Excitation of Waves in a Dispersive Medium. Example of Flow of a Bubbly Liquid

A. Perelomova*

Gdansk University of Technology, Faculty of Applied Physics and Mathematics, ul. Narutowicza 11/12, 80-233 Gdansk, Poland

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*e-mail: anna.perelomova@pg.edu.pl

The excitation of wave motion by an external source and the interaction of modes inherent to a flow in a dispersive medium are considered. Dispersion is caused by the presence of gaseous bubbles in a liquid. A large variety of steady excited waveforms is possible when the exciting wave is also steady and propagates at a constant velocity. The velocities of the exciter and forced waves may be different. This leads to a variety of non-stationary excitations.

topics: forced waves, dispersion, nonlinearity

1. Introduction

Dispersion as a physical phenomenon specifies the dependence of phase velocity on frequency. The physical reasons of dispersion are spatial and temporal inhomogeneities of different kinds, that is, the presence of boundaries, stratification of a medium, thermodynamic relaxation [1, 2]. The molecular structure and finite time of all relaxation processes are the general reasons for dispersion of small-scale waves. The dispersion of acoustic waves is a less common effect compared to the strong dispersion of light in optical media. Among acoustical media with strong dispersion, liquids involving sand or gaseous bubbles are of great importance. Traveling waves that have attained a stable form in the course of propagation are called stationary waves. Dispersion destroys the initial waveform and by itself can never support a stationary wave since any harmonic component in the initial spectrum propagates with its own velocity. When the effects of dispersion and nonlinearity balance each other, stationary waveforms traveling with different constant velocities may appear.

Waves in a liquid with gaseous bubbles are well-studied. Gaseous inclusions essentially increase the compressibility and thus reduce the speed of sound in a medium and have a key impact on the nonlinear properties of a medium. The nonlinear effects of a flow can exceed, by orders of magnitude, the nonlinear effects characterizing pure phases [2, 3]. This makes nonlinear distortion of sound and corresponding nonlinear phenomena essential even for moderate magnitudes of perturbations. Analytical models describing the propagation of finite-magnitude sound in a bubbly liquid vary from the simplest to fairly complex [2, 4–7]. They take into account thermodynamic processes in the bubble and liquid, the shape of the bubble during oscillations, gas composition, phase transitions of a vapor, and other phenomena. The character of sound propagation in a gassy liquid depends strongly on the ratio of the bubbles radii, the average distance between the bubbles (this in turn is determined by the volume fraction of the gaseous phase), and the characteristic wavelength of the primary wave [8, 9]. In this study, we use the simplest model, aiming to focus only on the dispersive and nonlinear properties of a flow.

Preliminary remarks about the excitation of waves are necessary. Despite the fact that the theory of inhomogeneous wave equations is well-developed, especially in the field of linear dynamics (e.g., [10]), interest in the dynamics of perturbations forced by wave exciters has grown in the last decades. "Wave resonance" is important in the process of excitation of waves by sources moving with a speed close to the eigen velocity of sound, that is, up to the velocity of perturbations in unforced oscillations without the impact of effects connected with nonlinearity and dispersion [11]. This phenomenon resembles mechanical resonance. Simple evaluations that explain the phenomenon are as follows [12, 13]: Since wave resonance is possible only with the eigen wave propagating in the same direction as the exciter, the simple inhomogeneous equation of the first order may be used as a starting point,

$$\frac{\partial U}{\partial t}+c_0\frac{\partial U}{\partial x}=\frac{1}{2}\frac{\partial}{\partial t}f\left(t-\frac{x}{c}\right)\equiv \Phi\left(t-\frac{x}{c}\right). \quad (1)$$

It does not account for nonlinear and dissipative effects and describes the excitation of the wave perturbation U in a wave with the eigen velocity c_0 by the primary steady perturbation f propagating with the speed c. Its solution, satisfying the boundary condition $U(t,x{=}0) = 0$, i.e.,

$$U = \frac{f\left(t-\frac{x}{c}\right) - f\left(t-\frac{x}{c_0}\right)}{1-\frac{c_0^2}{c^2}}, \tag{2}$$

contains uncertainty of the type $\frac{0}{0}$ if $c \to c_0$. Evaluating the limiting value, one arrives at

$$U = \frac{x}{2c_0}\frac{\partial}{\partial t} f\left(t-\frac{x}{c_0}\right). \tag{3}$$

An excited perturbation increases linearly with the distance traveled by the wave x. Wave resonance concerns different kinds of waves, including surface waves that may be excited by wind or by a pressure wave [14], Newtonian flows, and flows different from Newtonian. The analytical description of resonant and non-resonant interactions in connection with the Newtonian flows and external exciters, such as laser beams, was reviewed by Rudenko and Hedberg [12].

Authors of studies [15, 16] indicated the external force caused by the stimulated Mandelshtam–Brillouin scattering (SMBS) as an option. The electric field consists of a laser pump wave of magnitude E_p and a Stokes wave of magnitude E_s, which appears in the course of scattering of the laser pumping wave by sound of a frequency ω, which equals the difference between the pumping frequency and the Stokes wave frequency. The volume force due to SMBS contributes to the momentum equation [17]. Scattering is the most effective if the pump and Stokes waves propagate in the opposite direction, that is, during backward scattering. The leading-order averaged force takes the form

$$\Phi = \frac{Y\omega}{16\pi\, c_0^2 \rho_0} E_p E_s \sin\left(\omega\left(t-\frac{x}{c}\right)\right), \tag{4}$$

where Y is the coefficient of optical–acoustic coupling, U denotes the velocity, and ρ_0 is the equilibrium density of a medium. The harmonic force is a particular case of Brillouin scattering. As for wave–wave excitation in dispersive flows — as far as the author knows, it has not been studied. The standalone problem is the interaction of modes inherent to a flow, that is, specified by means of perturbations that satisfy the system of conservation equations. The joint impact of nonlinearity and dispersion leads to stationary waveforms (including solitons) that may propagate with various speed. In this case, the wave or non-wave mode inherent to a flow applies as the exciter.

The study is organized as follows. The conservation equations describing the fluid dynamic of gassy liquid are reminded in Sect. 2. Excitation of sound by wave exciters is considered in Sect. 3. Section 3.2 is devoted to the interaction of modes inherent to a flow (i.e., internal interaction). For this purpose, decomposition of the conservation equations is undertaken, which leads to the coupling of a weakly nonlinear system of equations for the interacting modes [18]. Section 3.2 considers the exact solution to the Korteweg–De Vries equation [4] as an internal exciter, and the non-stationary excitation is discussed in Sect. 4.

2. Conservation equations and unforced wave dynamics

One-dimensional motion (along axis OX) of the mixture, which consists of compressible liquid including identical spherical bubbles, is considered. All bubbles contain an ideal gas, at equilibrium they are of the same radii, and they are well separated and oscillate as spheres. Heat and mass transfer between the liquid and gas phases and viscous and thermal losses are not taken into account. The bubbly liquid is an acoustic medium, which is treated as a homogeneous continuum due to comparatively large characteristic wavelength. The pressure in the mixture is associated with the pressure in the liquid phase [4, 19]. Quantities related to gas or liquid, are denoted by the indices g or l, respectively, perturbations are primed, and unperturbed quantities are marked with an additional zero. The density of the mixture, ρ, relates to the densities of the gas and liquid in the following manner

$$\rho = \frac{\rho_g \rho_l}{\alpha_0 \frac{\rho_{g0}}{\rho_0}\rho_l + \left(1-\alpha_0\frac{\rho_{g0}}{\rho_0}\right)\rho_g}, \tag{5}$$

where α_0 is the initial volume fraction of gas in the mixture. The wave motion of incompressible liquids, including bubbles, was originally studied by van Wijngaarden [4]. The equations for the conservation of momentum, energy, and mass of a mixture take the forms

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0,$$

$$\frac{\partial p'}{\partial t} - c_l^2 \frac{\partial \rho_l'}{\partial t} - \frac{c_l^2(\gamma_l - 1)}{\rho_{l0}}\rho_l' \frac{\partial \rho_l'}{\partial t} = 0,$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial (v\rho')}{\partial x} = 0, \tag{6}$$

where v, p denote the velocity and pressure of the mixture. The second equation in (6) follows from the continuity and energy equations for a pure liquid, where $\gamma_l = \frac{C_{p,l}}{C_{V,l}}\frac{\rho_{l0}}{p_{l0}}(\frac{\partial p_l}{\partial \rho_l})_{T=\mathrm{const}}$, and C_p and C_V denote the heat capacities at constant pressure and density, respectively. Mass of gas inside a spherical bubble of a radius R remains constant and it is disturbed evenly in the bubble's volume, so that

$$R^3 \rho_g = R_0^3 \rho_{g0}, \tag{7}$$

and there is no energy exchange between the bubbles and the surrounding liquid, i.e.,

$$p_g \rho_g^{-\gamma_g} = p_{g0}\ \rho_{g0}^{-\gamma_g}, \tag{8}$$

where $\gamma_g = \frac{C_{p,g}}{C_{V,g}}$. The leading-order form of the Rayleigh–Plesset equation, which accounts for compressibility of a liquid with finite speed of sound c_l, completes the system [19, 20]

$$R\frac{\partial^2 R}{\partial t^2} + \frac{3}{2}\left(\frac{\partial R}{\partial t}\right)^2 - \frac{2}{c_l^2}\left(\frac{\partial R}{\partial t}\right)^2 = \frac{p_g' - p_l'}{\rho_l}. \tag{9}$$

In dimensionless variables

$$v^d = \frac{v}{c},\ p^d = \frac{p'}{c^2\rho_0},\ \rho^d = \frac{\rho'}{\rho_0},\ x^d = \frac{x}{\lambda},\ t^d = \frac{tc}{\lambda}, \tag{10}$$

equation (6) involving (7)–(9) may be readily rearranged in the leading-order system that contains quadratic nonlinear terms on its right as [21]

$$\begin{aligned}
&\frac{\partial v^d}{\partial t} + \frac{\partial p^d}{\partial x} = -v^d\frac{\partial v^d}{\partial x} + \rho^d\frac{\partial p^d}{\partial x},\\
&\frac{\partial p^d}{\partial t} + \frac{\partial v^d}{\partial x} - \frac{\alpha_0(1-\alpha_0)R_0^2\rho_{l0}^2 c^4}{3(\gamma_g p_{g0})^2\lambda^2}\frac{\partial^3 p^d}{\partial t^3} = \\
&\quad -c^4\frac{\alpha_0(1-\alpha_0)^2\rho_{l0}^2(\gamma_g+1)}{(\gamma_g p_{g0})^2}p^d\frac{\partial v^d}{\partial x}\\
&\quad -c^2\frac{(1-\alpha_0)(\gamma_l+1)}{c_l^2}\rho^d\frac{\partial v^d}{\partial x} - v^d\frac{\partial \rho^d}{\partial x} + \rho^d\frac{\partial v^d}{\partial x},\\
&\frac{\partial \rho^d}{\partial t} + \frac{\partial v^d}{\partial x} = -v^d\frac{\partial \rho^d}{\partial x} - \rho^d\frac{\partial v^d}{\partial x}.
\end{aligned} \tag{11}$$

Here, λ denotes the characteristic scale of perturbation, and c is the speed of sound of an infinitely small magnitude in the bubbly liquid [4]

$$\frac{1}{c^2} = \frac{(1-\alpha_0)^2}{c_l^2} + \frac{\alpha_0(1-\alpha_0)\rho_{l0}}{\gamma_g p_{g0}}. \tag{12}$$

In the following formulas, the upper indices of dimensionless quantities will be dropped. Nonlinear and dispersive terms of the same order are considered. This ensures the possibility of a balance between nonlinearity and dispersion in the course of forced wave excitation. The analysis relies on the definition of modes in the small-magnitude flow in accordance to the three roots of the dispersion equation, i.e., two acoustic ones, specifying sound progressive in the positive and negative directions of an axis OX (marked by indices 1 and 2, respectively), and the non-wave type of motion, which is characterized by the stationary isobaric variation of temperature and zero velocity, called the entropy mode. They may be determined as links of specific perturbations of velocity, pressure, and density. Weak nonlinearity brings corrections to linear relations in wave modes [18, 21]. In particular, the links specifying the first wave mode with the eigen velocity c are as follows

$$\begin{aligned}
v_1 &= \rho_1 + \frac{1}{2}D\frac{\partial^2\rho_1}{\partial x^2} + \left(\frac{\varepsilon}{2} - 1\right)\rho_1^2,\\
p_1 &= \rho_1 + D\frac{\partial^2\rho_1}{\partial x^2} + (\varepsilon - 1)\rho_1^2,
\end{aligned} \tag{13}$$

with

$$D = \frac{\alpha_0(1-\alpha_0)R_0^2\rho_{l0}^2 c^4}{3(\gamma_g\, p_{g0})^2\lambda^2} \tag{14}$$

being a dimensionless parameter responsible for the dispersive properties of wave motion, which are enhanced for waves with short wave lengths. Weak dispersion relates to $D \ll 1$. The parameter of nonlinearity in a bubbly liquid

$$\varepsilon = \frac{(1-\alpha_0)c^2(\gamma_l+1)}{2c_l^2} + \frac{c^4\alpha_0(1-\alpha_0)^2\rho_{l0}^2(\gamma_g+1)}{2(\gamma_g p_{g0})^2} \tag{15}$$

measures nonlinear distortion of a waveform in finite-magnitude flows [2]. The leading-order equation governing the perturbation of the density in a wave that propagates in the positive direction of the OX axis takes the form

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\rho_1}{\partial x} + \varepsilon\rho_1\frac{\partial\rho_1}{\partial x} + \frac{D}{2}\frac{\partial^3\rho_1}{\partial x^3} = 0. \tag{16}$$

This is the famous Korteweg–De Vries equation (KdV), which has been discussed in details in connection with flow of a bubbly liquid [4, 6, 7] but has found application in many other important physical phenomena. Equation (16) imposes an intense rightwards propagating sound as compared to other internal modes of the flow. It contains a quadratically nonlinear term proportional to the parameter of nonlinearity ε and a term responsible for dispersion, proportional to D. The dimensionless eigen velocity of the wave equals 1.

3. Simple cases and excitation of the stationary wave

The external wave source F propagating with the speed C may be readily included in the right-hand part of the dynamic equation. The left-hand part relies on the wave equation for the density perturbation in the excited first acoustic mode (16). Seeking a solution to the equation

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\rho_1}{\partial x} + \varepsilon\rho_1\frac{\partial\rho_1}{\partial x} + \frac{D}{2}\frac{\partial^3\rho_1}{\partial x^3} = F(t - x/C), \tag{17}$$

depending only on two variables $\tau = t - x/C$ (retarded time) and x (this variable is "slow", expressing the slow variance of the profile with distance), we arrive at the leading-order equation

$$\frac{\partial\rho_1}{\partial x} + \frac{C - 1 - \varepsilon\rho_1}{C}\frac{\partial\rho_1}{\partial\tau} - \frac{D}{2C^3}\frac{\partial^3\rho_1}{\partial\tau^3} = F(\tau). \tag{18}$$

The general remark is that if the force is localized in space, then

$$\begin{aligned}
&\frac{\partial}{\partial x}\int_{-\infty}^{\infty} d\tau\ \rho_1 = -\int_{-\infty}^{\infty} d\tau\ \frac{C - 1 - \varepsilon\,\rho_1}{C}\frac{\partial\rho_1}{\partial\tau}\\
&+\frac{D}{2C^3}\int_{-\infty}^{\infty} d\tau\ \frac{\partial^3\rho_1}{\partial\tau^3} + \int_{-\infty}^{\infty} d\tau\ F(\tau) = \int_{-\infty}^{\infty} d\tau\ F(\tau),
\end{aligned} \tag{19}$$

i.e., $\int_{-\infty}^{\infty} \mathrm{d}\tau\ \rho_1$ does not remain constant as in the course of free oscillations and may constantly decrease or increase with distance from the transducer. If the boundary condition is zero at $x = 0$, then

$$\int\limits_{-\infty}^{\infty} \mathrm{d}\tau\ \rho_1 = x \int\limits_{-\infty}^{\infty} \mathrm{d}\tau\ F(\tau). \tag{20}$$

In the case of $C{\neq}1$, insignificant dispersion and nonlinear effects yield the dynamic equation

$$\frac{\partial\rho_1}{\partial x} + \frac{C-1}{C}\frac{\partial\rho_1}{\partial\tau} = F(\tau) \tag{21}$$

with the solution

$$\rho_1 = -\frac{C}{C-1}\int_t^{t-\frac{(C-1)}{C}x} \mathrm{d}z\ F(z), \tag{22}$$

if $C \neq 1$. The case $C = 1$ leads to the solution growing with the distance from a transducer,

$$\rho_1 = xF(\tau). \tag{23}$$

This reveals the resonant character of the interaction if the exciter and exciting wave velocities coincide in non-dispersive flow. Dispersion brings special features in resonant excitation. Equation (18), describing the stationary waveforms $\rho_1(\tau)$, is readily integrated with the result

$$\frac{\varepsilon}{2}\rho_1^2 + (1{-}C)\rho_1 + \frac{D}{2C^2}\frac{\partial^2\rho_1}{\partial\tau^2} = -C\int \mathrm{d}\tau\ F(\tau). \tag{24}$$

3.1. Harmonic excitation

We associate the source with the harmonic wave $F(\tau) = A\sin(\tau)$. The case with $D = 0$, $C = 1$ is readily analyzed. In particular, the exact solution corresponding to the integration constant $2A/\varepsilon$ is

$$\rho_1 = 2\sqrt{\frac{A}{\varepsilon}}\cos\left(\frac{\tau}{2}\right). \tag{25}$$

Also, with $\varepsilon = 0$ and $C = 1$, and by restricting to the expansion of the same integration constant, the explicit solution is obtained

$$\rho_1 = -\frac{16A}{D}\cos\left(\frac{\tau}{2}\right), \tag{26}$$

satisfying the same boundary conditions

$$\rho_1(-\pi) = \rho_1(\pi) = 0. \tag{27}$$

With the use of variables

$$\tilde{\rho}_1 = \frac{C^2\varepsilon}{D}\rho_1, \quad G_1 = \frac{2C^2(1{-}C)}{D}, \quad \tilde{A} = \frac{2C^5\varepsilon A}{D^2}, \tag{28}$$

(24) can be expressed as

$$\tilde{\rho}_1^2 + G_1\tilde{\rho}_1 + \frac{\partial^2\tilde{\rho}_1}{\partial\tau^2} = \tilde{A}\cos(\tau) + Q, \tag{29}$$

where Q is an integration constant.

Figure 1 shows solutions to (29) with boundary conditions (27) for different values of $\tilde{A}$ and G_1. The case of $\tilde{A} = 0$ corresponds to the stationary waveforms without external excitation.

There is a variety of excitation of stationary waveforms with speeds C different from 1.

3.2. Excitation of the stationary wave by steady-form wave inherent to a flow

The equation governing the perturbation of the density in the second mode with an account of the interaction with the first wave mode was obtained by Perelomova and Pelc-Garska [18]

$$\frac{\partial\rho_2}{\partial t} - \frac{\partial\rho_2}{\partial x} - \varepsilon\rho_2\frac{\partial\rho_2}{\partial x} - \frac{D}{2}\frac{\partial^3\rho_2}{\partial x^3} = F(x,t) = \frac{D(4{-}3\varepsilon)}{4}\rho_1\frac{\partial^3\rho_1}{\partial x^3}. \tag{30}$$

The right-hand side of (30) reflects two causes of the interaction of internal modes, namely nonlinearity and dispersion. The acoustic source is of the order of the square Mach number and it is proportional to the dispersion parameter. In contrast, an external exciter might have a much larger magnitude. Stationary excitation in the field of the first mode is possible when the first mode is stationary and propagates with the constant speed C, which may be different from 1 — hence, the source F is a function of τ. This is possible due to the joint impact of dispersion and nonlinearity. Now, (30) is converted as

$$\frac{C+1+\varepsilon\,\rho_2}{C}\frac{\partial\rho_2}{\partial\tau} + \frac{D}{2C^3}\frac{\partial^3\rho_2}{\partial\tau^3} = F(\tau) = \frac{D(3\varepsilon-4)}{4C^3}\rho_1\frac{\partial^3\rho_1}{\partial\tau^3} \tag{31}$$

and may be readily integrated into the result

$$\frac{\varepsilon}{2}\rho_2^2 + (1+C)\rho_2 + \frac{D}{2C^2}\frac{\partial^2\rho_2}{\partial\tau^2} = C\int \mathrm{d}\tau\ F(\tau) = \frac{D(3\varepsilon-4)}{4C^2}\left[\rho_1\frac{\partial^2\rho_1}{\partial\tau^2} - \frac{1}{2}\left(\frac{\partial\rho_1}{\partial\tau}\right)^2\right] + Q, \tag{32}$$

where Q is an integration constant. As internal exciters, we may consider exact solutions to the KdV equation in the form of acoustic solitons

$$\rho_1 = \rho_{1,1}(\tau) = \frac{3D}{\varepsilon\,(1+\cosh(C\tau))} + \frac{C-1-D/2}{\varepsilon},$$

$$\rho_1 = \rho_{1,2}(\tau) = \frac{3}{\cosh^2\left(C\sqrt{\frac{\varepsilon}{2D}}\tau\right)} + \frac{C-1-\varepsilon}{\varepsilon} \tag{33}$$

and the internal forces related to them, F_1, F_2, respectively. Note that C may differ from 1 and may take negative values. The first source mode is de facto determined by the links of specific perturbations (13). Variety of C yields different limits of

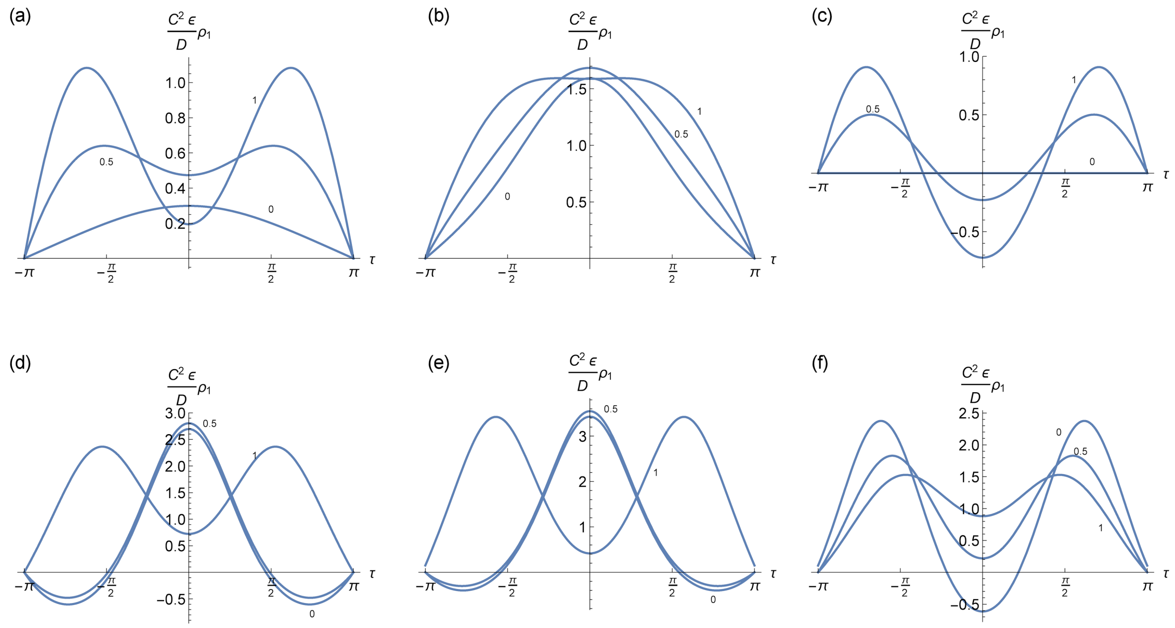

Fig. 1. Solutions to (29) for different values of $\tilde{A}$ $(0, 0.5, 1$, marked by numbers) and G_1 (0 in panel (a, d), -1 in panel (b, e), 1 in panel (c, f)). Upper row: zero integration constant Q, lower row: $Q = 2$.

density perturbation if $\tau \to \pm\infty$. In the terms of dimensionless variable $\tilde{\rho}_2 = \frac{C^2\varepsilon}{D}\rho_2$, (32) takes the form

$$\tilde{\rho}_2^2 + G_2\tilde{\rho}_2 + \frac{\partial^2\tilde{\rho}_2}{\partial\tau^2} = \frac{2C^5\varepsilon}{D^2}\int \mathrm{d}\tau\ F(\tau) = \frac{C^3\varepsilon(3\varepsilon-4)}{2D}\left[\rho_1\frac{\partial^2\rho_1}{\partial\tau^2} - \frac{1}{2}\left(\frac{\partial\rho_1}{\partial\tau}\right)^2\right] + Q,$$

$$G_2 = \frac{2C^2(1+C)}{D}. \tag{34}$$

Interestingly, (34) describes a large variety of interactions. In particular, if $C{=}1{+}D/2$ and $F{=}F_1$, (34) takes the form

$$\tilde{\rho}_2^2 + \frac{(D+2)^2(D+4)}{4D}\tilde{\rho}_2 + \frac{\partial^2\tilde{\rho}_2}{\partial\tau^2} = \frac{9D(D+2)^5(3\varepsilon-4)}{1024\varepsilon} \times\left(\cosh\left(\frac{(D+2)\tau}{2}\right) - 3\right)\cosh^{-6}\left(\frac{(D+2)\tau}{4}\right) + Q. \tag{35}$$

If $C = -1$ and $F = F_1$, this leads to

$$\tilde{\rho}_2^2 + \frac{\partial^2\tilde{\rho}_2}{\partial\tau^2} = \frac{3(3\varepsilon-4)}{64\varepsilon} \times\Big(3(5D-4) - 8(D+1)\cosh(\tau) + (D+4)\cosh(2\tau)\Big) \times\cosh^{-6}\left(\frac{\tau}{2}\right) + Q. \tag{36}$$

The case of $C = 1$ and $F = F_1$ brings us to

$$\tilde{\rho}_2^2 + \frac{4}{D}\tilde{\rho}_2 + \frac{\partial^2\tilde{\rho}_2}{\partial\tau^2} = \frac{3D(3\varepsilon-4)}{64\varepsilon} \times\Big(8\cosh(\tau) - \cosh(2\tau) - 15\Big)\cosh^{-6}\left(\frac{\tau}{2}\right) + Q. \tag{37}$$

Figure 2 illustrates the case of excitation by the force F_1 propagating with the speed $C = -1$ (see (36)) for different boundary conditions and integration constants. Usually, the parameter of nonlinearity in the presence of bubbles far exceeds the parameter of nonlinearity of the pure phases. For simple evaluations, we may use the following data: $\rho_{l0} = 10^3$ kg/m^3, $p_{g0} = 10^5$ Pa, $\gamma_g = 1.4$, $\gamma_l = 7$, $c_l = 1500$ m/s. The values of the initial volume fractions of gas in the mixture, i.e., $\alpha_0 = 10^{-3}$, $\alpha_0 = 10^{-4}$, and $\alpha_0 = 10^{-5}$, affect and set the values of $\varepsilon = 1064$, $\varepsilon = 4562$, and $\varepsilon = 1392$, respectively. Hence, in the evaluations of the right-hand side of (36), a limited expression describing the source (if $\varepsilon \to \infty$) is used

$$\frac{9(3(5D-4) - 8(D+1)\cosh(\tau) + (D+4)\cosh(2\tau))}{64} \times\cosh^{-6}\left(\frac{\tau}{2}\right) + Q. \tag{38}$$

Figure 2 shows the numerical solutions of (36) with the right-hand side (38) for some boundary conditions and integration constants Q.

The parameter D is small compared to unity, which at the same time indicates the small value of the characteristic inverse duration of perturbation, $\omega = 2\pi c/\lambda$. Evaluations — in accordance to (14) and the listed equilibrium parameters — yield $\omega \ll 66$ kHz for $\alpha_0 = 10^{-3}$, $\omega \ll 82$ kHz for $\alpha_0 = 10^{-4}$, and $\omega \ll 173$ kHz for $\alpha_0 = 10^{-5}$. All sources considered in this section are even functions. Therefore, the example ensures relation: even excited perturbations for even boundary conditions. Uneven boundary conditions would lead to uneven excited perturbations.

4. Non-stationary resonant and non-resonant excitation

4.1. Excitation with zero and non-zero detunings

When using $\tilde{\rho}_1$, $\tilde{A}$ from (28) and $\theta = t - x$, (17) in the case of an impulsive exciter takes the form

$$\frac{\partial \rho_1}{\partial x} + D\rho_1 \frac{\partial \rho_1}{\partial \theta} - \frac{D}{2}\frac{\partial^3 \rho_1}{\partial \theta^3} = \frac{D}{2}\tilde{A}\exp(-(\theta + \delta x)^2), \tag{39}$$

where $\delta = (1 - \frac{1}{C})$ denotes dimensionless detuning. The results of the numerical evaluation of forced oscillations are shown in Fig. 3. All evaluations were made with the help of Wolfram Mathematica. The initial perturbation at $z = 0$ is zero, $\tilde{\rho}(\theta, 0) = 0$, and the boundary conditions are considered zero at $\theta = \pm 1000$. Positive and negative detunings of the same absolute value yield indistinguishable curves.

The maximum magnitude in the course of excitation in Newtonian flows reveals beating if the detuning δ differs from zero [12, 22]. The period of these beatings depends on δ. The maximum magnitude in the dispersive flow increases in a continuous manner with x and tends to some limiting value. Zero detuning corresponds to resonant excitation. In this case, the linear growth of the maximum perturbation magnitude with distance agrees with (23).

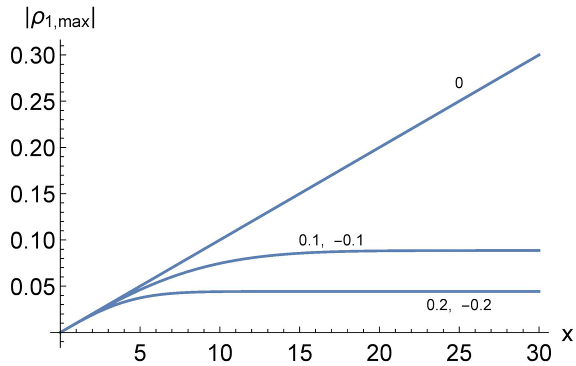

Fig. 3. Maximum absolute value of $\rho_1 = \frac{D}{\varepsilon}\tilde{\rho}_1$ over θ varying from -1000 to 1000 as a function of z in accordance to the numerical solution of (39) for different values of δ (0, ± 0.1, ± 0.2, marked by numbers). The case shown is for $D = 0.1$, $\varepsilon = 100$, $A = 0.01$.

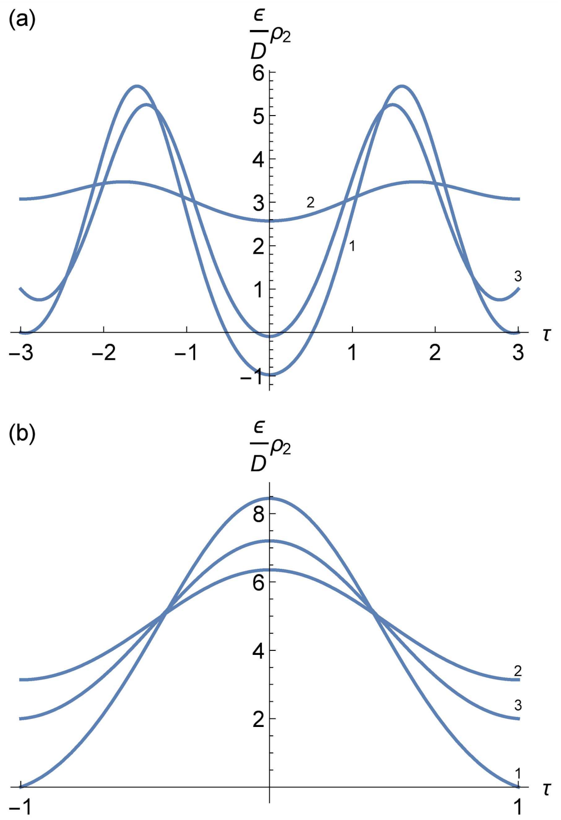

Fig. 2. (a) Solutions to (36) for $D = 0.1$, integration constant $Q = 10$ and different boundary conditions: $\tilde{\rho}_2(-3) = \tilde{\rho}_2(3) = 0$ (line 1), $\frac{\mathrm{d}\tilde{\rho}_2(-3)}{\mathrm{d}\tau} = \frac{\mathrm{d}\tilde{\rho}_2(3)}{\mathrm{d}\tau} = 0$ (line 2), $\tilde{\rho}_2(-3) = \tilde{\rho}_2(3) = 1$ (line 3). (b) Solutions to (36) for $D = 0.1$, $Q = 25$ and different boundary conditions: $\tilde{\rho}_2(-1) = \tilde{\rho}_2(1) = 0$ (1), $\frac{\mathrm{d}\tilde{\rho}_2(-1)}{\mathrm{d}\tau} = \frac{\mathrm{d}\tilde{\rho}_2(1)}{\mathrm{d}\tau} = 0$ (2), $\tilde{\rho}_2(-1) = \tilde{\rho}_2(1) = 2$ (3).

4.2. Resonant excitation of the stationary wave by the entropy mode

The standalone case is the excitation of the entropy mode by sound. The leading-order dynamic equation governing the excess density of the entropy mode in the field of the first dominant wave mode has been derived by Perelomova and Pelc-Garska [18]

$$\frac{\partial \rho_3}{\partial t} = -D(\varepsilon - 2)\rho_1 \frac{\partial^3 \rho_1}{\partial x^3}. \tag{40}$$

Dispersion has an impact only on the source right-hand part of (40). In fact, (40) imposes a variety of non-resonant interactions. It is readily integrated by assuming $\rho_1(x - Ct)$ and the onset of excitation at $t = 0$. As a result, one obtains

$$\rho_3 = \frac{D(\varepsilon - 2)}{C}\left[\rho_1 \frac{\partial^2 \rho_1}{\partial x^2} - \frac{1}{2}\left(\frac{\partial \rho_1}{\partial x}\right)^2\right]\Bigg|_x^{x-Ct}. \tag{41}$$

For any $C \neq 0$, a perturbation consists of two parts: one moving with the speed C, the other — stationary. The ideal resonant interaction are possible if $C \to 0$. In this case, ρ_3 increases infinitely with time

$$\lim_{C \to 0} \rho_3 = -D(\varepsilon - 2)t\rho_1 \frac{\partial^3 \rho_1}{\partial x^3} = D(\varepsilon - 2)t\phi(x). \tag{42}$$

The limiting expressions of ϕ_1 and ϕ_2 relating to $\rho_{1,1}$ and $\rho_{1,2}$ when $C \to 0$ (see (33)) take the respective forms

$$\phi_1 = \frac{9\big(\cosh(x) - 5\big)\tanh^3\left(\frac{x}{4}\right)}{8\varepsilon^2\cosh^4\left(\frac{x}{4}\right)}, \tag{43}$$

$$\phi_2 = \frac{3\sqrt{\varepsilon}\Big(1 - 5\varepsilon + (\varepsilon+1)\cosh\left(\sqrt{2\varepsilon}x\right)\Big)}{2\sqrt{2}\cosh^7\left(\sqrt{\frac{\varepsilon}{2}}x\right)} \times \left[\sinh\left(3\sqrt{\frac{\varepsilon}{2}}x\right) - 11\sinh\left(\sqrt{\frac{\varepsilon}{2}}x\right)\right]. \tag{44}$$

Since the exemplary perturbations in the density ρ_1 are even functions of x, an excited perturbation in density is an odd function of x. This ensures equal areas with positive and negative perturbations, making zero average perturbation in the density ρ_3 and zero average perturbation of temperature associated with the entropy mode, $T_3' = -\rho_3 T_0/\rho_0$.

5. Conclusions

The main goal of this study is to draw attention to the phenomenon of the excitation of waves in a weakly nonlinear dispersive fluid flow. In this study, dispersion specifying wave processes in a bubbly liquid is considered. Free perturbations in a flow are described by the KdV equation (16). This partial differential equation serves also as a mathematical model of waves on shallow water surfaces and has numerous connections to physical problems. It describes the evolution of one-dimensional long internal waves in the ocean with stratification of mass density, acoustic plasmic waves, and waves in a crystal lattice. In particular, the leading-order dynamic equation for the displacement of an atom, u, in the one-dimensional lattice consisting of atoms of equal mass M takes the form of the linearized KdV equation

$$\frac{\partial u}{\partial x} = \frac{a^2}{24c_0^3}\frac{\partial^3 u}{\partial \tau^3}, \tag{45}$$

where a denotes the distance between atoms in equilibrium, and $c_0 = a\sqrt{\kappa/M}$ designates the low-frequency sound speed. (The elastic interaction between closest neighbors is assumed, so that the force acting at atom number n is given by the formula $-\kappa(2u(n) - u(n-1) - u(n+1))$) Forced oscillations are described by the inhomogeneous KdV equation. As far as the author knows, neither external (i.e, caused by some external force) nor internal (caused by mode inherent to a flow) excitations have been considered in the previous studies. Rudenko and Hedberg [12] were the first to determine and discuss resonant excitation of waves by external wave exciters in connection with different kinds of nonlinearity (modular, quadratic, and quadratically-cubic) and Newtonian attenuation. The resonant interactions take place if the velocity of exciter coincides with the velocity of an excited wave. In the flow without attenuation, this leads to the linear growth of magnitudes of perturbations of the excited wave with the distance in the boundary excitation. Newtonian attenuation prevents this enlargement. Pure dispersion does not lead to attenuation of macroscopic energy. That is why the linear growth of the magnitude of the excited perturbations coincides with the formula (23).

Indeed, non-resonant and resonant interactions are also various in the flow with dispersion. The mode inherent to a flow may act as a stationary exciter, which is made possible due to the joint impact of nonlinearity and dispersion. This study considers also the resonant interactions of wave and entropy modes inherent to a flow, which occurs if an exciter represents stationary waveform with a speed approaching zero (Sect. 4.2). We do not consider the non-resonant excitation of the entropy mode in the field of intense harmonic sound. This kind of excitation was earlier discussed by Perelomova and Pelc-Garska in [18]. The initial points and simplifying conditions were the same as in the present study. In particular, the interaction reduces the density of the mixture and hence causes enlargement of bubbles. The reason for that is the nonlinear interaction of the entropy and sound modes, which may take place without diffusion and is not connected with the increase in the bubble mass. This phenomenon recalls isobaric acoustic heating in Newtonian flows that is followed by a decrease in the fluid density. The efficiency of the nonlinear generation of the entropy mode is proportional to the squared acoustic pressure and to the radius of a bubble R_0. This agrees with the features of the dynamics in the course of rectified diffusion [23], but reflects a different reason for this phenomenon.

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