unsolved. Many infectious disease models, heat movement, fluid flow, and many complicated advection issues all have this trait.[25,26] The exponential and Mittag-Leffler mappings in fractional calculus are unable to determine the timing of crossovers. As a result, one of the novel approaches of piecewise differentiation and integration has been developed in Ref. 27 to address such difficulties. They discussed the classical and global piecewise derivatives, as well as several applicable examples.

In this study, we reinterpret the model (1) for qualitative analysis as well as numerical iterative analysis in the sense of Caputo and Atangana–Baleanu piecewise derivative. The rest of this work is arranged as follows. The preliminaries are covered in Sec. 2, while the existence results are presented in Sec. 3. Section 4 presents the model analysis with the numerical approaches while simulations and discussion of the model are presented in Sec. 5. Section 6 concludes with a few concluding observations.

Equation (1) can be written in piecewise derivative in sense of singular and non-singular kernel as follows:

$$
\begin{aligned}
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathrm{S_H}(t)) \\
&\quad = \mathcal{B} - \frac{\vartheta_1 \mathrm{S_H}(t)\mathrm{I_H}(t) + \vartheta_2 \mathrm{S_H}(t)\mathrm{I_H}(t)}{\Lambda^H} \\
&\qquad - \nu_0 \mathrm{S_H}(t), \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathrm{I_H}(t)) \\
&\quad = -\frac{\vartheta_1 \mathrm{S_H}(t)\mathrm{I_H}(t) + \vartheta_2 \mathrm{S_H}(t)\mathrm{I_H}(t)}{\Lambda^H} \\
&\qquad - (\nu_0 + \nu_2 + \beta_1)\mathrm{I_H}(t), \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathrm{R_H}(t)) = \beta_1 \mathrm{I_H}(t) - \nu_0 \mathrm{R_H}(t), \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathcal{S_M}(t)) = \gamma - \frac{\vartheta_3 \mathcal{S_M}(t)\mathrm{I_H}(t)}{\Lambda^H} - \nu_1 \mathcal{S_M}(t), \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathcal{I_M}(t)) = \frac{\vartheta_3 \mathcal{S_M}(t)\mathcal{I_M}(t)}{\Lambda^H} - \nu_1 \mathcal{I_M}(t), \\
&\mathrm{S_H}(0) = \mathrm{S_{H0}} \geq 0, \quad \mathrm{I_H}(0) = \mathrm{I_{H0}} \geq 0, \\
&\mathrm{R_H}(0) = \mathrm{R_{H0}} \geq 0, \quad \mathcal{S_M}(0) = \mathcal{S_{M0}} \geq 0, \\
&\mathcal{I_M}(0) = \mathcal{I_{M0}} \geq 0, \quad 0 < \sigma \leq 1, \quad t \in [0, T]. \qquad (2)
\end{aligned}
$$

In detail, we can write Eq. (2) as

$$
\begin{aligned}
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathrm{S_H}(t)) \\
&\quad = \begin{cases} {}^{C}_{0}D_t^{\sigma}(\mathrm{S_H}(t)) = {}^{C}\mathcal{F}_1(\mathrm{S_H}, t), \\ {}^{ABC}_{\ 0}D_t^{\sigma}(\mathrm{S_H}(t)) = {}^{ABC}\mathcal{F}_1(\mathrm{S_H}, t), \end{cases} \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathrm{I_H}(t)) \\
&\quad = \begin{cases} {}^{C}_{0}D_t^{\sigma}(\mathrm{I_H}(t)) = {}^{C}\mathcal{F}_2(\mathrm{I_H}, t), \\ {}^{ABC}_{\ 0}D_t^{\sigma}(\mathrm{I_H}(t)) = {}^{ABC}\mathcal{F}_2(\mathrm{I_H}, t), \end{cases} \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathrm{R_H}(t)) \\
&\quad = \begin{cases} {}^{C}_{0}D_t^{\sigma}(\mathrm{R_H}(t)) = {}^{C}\mathcal{F}_3(\mathrm{R_H}, t), \\ {}^{ABC}_{\ 0}D_t^{\sigma}(\mathrm{R_H}(t)) = {}^{ABC}\mathcal{F}_3(\mathrm{R_H}, t), \end{cases} \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathcal{S_M}(t)) \\
&\quad = \begin{cases} {}^{C}_{0}D_t^{\sigma}(\mathcal{S_M}(t)) = {}^{C}\mathcal{F}_4(\mathcal{S_M}, t), \\ {}^{ABC}_{\ 0}D_t^{\sigma}(\mathcal{S_M}(t)) = {}^{ABC}\mathcal{F}_4(\mathcal{S_M}, t), \end{cases} \\
&{}^{CABC}_{\ 0}D_t^{\sigma}(\mathcal{I_M}(t)) \\
&\quad = \begin{cases} {}^{C}_{0}D_t^{\sigma}(\mathcal{I_M}(t)) = {}^{C}\mathcal{F}_5(\mathcal{I_M}, t), \\ {}^{ABC}_{\ 0}D_t^{\sigma}(\mathcal{I_M}(t)) = {}^{ABC}\mathcal{F}_5(\mathcal{I_M}, t), \end{cases} \qquad (3)
\end{aligned}
$$

with

$$0 < t \leq t_1, \quad t_1 < t \leq T,$$

where ${}^{C}_{0}D_t^{\sigma}$ and ${}^{C}_{0}D_t^{\sigma}$ are Caputo and ABC derivatives, respectively.

2. PRELIMINARIES

In this section, we will give some preliminary definition of Caputo and ABC fractional derivative and integrals.

Definition 1. The ABC derivative of a function $U(t)$ under the condition $U(t) \in \mathcal{H}^1(0, \tau)$ is defined as follows:

$$
\begin{aligned}
{}^{ABC}{}_0D_t^{\sigma}(U(t)) = \frac{\mathrm{ABC}(\sigma)}{1-\sigma} \int_0^t \frac{d}{d\zeta} U(\zeta) E_\sigma \\
\times \left[\frac{-\sigma}{1-\sigma}(t-\zeta)^\sigma\right] d\zeta. \qquad (4)
\end{aligned}
$$

Replace $E_\sigma[\frac{-\sigma}{1-\sigma}(t-\zeta)^\zeta]$ by $E_1 = \exp[\frac{-\sigma}{1-\sigma}(t-\zeta)]$, in (4), to get the Caputo–Fabrizio differential operator. Next, it is notified that

$$ {}^{ABC}{}_0D_t^{\sigma}[\text{constant}] = 0. $$

Here, $\mathcal{ABC}(\varrho)$ is called normalization operator which is formulated as $\mathrm{ABC}(0) = \mathrm{ABC}(1) = 1$. Also E_σ represents the special function known as Mittag-Leffler function, which is the generalization of the exponential function.

Definition 2. Let $U(t) \in L[0, T]$, then the fractional integral in $\mathbb{ABC}$ sense as

$$
\begin{aligned}
{}^{ABC}{}_0I_t^{\sigma}U(t) = \frac{1-\sigma}{\mathrm{ABC}(\sigma)} U(t) + \frac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)} \\
\times \int_0^t (t-\zeta)^{\sigma-1} U(\zeta) d\zeta. \qquad (5)
\end{aligned}
$$

Definition 3. Consider $U(t)$, for the definition of arbitrary order derivative in Caputo sense with respect to t as

$$ {}^{C}_{0}D_t^{\sigma}U(t) = \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-\zeta)^{n-\sigma-1}[U'(\zeta)] d\zeta. $$

Definition 4. Consider $U(t)$ for the definition Caputo integration with respect to t as

$$ {}^{C}_{0}I_t^{\sigma}U(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\zeta)^{\sigma-1} d\zeta, \quad \sigma > 0, $$

having converging integral.

Definition 5. Consider $U(t)$ differentiable and $g(t)$ is increasing function then for the definition of classical piecewise derivative[27] as

$$ {}_{0}^{PG}D_t U(t) = \begin{cases} U(t), & 0 < t \leq t_1, \\ \dfrac{U'(t)}{g'(t)}, & t_1 < t \leq T, \end{cases} $$

where ${}_{0}^{PG}D_t U(t)$ is for classical derivative for $0 < t \leq t_1$ and global derivative for $t_1 < t \leq T$.

Definition 6. Consider $U(t)$ differentiable and $g(t)$ is increasing function then for the definition of classical piecewise integration[27] as

$$ {}_{0}^{PG}I_t U(t) = \begin{cases} \displaystyle\int_0^t U(\tau)d\tau, & 0 < t \leq t_1, \\ \displaystyle\int_{t_1}^t U(\tau)g'(\tau)d(\tau), & t_1 < t \leq T, \end{cases} $$

where ${}_{0}^{PG}I_t U(t)$ is for classical integration for $0 < t \leq t_1$ and global integration for $t_1 < t \leq T$.

Definition 7. Consider $U(t)$ differentiable then for the definition of classical and fractional piecewise derivative[27] as

$$ {}_{0}^{PC}D_t^{\sigma} U(t) = \begin{cases} U'(t), & 0 < t \leq t_1, \\ {}_{0}^{C}D_t^{\sigma} U(t), & t_1 < t \leq T, \end{cases} $$

where ${}_{0}^{PC}D_t^{\sigma} U(t)$ is classical derivative for $0 < t \leq t_1$ and fractional derivative for $t_1 < t \leq T$

Definition 8. Consider $U(t)$ differentiable then for the definition of classical and fractional piecewise integration[27] as

$$ {}_{0}^{PC}I_t U(t) = \begin{cases} \displaystyle\int_0^t U(\tau)d\tau, & 0 < t \leq t_1, \\ \dfrac{1}{\Gamma\sigma} \displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1} \\ \quad \times U(\zeta)d(\zeta), & t_1 < t \leq T, \end{cases} $$

where ${}_{0}^{PC}I_t U(t)$ is for classical integration for $0 < t \leq t_1$ and Caputo integration for $t_1 < t \leq T$.

Definition 9. Consider $U(t)$ differentiable then for the definition of Caputo and ABC fractional piecewise derivative[27] as

$$ {}_{0}^{PC\mathrm{ABC}}D_t^{\sigma} U(t) = \begin{cases} {}_{0}^{C}D_t^{\sigma} U(t), & 0 < t \leq t_1, \\ {}_{0}^{\mathrm{ABC}}D_t^{\sigma} U(t), & t_1 < t \leq T, \end{cases} $$

where ${}_{0}^{PC\mathrm{ABC}}D_t^{\sigma} U(t)$ is Caputo derivative for $0 < t \leq t_1$ and fractional ABC derivative for $t_1 < t \leq T$.

Definition 10. Consider $U(t)$ differentiable then for the definition of fractional Caputo and fractional ABC piecewise integration[27] as

$$ {}_{0}^{PC\mathrm{ABC}}I_t U(t) = \begin{cases} \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1} \\ \quad \times U(\zeta)d(\zeta), & 0 < t \leq t_1, \\ \dfrac{1-\sigma}{\mathrm{ABC}(\sigma)} U(t) \\ \quad + \dfrac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)} \\ \displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1} U(\zeta)d(\zeta), & t_1 < t \leq T, \end{cases} $$

where ${}_{0}^{PC\mathrm{ABC}}I_t U(t)$ is for Caputo singular kernel integration for $0 < t \leq t_1$ and ABC integration for $t_1 < t \leq T$.

Lemma 11. *The solution of piecewise derivable equation is*

$$ {}_{0}^{PC\mathrm{ABC}}D_t^{\sigma} \mathrm{R_H}(t) = H(t, \mathrm{R_H}(t)), \quad 0 < \sigma \leq 1 $$

is

$$ \mathrm{R_H}(t) = \begin{cases} \mathrm{R_{H0}} + \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^t H(\zeta, \mathrm{R_H}(\zeta)) \\ \quad \times (t-\zeta)^{\sigma-1} d\zeta, \quad 0 < t \leq t_1, \\ \mathrm{R_H}(t_1) + \dfrac{1-\sigma}{\mathrm{ABC}(\sigma)} H(t, \mathrm{R_H}(t)) \\ \quad + \dfrac{\sigma}{\mathrm{ABC}\sigma\Gamma(\sigma)} \\ \displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1} H(\zeta \mathrm{R_H}(\zeta)) d(\zeta), \\ \quad t_1 < t \leq T \end{cases} $$

3. QUALITATIVE ANALYSIS

In this section, we find the existence as well as the uniqueness of the proposed model (3) in piecewise concept. Now we will find the existence of solution along with unique solution property of the considered piecewise derivable function. For this we can write the system (3) as given in Lemma 11 and by further description we write as follows:

$$ {}_{0}^{PC\mathrm{ABC}}D_t^{\varrho} \mathrm{Z}(t) = \mathcal{F}(t, \mathrm{Z}(t)), \quad 0 < \varrho \leq 1 $$

is

$$
\mathrm{Z}(t)=\begin{cases}\mathrm{Z}_0+\dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t \mathcal{F}(\mu,\mathrm{Z}(\mu,))\\ (t-\mu,)^{\mu,-1}d\mu, \quad 0<t\le t_1,\\ \mathrm{Z}(t_1)+\dfrac{1-\sigma}{\mathrm{ABC}(\sigma)}\mathcal{F}(t,\mathrm{Z}(t))\\ \quad +\dfrac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)}\\ \displaystyle\int_{t_1}^t (t-\mu)^{\mu-1}\mathcal{F}(\mu,\mathrm{Z}(\mu,))d\mu,\\ \quad t_1<t\le T,\end{cases} \tag{6}
$$

where

$$
\mathrm{Z}(t)=\begin{cases}\mathrm{S_H}(t),\\ \mathrm{I_H}(t),\\ \mathrm{R_H}(t),\\ \mathcal{S}_{\mathcal{M}}(t),\\ \mathcal{I}_{\mathcal{M}}(t),\end{cases} \quad \mathrm{Z}_0=\begin{cases}\mathrm{S_{H0}},\\ \mathrm{I_{H0}},\\ \mathrm{R_{H0}},\\ \mathcal{S}_{\mathcal{M}0},\\ \mathcal{I}_{\mathcal{M}0},\end{cases} \tag{7}
$$

$$
\mathrm{Z}(t_1)=\begin{cases}\mathrm{S_H}(t_1),\\ \mathrm{I_H}(t_1),\\ \mathrm{R_H}(t_1),\\ \mathcal{S}_{\mathcal{M}}(t_1),\\ \mathcal{I}_{\mathcal{M}}(t_1),\end{cases}
$$

$$
\mathcal{F}(t,\mathrm{Z}(t)) = \begin{cases}\mathcal{F}_1=\begin{cases}{}^{C}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\\ {}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\end{cases},\\ \mathcal{F}_2=\begin{cases}{}^{C}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\\ {}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\end{cases},\\ \mathcal{F}_3=\begin{cases}{}^{C}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\\ {}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\end{cases},\\ \mathcal{F}_4=\begin{cases}{}^{C}\mathcal{F}_4(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\\ {}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\end{cases},\\ \mathcal{F}_5=\begin{cases}{}^{C}\mathcal{F}_5(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t)\\ {}^{\mathrm{ABC}}\mathcal{F}_5(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_{\mathcal{M}},\mathcal{I}_{\mathcal{M}},t).\end{cases}\end{cases} \tag{8}
$$

We take $0<t\le T<\infty$ with a Banach space defined as $E_1=C[0,T]$ with a norm

$$
\|\mathrm{Z}\|=\max_{t\in[0,T]}|\mathrm{Z}(t)|.
$$

To obtain our result, we suppose growth condition on a nonlinear operator as follows:

(C1) $\exists\,\mathcal{L}_{\mathrm{Z}}>0;\ \forall\,\mathcal{F},\ \bar{\mathrm{Z}}\in E$ we have

$$
|\mathcal{F}(t,\mathrm{Z})-\mathcal{F}(t,\bar{\mathrm{Z}})|\le\mathcal{L}_{\mathcal{F}}|\mathrm{Z}-\bar{\mathrm{Z}}|,
$$

(C2) $\exists\,C_{\mathcal{F}}>0\,\&\,M_{\mathcal{F}}>0,$

$$
|\mathcal{F}(t,\mathrm{Z}(t))|\le C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}.
$$

If $\mathcal{F}$ be piecewise continuous on sub-interval $0<t\le t_1$ and $t_1<t\le T$ on $[0,T]$, also obeying $(C2)$, then piecewise problem (3) has ≥ 1 solution on each sub-interval.

Proof. Let us define a closed subset in both subintervals of $0,T$ as $\mathbb{B}$ and E as E using the Schauder fixed-point theorem.

$$
B=\{\mathrm{Z}\in E:\|\mathrm{Z}\|\le R_{1,2},R>0\}.
$$

Next consider an operator $\mathcal{T}:\mathbb{B}\to\mathbb{B}$ and applying (6) as

$$
\mathcal{T}(\mathrm{Z})=\begin{cases}\mathrm{Z}_0+\dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}\mathcal{F}(\mu,\mathbb{U}(\mu))(t-\mu)^{\mu-1}d\mu,\\ \quad 0<t\le t_1,\\ \mathrm{Z}(t_1)+\dfrac{1-\rho}{\mathrm{ABC}(\rho)}\mathcal{F}(t,\mathrm{Z}(t))\\ \quad +\dfrac{\rho}{\mathrm{ABC}(\rho)\Gamma(\rho)}\\ \displaystyle\int_{t_1}^t (t-\mu)^{\rho-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d(\mu),\\ \quad t_1<t\le T,\end{cases} \tag{9}
$$

On any $\mathrm{Z}\in B$, we get

$$
|\mathcal{T}(\mathrm{Z})(t)|\le\begin{cases}|\mathrm{Z}_0|+\dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\mu)^{\sigma-1}\\ \quad\times\ |\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu,\\ |\mathrm{Z}_{(t_1)}|+\dfrac{1-\sigma}{\mathrm{ABC}(\sigma)}\\ \quad\times\ |\mathcal{F}(t,\mathrm{Z}(t))|\\ \quad+\dfrac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)}\displaystyle\int_{t_1}^t(t-\mu)^{\sigma-1}\\ \quad\times\ |\mathcal{F}(\mu\mathrm{Z}(\mu))|d(\mu),\end{cases}
$$

$$\leq \begin{cases} |\mathrm{Z}_0| + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\mu)^{\sigma-1}[C_{\mathcal{F}}|\mathrm{Z}| \\ \quad + M_{\mathcal{F}}]d\mu, \\ |\mathrm{Z}_(t_1)| + \dfrac{1-\sigma}{\mathrm{ABC}(\sigma)} \\ \quad \times\ [C_{\mathcal{F}}|\mathrm{Z}| + M_{\mathcal{F}}] + \dfrac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)} \\ \quad \times \displaystyle\int_{t_1}^{t}(t-\mu)^{\sigma-1}[C_{\mathcal{F}}|\mathrm{Z}| + M_{\mathcal{F}}]d(\mu), \end{cases}$$

$$\leq \begin{cases} |\mathrm{Z}_0| + \dfrac{\mathbf{T}^{\sigma}}{\Gamma(\sigma+1)}[C_H|\mathbb{U}| \\ \quad + M_{\mathcal{F}}] = R_1, \quad 0 < t \leq t_1, \\ |\mathrm{Z}_(t_1)| + \dfrac{1-\sigma}{\mathrm{ABC}(\sigma)}[C_{\mathcal{F}}|\mathrm{Z}| + M_{\mathcal{F}}] \\ \quad + \ \dfrac{\sigma(T-\mathbf{T})^{\sigma}}{\mathrm{ABC}(\sigma)\Gamma\sigma+1}[C_{\mathcal{F}}|\mathrm{Z}| \\ \quad + M_{\mathcal{F}}]d(\mu) = R_2, \quad t_1 < t \leq T, \end{cases}$$

$$\leq \begin{cases} R_1, & 0 < t \leq t_1, \\ R_2, & t_1 < t \leq T. \end{cases}$$

From the above equation, as $\mathrm{Z} \in \mathbf{B}$. Thus, $\mathcal{T}(\mathbf{B}) \subset \mathbf{B}$. Hence, it proves that $\mathcal{T}$ is close and complete. Further to show the complete continuity we can write as we take $t_i < t_j \in [0, t_1]$ as first interval in Caputo sense, consider

$$\begin{aligned} &|\mathcal{T}(\mathrm{Z})(t_j) - \mathcal{T}(\mathrm{Z})(t_i)| \\ &= \left|\frac{1}{\Gamma(\sigma)}\int_0^{t_j}(t_j-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu,\right. \\ &\quad \left. -\frac{1}{\Gamma(\sigma)}\int_0^{t_i}(t_i-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu\right| \\ &\leq \frac{1}{\Gamma(\sigma)}\int_0^{t_i}[(t_i-\mu)^{\sigma-1} - (t_j-\mu)^{\sigma-1}], \\ &|\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu + \frac{1}{\Gamma(\sigma)}\int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}| \\ &\quad \times \mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\leq \frac{1}{\Gamma(\sigma)}\left[\int_0^{t_i}[(t_i-\mu)^{\sigma-1} - (t_j-\mu)^{\sigma-1}]d\mu\right. \\ &\quad \left. + \int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}d\mu\right](C_H|\mathrm{Z}| + M_{\mathcal{F}}) \\ &\leq \frac{(C_{\mathcal{F}}\mathrm{Z} + M_{\mathcal{F}})}{\Gamma(\sigma+1)}[t_j^{\sigma} - t_i^{\sigma} + 2(t_j-t_i)^{\sigma}]. \end{aligned}$$

Next (10), we obtain $t_i \to t_j$, then

$$|\mathcal{T}(\mathrm{Z})(t_j) - \mathcal{T}(\mathrm{Z})(t_i)| \to 0, \quad \text{as } t_i \to t_j.$$

So $\mathcal{T}$ is equicontinuous in $[0, t_1]$ interval. Next we consider other interval $t_i, t_j \in [t_1, T]$ in the ABC sense as

$$\begin{aligned} &|\mathcal{T}(\mathrm{Z})(t_j) - \mathcal{T}(\mathrm{Z})(t_i)| \\ &= \left|\frac{1-\sigma}{\mathrm{ABC}(\sigma)}\mathcal{F}(t,\mathrm{Z}(t)) + \frac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)}\right. \\ &\quad \times \int_{t_1}^{t_j}(t_j-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu, \\ &\quad - \frac{1-\sigma}{\mathrm{ABC}(\sigma)}\mathcal{F}(t,\mathrm{Z}(t)) + \frac{(\sigma)}{\mathrm{ABC}(\sigma)\Gamma(\sigma)} \\ &\quad \left. \times \int_{t_1}^{t_i}(t_i-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu\right| \\ &\leq \frac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)}\int_{t_1}^{t_i}[(t_i-\mu)^{\sigma-1} \\ &\quad - (t_j-\mu)^{\sigma-1}]|\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\quad + \frac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)}\int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1} \\ &\quad \times |\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\leq \frac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)}\left[\int_{t_1}^{t_i}[(t_i-\mu)^{\sigma-1}\right. \\ &\quad \left. - (t_j-\mu)^{\sigma-1}]d\mu + \int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}d\mu\right] \\ &\quad \times (C_{\mathcal{F}}|\mathrm{Z}| + M_{\mathcal{F}}) \\ &\leq \frac{\sigma(C_{\mathcal{F}}\mathrm{Z} + M_{\mathcal{F}})}{\mathrm{ABC}(\sigma)\Gamma(\sigma+1)}[t_j^{\sigma} - t_i^{\sigma} + 2(t_j-t_i)^{\sigma}]. \end{aligned}$$

Next as (10), we obtain $t_i \to t_j$, then

$$|\mathcal{T}(\mathrm{Z})(t_j) - \mathcal{T}(\mathrm{Z})(t_i)| \to 0, \quad \text{as } t_i \to t_j.$$

So $\mathcal{T}$ shows the equicontinuity in $[t_1, T]$ interval. Therefore, $\mathcal{T}$ is equicontinuous mapping. The operator $\mathcal{T}$ is completely continuous, uniformly continuous, and has bounds, according to the Arzel'a–Ascoli theorem. As a result, the piecewise derivable problem (3) has $a \geq 1$ solution on each sub-interval, according to Schauder's fixed-point Theorem. □

With $(C1)$, the proposed model has unique root if $\mathcal{T}$ be a contraction operator.

Proof. As we have taken an operator $\mathcal{T} : \mathbf{B} \to \mathbf{B}$ piecewise continuous, take Z and $\bar{\mathrm{Z}} \in B$ on $[0, t_1]$ in

Caputo sense as

$$\|\mathcal{T}(\mathrm{Z})-\mathcal{T}(\bar{\mathrm{Z}})\| = \max_{t\in[0,t_1]}\left|\frac{1}{\Gamma(\sigma)}\int_0^t (t-\mu)^{\sigma-1} \times \mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu - \frac{1}{\Gamma(\sigma)} \times \int_0^t (t-\mu)^{\sigma-1}\mathcal{F}(\mu,\bar{\mathrm{Z}}(\mu))d\mu\right| \leq \frac{\mathbf{T}^\sigma}{\Gamma(\sigma+1)}L_{\mathcal{F}}\|\mathrm{Z}-\bar{\mathrm{Z}}\|. \tag{10}$$

From (10), we have

$$\|\mathcal{T}(\mathrm{Z})-\mathcal{T}(\bar{\mathrm{Z}})\| \leq \frac{\mathbf{T}^\sigma}{\Gamma(\sigma+1)}L_{\mathcal{F}}\|\mathrm{Z}-\bar{\mathrm{Z}}\|. \tag{11}$$

So $\mathcal{T}$ is contracted. Therefore, finally in sense of Banach contraction theorem, the considered problem has unique solution in given sub-interval. Next for the other interval $t \in [t_1, T]$ in the sense of ABC derivative as

$$\|\mathcal{T}(\mathrm{Z})-\mathcal{T}(\bar{\mathrm{Z}})\| \leq \frac{1-\sigma}{\mathrm{ABC}(\sigma)}L_{\mathcal{F}}\|\mathrm{Z}-\bar{\mathrm{Z}}\| + \frac{\sigma(\mathbf{T}-T^\sigma)}{\mathrm{ABC}(\sigma)\Gamma(\sigma+1)}L_{\mathcal{F}}\|\mathrm{Z}-\bar{\mathrm{Z}}\| \tag{12}$$

or

$$\|\mathcal{T}(\mathrm{Z})-\mathcal{T}(\bar{\mathrm{Z}})\| \leq L_{\mathcal{F}}\left[\frac{1-\sigma}{\mathrm{ABC}(\sigma)} + \frac{\sigma(T-\mathbf{T})^\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma+1)}\right] \times \|\mathrm{Z}-\bar{\mathrm{Z}}\|. \tag{13}$$

Therefore, $\mathcal{T}$ is contracted. Therefore, finally in sense of Banach contraction theorem, the considered problem has unique solution in given sub-interval. So by Eqs. (11) and (13), the piecewise derivable problem has unique solution on each sub-intervals. □

4. NUMERICAL SCHEME FOR PIECEWISE DENGUE 5D DYNAMICAL MODEL WITH FRACTIONAL ORDER

Next we will establish a numerical scheme for the proposed piecewise differentiable problem (3). We will develop a numerical scheme for the two sub-interval of $[0, T]$, in Caputo and ABC sense. We will take help from the piecewise derivative integer order numerical scheme as in Ref. 27. We apply the piecewise integration to Eq. (3) for Caputo and ABC format as follows:

$$\mathrm{S_H}(t) = \begin{cases} \mathrm{S_H}(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\tau)^{\sigma-1c} \times \mathcal{F}_1(t,\mathrm{S_H})d\tau, \\ \mathrm{S_H}(t_1) + \dfrac{1-\sigma}{AB(\sigma)} \times \mathcal{F}_1(t,\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_\mathcal{M},\mathcal{I}_\mathcal{M}) + \dfrac{\sigma}{AB(\sigma)\Gamma(\sigma)}\displaystyle\int_{t_1}^{t}(t-\tau)^{\sigma-1} \times \mathcal{F}_1(t,\mathrm{S_H})d\tau \end{cases}$$

$$\mathrm{I_H}(t) = \begin{cases} \mathrm{I_H}(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\tau)^{\sigma-1c} \times \mathcal{F}_2(t,\mathrm{I_H})d\tau, \\ \mathrm{I_H}(t_1) + \dfrac{1-\sigma}{AB(\sigma)} \times \mathcal{F}_2(t,\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_\mathcal{M},\mathcal{I}_\mathcal{M}) + \dfrac{\sigma}{AB(\sigma)\Gamma(\sigma)}\displaystyle\int_{t_1}^{t}(t-\tau)^{\sigma-1} \times \mathcal{F}_2(t,\mathrm{I_H})d\tau \end{cases}$$

$$\mathrm{R_H}(t) = \begin{cases} \mathrm{R_H}(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\tau)^{\sigma-1c} \times \mathcal{F}_3(t,\mathrm{R_H})d\tau, \\ \mathrm{R_H}(t_1) + \dfrac{1-\sigma}{AB(\sigma)} \times \mathcal{F}_3(t,\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_\mathcal{M},\mathcal{I}_\mathcal{M}) + \dfrac{\sigma}{AB(\sigma)\Gamma(\sigma)}\displaystyle\int_{t_1}^{t}(t-\tau)^{\sigma-1} \times \mathcal{F}_3(t,\mathrm{R_H})d\tau \end{cases}$$

$$\mathcal{S}_\mathcal{M}(t) = \begin{cases} \mathcal{S}_\mathcal{M}(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\tau)^{\sigma-1c} \times \mathcal{F}_4(t,\mathcal{S}_\mathcal{M})d\tau, \\ \mathcal{S}_\mathcal{M}(t_1) + \dfrac{1-\sigma}{AB(\sigma)} \times \mathcal{F}_4(t,\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S}_\mathcal{M},\mathcal{I}_\mathcal{M}) + \dfrac{\sigma}{AB(\sigma)\Gamma(\sigma)}\displaystyle\int_{t_1}^{t}(t-\tau)^{\sigma-1} \times \mathcal{F}_4(t,\mathcal{S}_\mathcal{M})d\tau \end{cases}$$

$$\mathcal{I}_{\mathcal{M}}(t) = \begin{cases} \mathcal{I}_{\mathcal{M}}(0) + \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^{t_1} (t-\tau)^{\sigma-1c} \\ \qquad \times \mathcal{F}_5(t, \mathcal{I}_{\mathcal{M}}) d\tau, \\ \mathcal{I}_{\mathcal{M}}(t_1) + \dfrac{1-\sigma}{AB(\sigma)} \\ \qquad \times \mathcal{F}_5(t, \mathrm{S_H}, \mathrm{I_H}, \mathrm{R_H}, \mathcal{S}_{\mathcal{M}}, \mathcal{I}_{\mathcal{M}}) \\ \quad + \dfrac{\sigma}{AB(\sigma)\Gamma(\sigma)} \displaystyle\int_{t_1}^{t} (t-\tau)^{\sigma-1} \\ \qquad \times \mathcal{F}_5(t, \mathcal{I}_{\mathcal{M}}) d\tau \end{cases} \tag{14}$$

with $0 < t \le t_1$, $t_1 < t \le T$ where ${}^{C}\mathcal{F}_i(t) = {}^{C}\mathcal{F}_i(\mathrm{S_H}, \mathrm{I_H}, \mathrm{R_H}, \mathcal{S}_{\mathcal{M}}, \mathcal{I}_{\mathcal{M}}, t)$ and ${}^{\mathrm{ABC}}\mathcal{F}_i(t) = {}_i^{\mathrm{ABC}}\mathcal{F}(\mathrm{S_H}, \mathrm{I_H}, \mathrm{R_H}, \mathcal{S}_{\mathcal{M}}, \mathcal{I}_{\mathcal{M}}, t)$ are the left-hand side of Eq. (14) for $i = 1, 2, 3, 4$, also given in Eq. (3). We will derive the scheme for system (14) and the same procedure will be for the rest of the compartments.

At $t = t_{n+1}$

$$\mathrm{S_H}(t_{n+1})) = \begin{cases} \mathrm{S_{H0}} + \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^{t_1} (t-\zeta)^{\sigma-1} \\ \qquad \times {}^{C}\mathcal{F}_1(\mathrm{S_H}, t) d\zeta, \\ \mathrm{S_H}(t_1) + \dfrac{1-\sigma}{\mathrm{ABC}(\sigma)} {}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}, t_n) \\ \qquad + \dfrac{\sigma}{\mathrm{ABC}(\sigma)\Gamma(\sigma)} \\ \qquad \times \displaystyle\int_{t_1}^{t_{n+1}} (t-\zeta)^{\sigma-1} \\ \qquad \times {}^{\mathrm{ABC}}\mathcal{F}_1(\zeta) d\zeta, \quad t_1 < t \le T, \end{cases} \tag{15}$$

We write Eq. (15) in the Newton interpolation approximation given in Ref. 27 as follows:

$$\mathrm{S_H}(t_{n+1})) = \begin{cases} \mathrm{S_{H0}} + \left\{ \begin{aligned} & \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \\ & \quad \times \Pi + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}\mathrm{K}-1, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\ & \quad \left. - {}^{C}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \\ & \quad \times \sum + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\ & \quad - 2{}^{C}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ & \quad \left. + {}^{C}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta \end{aligned} \right\}. \\ \mathrm{S_H}(t_1) + \left\{ \begin{aligned} & \frac{1-\sigma}{\mathrm{ABC}(\sigma)} {}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, \mathcal{S}_{\mathcal{M}}{}^{n}, \mathcal{I}_{\mathcal{M}}{}^{n}, t_n) \\ & \quad + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \right. \\ & \quad \left. \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-1}, \right. \\ & \quad \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ & \quad \left. + \mathrm{ABC}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \\ & \quad \times \sum + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\ & \quad - 2{}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ & \quad \left. + {}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta \end{aligned} \right\}. \end{cases}$$

For the remaining three compartments, we can write the Newton interpolation approximation as follows:

$$
\mathbb{I}(t_{n+1})) = \begin{cases}
\mathbb{I}_0 + \left\{ \begin{aligned}
&\frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \\
&\quad \times \Pi + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\
&\quad \left. - {}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathbf{k}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \\
&\quad \times \sum + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S_M}^{\mathrm{K}}, \mathcal{I_M}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\
&\quad - 2{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
&\quad \left. + {}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \Delta
\end{aligned} \right\}, \\[2ex]
\mathrm{I_H}(t_1) + \left\{ \begin{aligned}
&\frac{1-\sigma}{\mathrm{ABC}(\sigma)} {}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, \mathcal{S_M}^{n}, \mathcal{I_M}^{n}, t_n) \\
&\quad + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \right. \\
&\quad \left. \times\, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \Pi + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \right. \\
&\qquad \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
&\quad \left. + \mathrm{ABC}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \\
&\quad \times \sum + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S_M}^{\mathrm{K}}, \mathcal{I_M}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\
&\quad - 2^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
&\quad \left. + {}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \Delta
\end{aligned} \right\},
\end{cases}
$$

$$
\mathrm{R_H}(t_{n+1})) = \begin{cases}
\mathrm{R_{H0}} + \left\{ \begin{aligned}
&\frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \\
&\quad \times \Pi + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\
&\left. -{}^{C}\, \mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \\
&\quad \times \sum + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S_M}^{\mathrm{K}}, \mathcal{I_M}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\
&\quad - 2{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
&\quad \left. + {}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \Delta
\end{aligned} \right\}, \\[2ex]
\mathrm{R_H}(t_1) + \left\{ \begin{aligned}
&\frac{1-\sigma}{\mathrm{ABC}(\sigma)} {}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, \mathcal{S_M}^{n}, \mathcal{I_M}^{n}, t_n) \\
&\quad + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \right. \\
&\quad\ \left. \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \Pi + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \right. \\
&\quad\ \left. \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) + \mathrm{ABC}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \\
&\quad \times \sum + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S_M}^{\mathrm{K}}, \mathcal{I_M}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\
&\quad - 2^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
&\quad \left. + {}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2})\right] \Delta
\end{aligned} \right\},
\end{cases}
$$

$$
\mathcal{S}_{\mathcal{M}}(t_{n+1})) = \begin{cases}
\mathcal{S}_{\mathcal{M}0} + \left\{ \begin{array}{l}
\frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \\
\quad \times\ Pi + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathbf{k}=2}^{i} \left[{}^{C}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\
\quad \left. -\ {}^{C}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathbf{k}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \\
\quad \times \sum + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\
\quad -\ 2{}^{C}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
\quad \left. +\ {}^{C}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta
\end{array} \right\}, \\[2ex]
\mathcal{S}_{\mathcal{M}}(t_1) + \left\{ \begin{array}{l}
\frac{1-\sigma}{\mathrm{ABC}(\sigma)} {}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, \mathcal{S}_{\mathcal{M}}{}^{n}, \mathcal{I}_{\mathcal{M}}{}^{n}, t_n) \\
\quad +\ \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \right. \\
\quad \left. \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\sigma t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \right. \\
\quad \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) + \mathrm{ABC}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \\
\quad \left. \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \sum + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \\
\quad \times \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}}, t_{\mathbf{k}}) \right. \\
\quad -\ 2{}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
\quad \left. +\ {}^{\mathrm{ABC}}\mathcal{F}_4(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta
\end{array} \right\},
\end{cases}
$$

$$
\mathcal{I}_{\mathcal{M}}(t_{n+1})) = \begin{cases}
\mathcal{I}_{\mathcal{M}0} + \left\{ \begin{array}{l}
\frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \\
\quad \times \Pi + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathbf{k}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\
\quad \left. -\ {}^{C}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathbf{k}-2}) \right] \\
\quad \times \sum + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\
\quad -\ 2{}^{C}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\
\quad \left. +\ {}^{C}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta
\end{array} \right\}, \\[2ex]
\mathcal{I}_{\mathcal{M}}(t_1) + \left\{ \begin{array}{l}
\frac{1-\sigma}{ABC(\sigma)} {}^{ABC}\mathcal{F}_5(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, \mathcal{S}_{\mathcal{M}}{}^{n}, \mathcal{I}_{\mathcal{M}}{}^{n}, t_n) \\
\quad +\ \frac{\sigma}{ABC(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \right. \\
\quad \left. \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \right. \\
\quad \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) + \mathrm{ABC}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \\
\quad \left. \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \sum + \frac{\sigma}{\mathrm{ABC}(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{\mathrm{ABC}}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}}, \right. \\
\quad \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}}, t_{\mathrm{K}}) - 2{}^{\mathrm{ABC}}\mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \\
\quad \left. \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-1}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) +{}^{rmABC}\ \mathcal{F}_5(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S}_{\mathcal{M}}{}^{\mathrm{K}-2}, \mathcal{I}_{\mathcal{M}}{}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta
\end{array} \right\}.
\end{cases}
$$

Here

$$\Delta = \begin{bmatrix} (1+n-\mathrm{K})^{\sigma}(2(n-\mathrm{K})^2+(3\sigma+10) \\ \times (n-\mathrm{K})+2\sigma^2+9\sigma+12) \\ -(n-\mathrm{K})(2(n-\mathrm{K})^2+(5\sigma+10) \\ \times(-\mathrm{K}+n)+6\sigma^2+18\sigma+12) \end{bmatrix},$$

$$\sum = \begin{bmatrix} (1+n-\mathrm{K})^{\sigma}(3+2\sigma-\mathrm{K}+n) \\ -(n-\mathrm{K})(n-\mathrm{K}+3\sigma+3) \end{bmatrix},$$

$$\Delta = [(1+n-\mathrm{K})^{\sigma}-(n-\mathrm{K})^{\sigma}]$$

and

$$\begin{aligned} {}^{C}\mathcal{F}_1(\mathrm{S_H},t) &= {}^{\mathrm{ABC}}\mathcal{F}_1(\mathrm{S_H},t) \\ &= \mathcal{B} - \frac{\vartheta_1\mathrm{S_H}(t)\mathrm{I_H}(t)+\vartheta_2\mathrm{S_H}(t)\mathrm{I_H}(t)}{\Lambda^H} - \nu_0\mathrm{S_H}(t), \end{aligned}$$

$$\begin{aligned} {}^{C}\mathcal{F}_2(\mathrm{I_H},t) &= {}^{\mathrm{ABC}}\mathcal{F}_2(\mathrm{I_H},t) \\ &= -\frac{\vartheta_1\mathrm{S_H}(t)\mathrm{I_H}(t)+\vartheta_2\mathrm{S_H}(t)\mathrm{I_H}(t)}{\Lambda^H} - (\nu_0+\nu_2+\beta_1)\mathrm{I_H}(t), \end{aligned}$$

$$\begin{aligned} {}^{C}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S_M},\mathcal{I_M},t) &= {}^{\mathrm{ABC}}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S_M},\mathcal{I_M},t) \\ &= \beta_1\mathrm{I_H}(t)-\nu_0\mathrm{R_H}(t), \end{aligned}$$

$$\begin{aligned} {}^{C}\mathcal{F}_4(\mathcal{S_M},t) &= {}^{\mathrm{ABC}}\mathcal{F}_4(\mathcal{S_M},t) \\ &= \gamma - \frac{\vartheta_3\mathcal{S_M}(t)\mathrm{I_H}(t)}{\Lambda^H} - \nu_1\mathcal{S_M}(t), \end{aligned}$$

$$\begin{aligned} {}^{C}\mathcal{F}_5(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S_M},\mathcal{I_M},t) &= {}^{\mathrm{ABC}}\mathcal{F}_5(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},\mathcal{S_M},\mathcal{I_M},t) \\ &= \frac{\vartheta_3\mathcal{S_M}(t)\mathcal{I_M}(t)}{\Lambda^H} - \nu_1\mathcal{I_M}(t). \end{aligned}$$

5. SIMULATIONS AND DISCUSSION

This section presents the simulations of model (2), which is considered with the Caputo and ABC piecewise operator. For the simulation of the approximate solution, we consider the initial values as $\mathrm{S_H}(0) = 7.4$, $\mathrm{I_H}(0) = 0.02$, $\mathrm{R_H}(0) = 0$, $\mathcal{S_M}(0) = 0$, $\mathcal{I_M}(0) = 0$. The parameters used here for the simulation of our results are presented in Table 1. To study the affects of the piecewise operator, we divide the interval $[0,T]$ into two sub-intervals, which are $(0,t_1] = (0,17]$ and $(t_1,T] = (17,60]$. In first sub-interval, we consider the Caputo operator while in the second interval the fractional order ABC derivative is considered. So, the first sub-interval demonstrates the dynamics of the model (2) in the Caputo's sense while the second sub-interval shows the behavior of the model with different fractional orders in ABC sense. In Figs. 1–5, the fractional orders are considered as (blue,0.99), (red,0.97), (black,0.95), (purple,0.93) and (green,0.91).

In Fig. 1, we present the dynamics of susceptible human population. Figures 2 and 3 depict the population dynamics of infected and recovered human populations, respectively. Similarly, Figs. 4 and 5 are projected to observe the dynamics of susceptible and infected mosquitoes populations. In Fig. 1, we observe that with time the susceptible humans $\mathbb{S}$ will decrease and then after $t = 10$ the susceptible humans again will increase. After advancing

Table 1 **Parameters, Description and their Values of Model 2.**[28]

Symbols	Description	Value
$\mathcal{B}$	Human birth rate	10
ϑ_1	Human infection rate	1
ϑ_2	Infected human rate with mosquitoes	0.025
ϑ_3	Infected mosquitoes rate with human	0.02
ν_0	Human death rate	0.1
ν_1	Mosquitoes death rate	0.1
ν_2	Rate of deaths with disease	0.2
γ	Mosquitoes birth rate	9.08

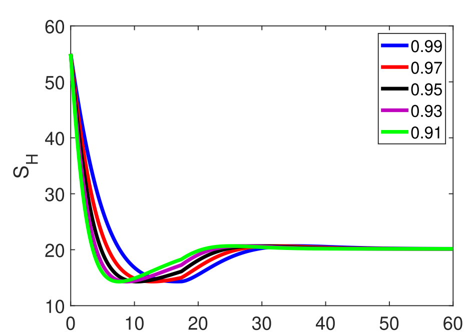

Fig. 1 The dynamics of susceptible human $\mathbb{S}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

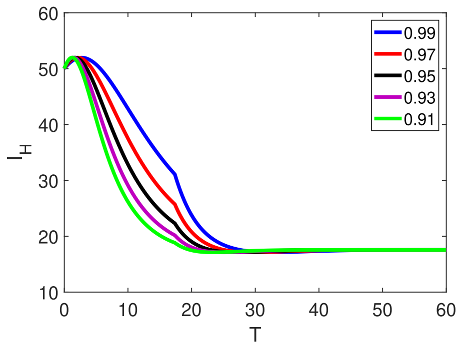

Fig. 2 The population dynamics of infected human $\mathbb{E}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

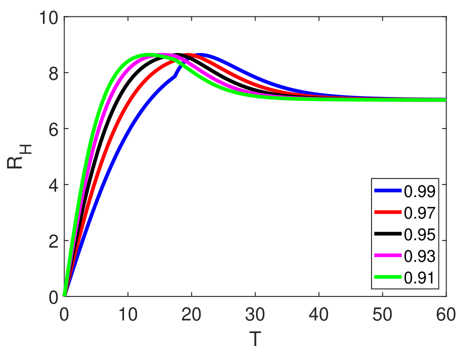

Fig. 3 The population behavior of the recovered human $\mathbb{I}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

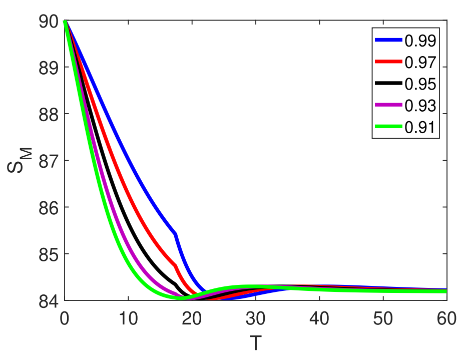

Fig. 4 The dynamics of susceptible mosquitoes $\mathbb{V}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

towards the second interval where the ABC operator used the population in S_h increases and becomes stable at $t = 30$. Similarly, from Fig. 2, it is observed that the infected human population I_h decreases with the passage of time where the rapid decrease in I_h can be seen after $t = t_1$, shows that the transfer of individuals to the recovered class is faster as compared to the first interval and becomes stable after $t = 25$ at fractional order 0.91. Further in Fig. 3, the recovered humans from the dengue fever R_h will increase reaching their peak values around $t = 20$,

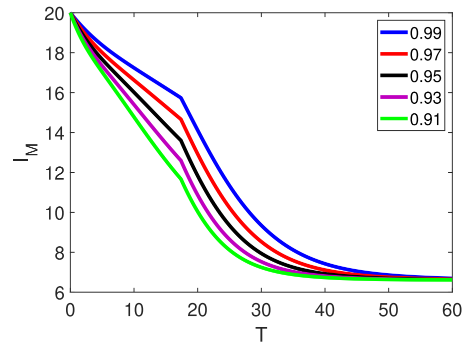

Fig. 5 The dynamics of infected mosquitoes $\mathbb{S}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

then after a little decrease of the individuals in R_h, will become stable at $t = 40$. In Fig. 4, it is observed that the population of the susceptible mosquitoes decreases with time which becomes stable when $t = 35$. Furthermore, Fig. 5 demonstrates that the infected mosquitoes decrease with time which shows faster decrease in the number of infected mosquitoes in the second interval, which become stable at $t = 50$. We have considered the two subintervals $[0, t_1] = [0, 18]$ and $[t_1, T] = [18, 40]$ for the simulation purpose of Figs. 6–10. In Fig. 6, the

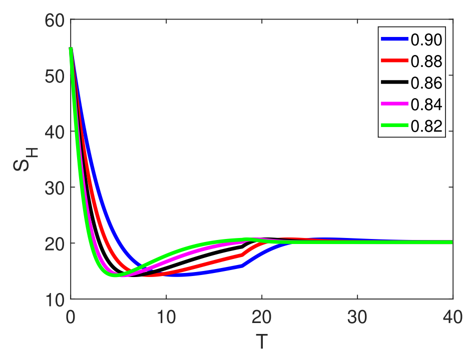

Fig. 6 The dynamics of susceptible human $\mathbb{S}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

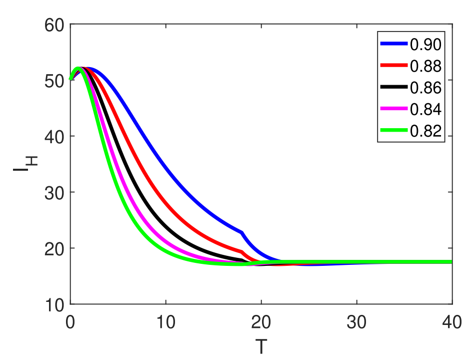

Fig. 7 The population dynamics of infected human $\mathbb{E}$ in the C-ABC piecewise model (2) with $t_1 = 18$.

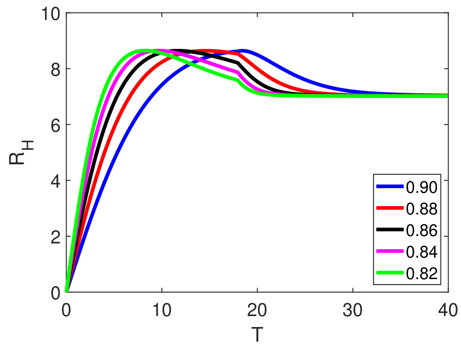

Fig. 8 The population behavior of recovered human 𝕀 in the C-ABC piecewise model (2) with $t_1 = 18$.

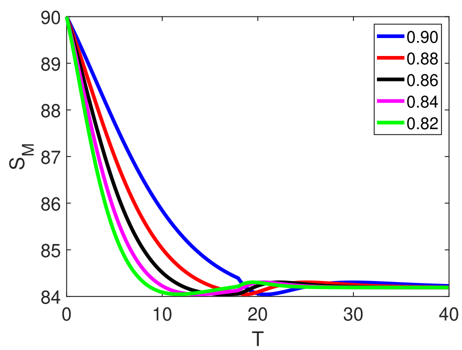

Fig. 9 The dynamics of susceptible mosquitoes 𝕍 in the C-ABC piecewise model (2) with $t_1 = 18$.

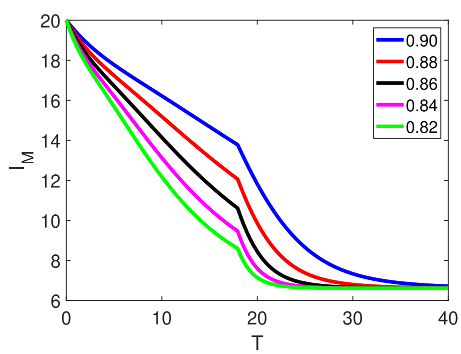

Fig. 10 The dynamics of infected mosquitoes 𝕊 in the C-ABC piecewise model (2) with $t_1 = 18$.

dynamical behavior of the susceptible individuals is demonstrated. Figure 7 shows the evolution of the infected population. Similarly in Figs. 8 and 9, the population behaviors of the recovered human and susceptible mosquitoes are projected, while Fig. 10 demonstrates the dynamics of the infected mosquitoes. Here the fractional orders are considered to be rather lower as compared to Figs. 1–5. The fractional orders for Figs. 6–10 are considered as (blue,0.90), (red,0.88), (black,0.86), (purple,0.84) and (green,0.82). From the figures simulated with lower fractional orders, it is observed that the disease dies out soon at lower fractional orders and the system state variables become stable even at $t = 23$.

6. CONCLUSION

In this paper, we have analyzed the dynamics of dengue epidemic model with novel piecewise derivative in the sense of Caputo and Atangana–Baleanu Caputo operator. The existence and uniqueness of a solution with piecewise derivative is examined for the aforesaid disease model. The suggested problem's approximate solution is obtained using the piecewise approach Newton polynomial approach. In terms of singular and non-singular kernels, a numerical scheme for piecewise derivatives has been established. The numerical simulation for the piecewise dengue model is presented for various fractional orders. We observe that piecewise operators present better dynamics of the models as compared to the classical ones. This work advances the idea of the piecewise derivatives and presents the dynamics of the crossover behavior more clear.

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