

## A GENERALIZED VERSION OF THE LIONS-TYPE LEMMA

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**Abstract.** In this short paper, I recall the history of dealing with the lack of compactness of a sequence in the case of an unbounded domain and prove the vanishing Lions-type result for a sequence of Lebesgue-measurable functions. This lemma generalizes some results for a class of Orlicz–Sobolev spaces. What matters here is the behavior of the integral, not the space.

### 1. Introduction

In 1984 P.L. Lions published his celebrated article [10], in which he introduced a concentration-compactness method for solving minimization problems on unbounded domains. One of the main tools provided by [10] is lemma I.1. A variety of formulations of this lemma has been widely used to deal with the lack of compactness on unbounded domains for different types of equations. In [7, p. 102] we can find the following version of the Lions Lemma:

LEMMA 1. *Suppose  $\{u_n\} \in \mathbf{H}^1(\mathbb{R}^N)$  is a bounded sequence satisfying*

$$\lim_{n \rightarrow \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^p \right) = 0$$

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for some  $p \in [2, 2^*]$  and  $r > 0$ , where  $B_r(y)$  denotes the open ball of radius  $r$  centered at  $y \in \mathbb{R}^N$ . Then  $u_n \rightarrow 0$  strongly in  $\mathbf{L}^q(\mathbb{R}^N)$  for all  $2 < q < 2^*$ , where  $2^*$  is the limiting exponent in the Sobolev embedding  $\mathbf{H}^1(\mathbb{R}^N) \hookrightarrow \mathbf{L}^p(\mathbb{R}^N)$ .

This version of lemma has been used for solving semilinear elliptic equation in the whole space  $\mathbb{R}^N$ , i.e.

$$-\Delta u + u = h(u), \quad u \in \mathbf{H}^1(\mathbb{R}^N).$$

In [8] and [12] you can find a comprehensive description of lack of compactness in Sobolev spaces

The Lions Lemma has been generalized in some ways, for example in [3] we can find the formulation of the lemma for isotropic Orlicz–Sobolev spaces  $\mathbf{W}_0^1 \mathbf{L}^A(\mathbb{R}^N)$ , i.e. spaces obtained by the completion of  $C_0^\infty(\mathbb{R}^N)$  with respect to the norm  $\|u\|_{\mathbf{W}^1 \mathbf{L}^A(\mathbb{R}^N)} = \|\|\nabla u\|\|_{\mathbf{L}^A(\mathbb{R}^N)} + \|u\|_{\mathbf{L}^A(\mathbb{R}^N)}$ , where

$$\|u\|_{\mathbf{L}^A(\mathbb{R}^N)} = \inf \left\{ k > 0 : \int_{\mathbb{R}^N} A\left(\frac{|u|}{k}\right) dt \leq 1 \right\}$$

is a Luxemburg norm,  $A: \mathbb{R} \rightarrow [0, \infty)$  is an  $N$ -function (i.e. is convex, even, coercive and vanishes only at 0) satisfying  $\Delta_2 \nabla_2$  condition (i.e. there exist  $K_1, K_2 > 0$ , such that  $K_1 A(v) \leq A(2v) \leq K_2 A(v)$  for all  $v \in \mathbb{R}^n$ ).

LEMMA 2 ([3, Theorem 1.3]). Assume that  $a(t)t$  is increasing in  $(0, +\infty)$  and that there exist  $l, m \in (1, N)$  such that

$$(1) \quad l \leq \frac{a(|t|)t^2}{A(t)} \leq m \quad \text{for all } t \neq 0,$$

where  $A(t) = \int_0^{|t|} a(s)s ds$ ,  $l \leq m < l^* = \frac{lN}{N-l}$ . Let  $\{u_n\} \subset \mathbf{W}^1 \mathbf{L}^A(\mathbb{R}^N)$  be a bounded sequence such that there exists  $r > 0$  satisfying:

$$(L_1) \quad \lim_{n \rightarrow \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} A(|u_n|) \right) = 0.$$

Then, for any  $N$ -function  $B$  verifying  $\Delta_2$ -condition (i.e. there exists  $K > 0$  such that  $B(2t) \leq KB(t)$  for all  $t > 0$ ) and satisfying

$$\lim_{t \rightarrow 0} \frac{B(t)}{A(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{B(t)}{A^*(t)} = 0,$$

where  $A^*$  is a Sobolev measurete of  $A$  (defined by  $(A^*)^{-1}(t) = \int_0^t \frac{A^{-1}(s)}{s^{(N+1)/N}} ds$ ), we have

$$u_n \rightarrow 0 \quad \text{in } \mathbf{L}^B(\mathbb{R}^N).$$



In [3] the authors use Lemma 2 to prove the existence of solutions to some isotropic quasilinear problems.

It is worth noticing, that in the proof of the lemma above authors essentially use the fact that function  $A$  satisfies  $\Delta_2 \nabla_2$  condition, which is guaranteed by condition (1). Isotropic Young function satisfying globally  $\Delta_2 \nabla_2$  condition is bounded by some power functions with power  $1 < p < \infty$  (see e.g [6, Lemma C.4]). If  $A$  satisfies  $\Delta_2 \nabla_2$  then  $\mathbf{W}^1 \mathbf{L}^A$  is a reflexive, separable Banach space (see e.g. [1, Theorem 8.31]).

There are also papers, where authors consider non-reflexive spaces, e.g. [2]. In this case instead of condition  $(L_1)$  authors use the assumption  $(L_2)$  (see [9]) and assume that the sequence  $\{\int_{\mathbb{R}^N} A^*(|u_n|) dx\}$  is bounded.

LEMMA 3 ([2, Theorem 1.3]). *Let  $A, B$  be an  $N$ -functions,  $A^*$  be a Sobolev conjugate of  $A$  and*

$$\lim_{t \rightarrow 0} \frac{B(t)}{A(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{B(t)}{A^*(t)} = 0.$$

*If  $\{u_n\} \subset \mathbf{W}^1 \mathbf{L}^A(\mathbb{R}^N)$  is a sequence such that  $\{\int_{\mathbb{R}^N} A(|u_n|) dx\}$  and  $\{\int_{\mathbb{R}^N} A^*(|u_n|) dx\}$  are bounded, and for each  $\varepsilon > 0$  we have*

$$(L_2) \quad \text{meas}(|u_n| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

*then*

$$\int_{\mathbb{R}^N} B(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In [13] the author uses the lemma similar to Lemma 2, but for sequences from anisotropic Orlicz–Sobolev spaces, to find solutions of the anisotropic quasilinear problem

$$-\text{div}(\nabla \Phi(\nabla u)) + V(x)N'(u) = f(u), \quad \text{where } u \in \mathbf{W}^1 \mathbf{L}^\Phi(\mathbb{R}^n),$$

where  $\Phi$  is an anisotropic  $n$ -dimensional  $N$ -function (see more in [5]), satisfying  $\Delta_2 \nabla_2$  condition and  $N$  is a differentiable  $N$ -function, such that  $N \approx \Phi_0$ , where  $\Phi_0: [0, \infty) \rightarrow [0, \infty)$  is the left-continuous increasing function obeying

$$|\{v \in \mathbb{R}^n: \Phi_0(|v|) \leq t\}| = |\{v \in \mathbb{R}^n: \Phi(v) \leq t\}| \quad \text{for } t \geq 0,$$

where  $|\cdot|$  stands for Lebesgue measure.

In [11] the authors prove the Lions type lemma for reflexive fractional Orlicz–Sobolev spaces, while in [4] the authors prove it for non-reflexive fractional Orlicz–Sobolev spaces.



## 2. Main Theorem

In this paper, we generalize the Lions-type Lemmas 1, 2, 3, we mentioned in the introduction. We do not assume that functions  $\Psi$ ,  $\Phi_1$ , and  $\Phi_2$ , from the theorem below, are  $N$ -functions.

We need only the fact that they are locally essentially bounded, non-negative, essential supremum of  $\Psi$  is greater than zero and  $\Phi_1$  vanishes only at zero (assumption (2)). It is worth noticing that they can have growth not bounded by polynomials, so it will be possible to use this lemma in non-reflexive spaces. In the proof of the following lemma we will use some techniques from [11].

THEOREM 4. Assume that  $\Phi_1, \Phi_2, \Psi \in \mathbf{L}_{loc}^\infty(\mathbb{R}^n, [0, \infty))$ ,

$$(2) \quad \begin{aligned} \forall_{R>0} \quad \text{ess sup}_{B_R(0)} \Psi &> 0, \\ \Phi_1(x) = 0 &\iff x = 0, \end{aligned}$$

$$(\Psi_1) \quad \lim_{|v| \rightarrow 0} \frac{\Psi(v)}{\Phi_1(v)} = 0,$$

$$(\Psi_2) \quad \lim_{|v| \rightarrow \infty} \frac{\Psi(v)}{\Phi_2(v)} = 0.$$

Let  $\{u_k\}$  be a sequence of Lebesgue-measurable functions  $u_k: \mathbb{R}^N \rightarrow \mathbb{R}^n$  such that  $\int_{\mathbb{R}^N} \Phi_1(u_k), \int_{\mathbb{R}^N} \Phi_2(u_k)$  exist,

$$M_1 = \sup_k \int_{\mathbb{R}^N} \Phi_1(u_k) < \infty, \quad M_2 = \sup_k \int_{\mathbb{R}^N} \Phi_2(u_k) < \infty,$$

and

$$(3) \quad \lim_{k \rightarrow \infty} \left[ \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \Phi_1(u_k) \right] = 0,$$

for some  $r > 0$ . Then

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(u_k) = 0.$$



PROOF. We let  $|A|$  denote the Lebesgue measure of a subset  $A$ . Let  $\{u_k\}$ ,  $\Phi_1, \Phi_2, \Psi$  satisfy the above assumptions.

Fix  $\varepsilon > 0$ . From  $(\Psi_1)$ , there exists  $\delta > 0$ , such that

$$(4) \quad \frac{\Psi(v)}{\Phi_1(v)} < \frac{\varepsilon}{3M_1}$$

for all  $|v| \leq \delta$ .

Similarly from  $(\Psi_2)$ , there exists  $T > 0$ , such that

$$(5) \quad \frac{\Psi(v)}{\Phi_2(v)} < \frac{\varepsilon}{3M_2}$$

for all  $|v| \geq T$ . Let us denote:

$$A_k = \{x \in \mathbb{R}^N : |u_k(x)| < \delta\}, \quad B_k = \{x \in \mathbb{R}^N : \delta \leq |u_k(x)| \leq T\}, \\ C_k = \{x \in \mathbb{R}^N : |u_k(x)| > T\}.$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_k) = \int_{A_k} \Psi(u_k) + \int_{B_k} \Psi(u_k) + \int_{C_k} \Psi(u_k).$$

By (4), we obtain

$$\int_{A_k} \Psi(u_k) \leq \frac{\varepsilon}{3M_1} \int_{\mathbb{R}^N} \Phi_1(u_k) < \frac{\varepsilon}{3}$$

and by (5), we get

$$\int_{C_k} \Psi(u_k) \leq \frac{\varepsilon}{3M_2} \int_{\mathbb{R}^N} \Phi_2(u_k) < \frac{\varepsilon}{3}.$$

We need to show that

$$\int_{B_k} \Psi(u_k) < \frac{\varepsilon}{3}.$$

We will show that  $|B_k| \rightarrow 0$  as  $k \rightarrow \infty$ .

Assume, by contradiction, that (up to subsequence)

$$(6) \quad |B_k| \rightarrow L > 0.$$



First of all, we will show (just as in [11, p. 506]), that for some subsequence  $\{u_k\}$ , there exist  $y_0 \in \mathbb{R}^N$  and  $\sigma > 0$ , such that

$$(7) \quad |B_k \cap B_r(y_0)| \geq \sigma > 0.$$

Assume, again by contradiction, that for all  $\varepsilon > 0$ ,  $m \in \mathbb{N}$ ,  $y \in \mathbb{R}^N$  we have

$$(8) \quad |B_k \cap B_r(y)| < \frac{\varepsilon}{2^m}.$$

The last estimate holds for any subsequence of  $\{u_k\}$ , and WLOG we can take just  $\{u_k\}$ . Let us choose  $\{y_m\} \subset \mathbb{R}^N$ , such that

$$B := \bigcup_{m=1}^{\infty} B_r(y_m) = \mathbb{R}^N.$$

Using (8), for arbitrary  $\varepsilon$  we obtain

$$|B_k| = |B_k \cap B| \leq \sum_{m=1}^{\infty} |B_k \cap B_r(y_m)| < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon,$$

which contradicts (6).

Let

$$C_{\Psi} = \operatorname{ess\,sup}_{\delta \leq |v| \leq T} \Psi(v), \quad c_{\Phi} = \operatorname{ess\,inf}_{\delta \leq |v| \leq T} \Phi_1(v),$$

$$C_{\Phi} = \operatorname{ess\,sup}_{\delta \leq |v| \leq T} \Phi_1(v).$$

We observe that

$$\int_{B_r(y_0)} \Phi_1(u_k) \geq \int_{B_r(y_0) \cap B_k} \Phi_1(u_k) \geq c_{\Phi} |B_k \cap B_r(y_0)|.$$

Hence, by assumption (3), we have that

$$|B_k \cap B_r(y_0)| \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

which contradicts (7) and as a result  $|B_k| \rightarrow 0$  as  $k \rightarrow \infty$ . Hence we have that there exists  $k_0$  such that for all  $k \geq k_0$

$$|B_k| < c_{\Phi} (3C_{\Phi}C_{\Psi})^{-1} \varepsilon.$$



Then

$$|B_k| \leq (c_\Phi)^{-1} \int_{B_k} \Phi_1(u_k) \leq C_\Phi (c_\Phi)^{-1} |B_k|$$

and

$$\int_{B_k} \Psi(u_k) \leq C_\Psi (c_\Phi)^{-1} \int_{B_k} \Phi_1(u_k) \leq C_\Psi C_\Phi (c_\Phi)^{-1} |B_k| < \frac{\varepsilon}{3}. \quad \square$$

REMARK 5. Note that what matters in this theorem (just as in the concentration-compactness lemma of Lions in [9]) is the behavior of the integral, not the space.

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