



# Periodic Solutions of Generalized Lagrangian Systems with Small Perturbations

Joanna Janczewska<sup>1</sup>

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## Abstract

In this paper we study the generalized Lagrangian system with a small perturbation. We assume the main term in the system to have a maximum, but do not suppose any condition for perturbation term. Then we prove the existence of a periodic solution via Ekeland's principle. Moreover, we prove a convergence theorem for periodic solutions of perturbed systems.

**Keywords** Periodic solution · Trudinger's function · Ekeland's variational principle · Palais–Smale condition · Lagrangian system · Orlicz–Sobolev space

**AMS Subject Classification** Primary 34C25; Secondary 37J46 · 49J35

## 1 Introduction and Main Results

In this paper we prove the existence of periodic solutions for the second order Hamiltonian systems

$$\begin{cases} \frac{d}{dt} (\nabla \Phi(\dot{q}(t))) + V_q(t, q(t)) = \lambda W_q(t, q(t)), & t \in [0, T], \\ q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0, \end{cases} \quad (1)$$

where  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $W : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $C^1$ -smooth,  $T$ -periodic with respect to  $t \in \mathbb{R}$ ,  $n \geq 1$ ,  $T > 0$ ,  $\lambda$  is a real small parameter and  $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$  is a  $G$ -function in the sense of Trudinger, i.e.  $\Phi(0) = 0$ ,  $\Phi$  is  $C^1$ -smooth, coercive, convex and symmetric, and  $\nabla \Phi \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ . Here and subsequently  $V_q : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $W_q : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the gradient maps of  $V$  and  $W$ , respectively, with respect to  $q \in \mathbb{R}^n$ . From now on  $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  stands for the standard inner product in  $\mathbb{R}^n$  and  $|\cdot| : \mathbb{R}^n \rightarrow [0, \infty)$  is the Euclidean norm. We assume the conditions below:

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✉ Joanna Janczewska  
joanna.janczewska@pg.edu.pl

<sup>1</sup> Institute of Applied Mathematics, Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

(a) there exists a constant  $\alpha > 0$  such that

$$V(t, q) + \alpha|q|^2 \leq V(t, 0)$$

for all  $t \in [0, T]$  and  $q \in \mathbb{R}^n$ ;

( $\Delta_2$ ) there is a constant  $L > 0$  such that

$$\Phi(2q) \leq L\Phi(q)$$

for each  $q \in \mathbb{R}^n$ ;

( $\nabla_2$ ) there exists a constant  $l > 0$  such that

$$\Phi(lq) \geq 2l\Phi(q)$$

for each  $q \in \mathbb{R}^n$ .

Our assumptions imply that the action functional corresponding to the system (1) with  $\lambda = 0$  satisfies the Palais–Smale condition (Lemma 2.1 in Sect. 2). Let us also remark that  $q \equiv 0$  is a solution of (1) for  $\lambda = 0$ . Our aim is to prove the existence of periodic solutions of (1) for  $|\lambda|$  small enough without any extra conditions on  $W$ .

Let us consider the Orlicz space

$$L^\Phi(0, T; \mathbb{R}^n) = \left\{ q: \mathbb{R} \rightarrow \mathbb{R}^n : q \text{ is } T\text{-periodic, measurable, } \int_0^T \Phi(q(t))dt < \infty \right\}$$

with the Luxemburg norm

$$\|q\|_\Phi = \inf \left\{ v > 0 : \int_0^T \Phi\left(\frac{q(t)}{v}\right) dt \leq 1 \right\}.$$

It is well-known that  $L^\Phi(0, T; \mathbb{R}^n)$  is a Banach space (cf. [11]). As  $\Phi$  is  $\Delta_2$ -regular and  $\nabla_2$ -regular,  $L^\Phi(0, T; \mathbb{R}^n)$  is separable and reflexive (cf. [1]). Moreover, it is not difficult to show that

$$\|q\|_\Phi \leq 1 + \int_0^T \Phi(q(t))dt, \quad q \in L^\Phi(0, T; \mathbb{R}^n). \quad (2)$$

**Proposition 1.1** (cf. [3], Lem. 3.16) *Let  $q_k$  be a sequence in  $L^\Phi(0, T; \mathbb{R}^n)$  and  $q \in L^\Phi(0, T; \mathbb{R}^n)$ . If  $q_k \rightarrow q$  almost everywhere in  $(0, T)$  and  $\int_0^T \Phi(q_k(t))dt \rightarrow \int_0^T \Phi(q(t))dt$  then  $q_k \rightarrow q$  in  $L^\Phi(0, T; \mathbb{R}^n)$ .*

The mixed Orlicz–Sobolev space  $W_T^{1,\Phi}$  is the space of functions  $q \in L^2(0, T; \mathbb{R}^n)$  having a weak derivative  $\dot{q} \in L^\Phi(0, T; \mathbb{R}^n)$ . Let us recall that, if  $q \in W_T^{1,\Phi}$ ,

$$q(t) = \int_0^t \dot{q}(s)ds + c$$



and  $q(0) = q(T)$ . The norm over  $W_T^{1,\Phi}$  is defined by

$$\|q\|^2 = \|q\|_2^2 + \|\dot{q}\|_\Phi^2,$$

where

$$\|q\|_2 = \left( \int_0^T |q(t)|^2 dt \right)^{\frac{1}{2}}.$$

It is easy to verify that  $W_T^{1,\Phi}$  is a reflexive Banach space.

**Proposition 1.2** (cf. [8], Prop. 2.1) *There exists a positive constant  $C_\Phi$  such that for  $q \in W_T^{1,\Phi}$ ,*

$$\|q\|_\infty \leq C_\Phi \|q\|, \tag{3}$$

where  $\|q\|_\infty = \max_{t \in [0, T]} |q(t)|$ .

By Proposition 2.3 of [8], the imbedding of  $W_T^{1,\Phi}$  in  $C(0, T; \mathbb{R}^n)$ , with its natural norm  $\|\cdot\|_\infty$ , is compact. We are now ready to state the announced result.

**Theorem 1.3** *Let  $V(t, q)$  and  $W(t, q)$  be  $C^1$ -smooth on  $\mathbb{R} \times \mathbb{R}^n$ ,  $T$ -periodic in  $t$ , and  $\Phi(q)$  be a  $G$ -function. Under the assumptions (a),  $(\Delta_2)$ ,  $(\nabla_2)$ , the following assertions hold.*

- (i) *There is a positive number  $\lambda_0$  such that the system (1) has a solution  $q_\lambda$  when  $|\lambda| \leq \lambda_0$ .*
- (ii) *For any sequence  $\lambda_j$  converging to zero, along a subsequence  $q_{\lambda_j}$  converges to zero in  $W_T^{1,\Phi}$ .*

Let us emphasize that we mean by solution of (1) an absolutely continuous function in  $L^2(0, T; \mathbb{R}^n)$  that satisfies (1) weakly. If we require that  $\Phi$  is not only convex but strictly convex, then  $q_\lambda$  has a classical first derivative. There are many important examples of  $\Phi$  satisfying our assumptions. If we set  $\Phi(q) = \frac{1}{2}|q|^2$ ,  $q \in \mathbb{R}^n$ , we obtain the classical second order Hamiltonian systems. Applications of fundamental techniques of critical point theory to the existence of periodic solutions of second order Hamiltonian systems were presented e.g. in [9]. If we set  $\Phi(q) = \frac{1}{p}|q|^p$ ,  $q \in \mathbb{R}^n$ ,  $1 < p < \infty$ , we get the one-dimensional  $p$ -Laplacian. Nonlinear perturbations of this operator have been studied recently e.g. in [2, 5, 6]. Variational systems involving  $p$ -Laplacian occur naturally in a variety of settings in physics and engineering [2]. Moreover, let us remind an anisotropic example  $\Phi(q) = \sum_{i=1}^n a_i |q_i|^{p_i}$ ,  $1 < p_i < \infty$ ,  $a_i > 0$ ,  $q = (q_1, q_2, \dots, q_n)$ , which has been investigated e.g. in [4, 10].



## 2 Proof of Theorem 1.3

We shall prove Theorem 1.3. Our approach is based on Ekeland's variational principle. For (1) with  $\lambda = 0$ , we define the Lagrangian functional by

$$I_0(q) = \int_0^T (\Phi(\dot{q}(t)) - V(t, q(t))) dt, \quad (4)$$

where  $\Phi$  and  $V$  satisfy our assumptions. Then  $I_0$  is well-defined in  $W_T^{1,\Phi}$  and becomes a  $C^1$ -functional (cf. [8], Prop. 2.10). Moreover,  $I_0$  is bounded from below. Using (a), we get

$$I_0(q) \geq \int_0^T -V(t, q(t)) dt \geq \int_0^T -V(t, 0) dt =: V_0. \quad (5)$$

From an easy calculation, we also see that

$$I'_0(q)v = \int_0^T ((\nabla\Phi(\dot{q}(t)), \dot{v}(t)) - (V_q(t, q(t)), v(t))) dt, \quad (6)$$

where  $q, v \in W_T^{1,\Phi}$ .

**Lemma 2.1**  $I_0$  satisfies the Palais–Smale condition.

**Proof** Let  $q_k$  be any sequence in  $W_T^{1,\Phi}$  such that  $I_0(q_k)$  is bounded and  $I'_0(q_k)$  converges to zero in  $(W_T^{1,\Phi})^*$ . By (a) and (2), we obtain

$$\begin{aligned} I_0(q) &\geq \|\dot{q}\|_\Phi - 1 + \alpha \int_0^T |q(t)|^2 dt + \int_0^T -V(t, 0) dt \\ &= \|\dot{q}\|_\Phi - 1 + \alpha \|q\|_2^2 + V_0. \end{aligned} \quad (7)$$

As  $I_0(q_k)$  is bounded, there is  $C > 0$  such that  $|I_0(q_k)| \leq C$  for each  $k \in \mathbb{N}$ . We thus get

$$\|\dot{q}_k\|_\Phi - 1 + \alpha \|q_k\|_2^2 + V_0 \leq C \quad (8)$$

for each  $k \in \mathbb{N}$ . Hence  $q_k$  is bounded in  $W_T^{1,\Phi}$ . Since  $W_T^{1,\Phi}$  is reflexive, there is a subsequence of  $q_k$  that converges weakly to some  $q \in W_T^{1,\Phi}$ . We keep denoting this subsequence by  $q_k$ . By the compact imbedding,  $q_k$  converges to  $q$  in  $C(0, T; \mathbb{R}^n)$  and, in consequence,  $q_k$  converges to  $q$  in  $L^2(0, T; \mathbb{R}^n)$ . Moreover, since the modulus function increases essentially more slowly than  $\Phi$  near infinity  $\dot{q}_k$  goes to  $\dot{q}$  in  $L^1(0, T; \mathbb{R})$ , and hence, along a subsequence  $\dot{q}_k$  goes to  $\dot{q}$  almost everywhere in  $(0, T)$ . Without loss of generality we denote this subsequence by  $q_k$ . According to the above remarks, we have

$$|I'_0(q_k)(q_k - q)| \leq \|I'_0(q_k)\|_{(W_T^{1,\Phi})^*} \|q_k - q\| \rightarrow 0,$$



$$\int_0^T (V_q(t, q_k(t)), q_k(t) - q(t)) dt \rightarrow 0,$$

and consequently,

$$\begin{aligned} \int_0^T (\nabla\Phi(\dot{q}_k(t)), \dot{q}_k(t) - \dot{q}(t)) dt &= I'_0(q_k)(q_k - q) \\ &+ \int_0^T (V_q(t, q_k(t)), q_k(t) - q(t)) dt \rightarrow 0 \end{aligned} \tag{9}$$

as  $k \rightarrow \infty$ . As  $\Phi$  is convex,

$$\Phi(x) - \Phi(x - y) \leq (\nabla\Phi(x), y)$$

for each  $x, y \in \mathbb{R}^n$ . From this it follows that

$$\begin{aligned} \int_0^T \Phi(\dot{q}_k(t)) dt - \int_0^T \Phi(\dot{q}(t)) dt &\leq \int_0^T (\nabla\Phi(\dot{q}_k(t)), \dot{q}_k(t) - \dot{q}(t)) dt, \\ \int_0^T \Phi(\dot{q}_k(t)) dt &\leq \int_0^T \Phi(\dot{q}(t)) dt + \int_0^T (\nabla\Phi(\dot{q}_k(t)), \dot{q}_k(t) - \dot{q}(t)) dt. \end{aligned}$$

Letting  $k \rightarrow \infty$  we obtain

$$\limsup_{k \rightarrow \infty} \int_0^T \Phi(\dot{q}_k(t)) dt \leq \int_0^T \Phi(\dot{q}(t)) dt.$$

On the other hand, by Fatou's lemma

$$\liminf_{k \rightarrow \infty} \int_0^T \Phi(\dot{q}_k(t)) dt \geq \int_0^T \Phi(\dot{q}(t)) dt.$$

Therefore

$$\lim_{k \rightarrow \infty} \int_0^T \Phi(\dot{q}_k(t)) dt = \int_0^T \Phi(\dot{q}(t)) dt,$$

and finally, by Proposition 1.1,  $\dot{q}_k \rightarrow \dot{q}$  in  $L^\Phi(0, T; \mathbb{R}^n)$ . Since  $q_k \rightarrow q$  in  $L^2(0, T; \mathbb{R}^n)$  and  $\dot{q}_k \rightarrow \dot{q}$  in  $L^\Phi(0, T; \mathbb{R}^n)$ , we have  $q_k \rightarrow q$  in  $W_T^{1,\Phi}$ , which completes the proof.  $\square$

We now choose a function such that  $0 \leq h(x) \leq 1$  in  $\mathbb{R}^n$ ,  $h(x) = 1$  for  $|x| \leq C_\Phi$  and  $h(x) = 0$  for  $|x| \geq 2C_\Phi$ , where  $C_\Phi$  is given by (3). We define

$$I_\lambda(q) = \int_0^T (\Phi(\dot{q}(t)) - V(t, q(t)) + \lambda h(q(t))W(t, q(t))) dt, \tag{10}$$



where  $q \in W_T^{1,\Phi}$ . Then a critical point of  $I_\lambda$  is a solution of

$$\begin{cases} \frac{d}{dt} (\nabla \Phi(\dot{q}(t))) + V_q(t, q(t)) = \lambda h(q(t)) W_q(t, q(t)) + \lambda \nabla h(q(t)) W(t, q(t)) \\ q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0. \end{cases} \quad (11)$$

Our plan to prove Theorem 1.3 is as follows. First, we find a critical point  $q_\lambda$  of  $I_\lambda$ . Next, we show that  $\|q_\lambda\|_\infty \leq C_\Phi$  for  $|\lambda|$  small enough. Then  $h(q_\lambda) = 1$ ,  $\nabla h(q_\lambda) = 0$  and therefore  $q_\lambda$  becomes a solution of (1). Set

$$C_0 = \max\{W(t, q) : t \in [0, T] \wedge |q| \leq 2C_\Phi\}.$$

We have

$$I_\lambda(q) = I_0(q) + \lambda \int_0^T h(q(t)) W(t, q(t)) dt \geq V_0 - |\lambda| T C_0,$$

and so  $I_\lambda$  is bounded from below. Using the same arguments as in Lemma 2.1 with the fact that  $h(q)W(t, q)$  and its gradient with respect to  $q$  are bounded, we get the next lemma.

**Lemma 2.2** *For each  $\lambda \in \mathbb{R}$ ,  $I_\lambda$  satisfies the Palais–Smale condition.*

Applying Ekeland's variational principle we conclude that  $I_\lambda$  has a minimum on  $W_T^{1,\Phi}$ . It follows that there is  $q_\lambda \in W_T^{1,\Phi}$  such that

$$I_\lambda(q_\lambda) = \inf_{q \in W_T^{1,\Phi}} I_\lambda(q) \wedge I'_\lambda(q_\lambda) = 0.$$

Since

$$I_0(q) - |\lambda| T C_0 \leq I_\lambda(q) \leq I_0(q) + |\lambda| T C_0$$

for each  $q \in W_T^{1,\Phi}$ , we obtain  $I_\lambda(q_\lambda) \rightarrow V_0$  as  $\lambda \rightarrow 0$ .

**Lemma 2.3** *Let  $\lambda_m$  be a sequence converging to zero and let the functional  $I_{\lambda_m}$  reach a minimum at the point  $q_{\lambda_m}$ . Then a subsequence of  $q_{\lambda_m}$  converges to zero in  $W_T^{1,\Phi}$ .*

**Proof** By definition,

$$I_{\lambda_m}(q_{\lambda_m}) = \inf_{q \in W_T^{1,\Phi}} I_{\lambda_m}(q) \wedge I'_{\lambda_m}(q_{\lambda_m}) = 0,$$

and hence  $q_{\lambda_m}$  is a solution of (11) with  $\lambda$  replaced by  $\lambda_m$ . Using the same argument as in the proof of Lemma 2.1, by the boundedness of  $I_{\lambda_m}(q_{\lambda_m})$ , we can conclude that  $q_{\lambda_m}$  is bounded in  $W_T^{1,\Phi}$  and a subsequence of  $q_{\lambda_m}$  converges to a limit  $q_0$  in  $W_T^{1,\Phi}$ . Then  $q_0$  satisfies that  $I_0(q_0) = V_0$  and  $I'_0(q_0) = 0$ , i.e.  $q_0 \equiv 0$ .  $\square$



**Lemma 2.4** *There is  $\lambda_0 > 0$  such that for  $|\lambda| \leq \lambda_0$  we have  $\|q_\lambda\|_\infty \leq C_\Phi$ .*

**Proof** Suppose on the contrary to our claim that there is a sequence  $\lambda_m$  converging to zero such that  $\|q_{\lambda_m}\|_\infty > C_\Phi$ . By Lemma 2.3 it follows that there is a subsequence of  $q_{\lambda_m}$  going to zero in  $W_T^{1,\Phi}$ . Without loss of generality we will denote this subsequence by  $q_{\lambda_m}$ . Thus for  $m$  large enough,  $\|q_{\lambda_m}\| \leq 1$ , and consequently  $\|q_{\lambda_m}\|_\infty \leq C_\Phi$ , by (3). A contradiction occurs.  $\square$

The lemma above will be used to find a solution of (1). We are now in a position to prove Theorem 1.3.

**Proof (Proof of Theorem 1.3)** Choose  $\lambda_0 > 0$  that satisfies Lemma 2.4. Let  $I_\lambda$  reach a minimum at  $q_\lambda$  with  $|\lambda| \leq \lambda_0$ . Then  $\|q_\lambda\|_\infty \leq C_\Phi$ . For this reason  $h(q_\lambda) = 1$ ,  $\nabla h(q_\lambda) = 0$ , and consequently  $q_\lambda$  becomes a solution of (1). Let  $\lambda_j$  be a sequence converging to zero. From Lemma 2.3 it follows that a subsequence of  $q_{\lambda_j}$  converges to zero in  $W_T^{1,\Phi}$ , which completes the proof.  $\square$

We conclude our work by explaining the regularity of solutions of (1) in case that  $\Phi$  is strictly convex. We set for  $|\lambda| \leq \lambda_0$  and  $t \in [0, T]$ ,

$$x_\lambda(t) = \nabla \Phi(\dot{q}_\lambda(t)).$$

Let us note that

$$\dot{x}_\lambda(t) = \frac{d}{dt}(\nabla \Phi(\dot{q}_\lambda(t))) = -V_q(t, q_\lambda(t)) + \lambda W_q(t, q_\lambda(t)),$$

and so it is continuously differentiable. It is known that if  $\Phi$  is strictly convex then  $\nabla \Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is invertible and its inverse map  $(\nabla \Phi)^{-1} = \nabla \Phi^*$  is continuous (Corollary 4.1.3 in [7]), where  $\Phi^*$  denotes the Fenchel transform of  $\Phi$  defined by

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^n} ((x, y) - \Phi(x)).$$

Hence  $\dot{q}_\lambda(t) = (\nabla \Phi)^{-1}(x_\lambda(t))$  is continuously differentiable too. Finally, if  $\nabla \Phi^*$  is  $C^1$  then  $q_\lambda$  is  $C^2$ , i.e. a classical solution. These additional assumptions are satisfied for  $\Phi(x) = \frac{1}{p}|x|^p$ ,  $1 < p \leq 2$ .

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### Declarations

**Conflict of interest** The author has no conflict of interest to declare that are relevant to the content of this article.



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