INFORMATIONAL ENTROPY IN SIMULATION OF ONE-DIMENSIONAL RANDOM FIELDS

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Abstract: The entropy H of a continuous distribution with probability density function $f(\bullet)$ is defined as a function of the number of nodes (n) in a one-dimensional scalar random field. For the second order theory this entropy is expressed by the determinants of the covariance matrices and simulated for several types of correlation functions. In the numerical example the propagation of the entropy for the static response of linear elastic, randomly loaded beam has been considered. Two unexpected results have been observed:

- function $H(n)$ is entirely different for differentiable (m. s.) and non-differentiable fields, with the same parameters in the correlation functions,
- in some cases, the greater randomness at the input (measured by the entropy) does not lead to the greater randomness at the output.

Keywords: random field, information entropy, beam structure, discretization

1. Introduction

In this paper we investigate simulations in a probabilistic model of a structure, with the help of the entropy concept. One can understand the essence of the informational entropy taking into consideration a random experiment with two mutually exclusive events. Let us consider only two cases. In the first case the corresponding probabilities are $p_1 = 0.99$, $p_2 = 0.01$ and in the second case $p_1 = p_2 = 0.5$. Making a comparison between the two cases, one can state that in the first case the situation is clear: the event with probability 0.99 will occur more frequently and one can assume that in a single trial just this event will happen. In the second case the result of a single trial is not so easily foreseeable.

A quantity which is a measure of the uncertainty in a discrete distribution is the informational entropy:

$$H = -\sum_{i=1}^{k} p_i \ln p_i, \tag{1}$$

where k denotes the number of values of the random variable.

The entropy of a continuous distribution with n-dimensional density function is defined by quite a different formula:

$$H = -\int_{\mathscr{D}} \dots \int f(\mathbf{u}) \ln f(\mathbf{u}) d\mathbf{u}, \tag{2}$$

where $\mathscr{D} \subseteq R^n$, and $f(\mathbf{u})$ is the probability density function of n-dimensional random variable $\mathbf{U}$. The interpretation of this entropy is that it is an objective measure of the uncertainty in a continuous distribution.

Until now, the most significant application of this idea is the maximum entropy principle [1]. Sobczyk and others have advocated maximization of this entropy as a criterion for determining probability distribution in a system described by random variables. The available countable information is expressed as statistical moments and utilized as constraints. Some determined from the principle approximations of $f(\mathbf{u})$ in the form of sequences of functions $f_N(\mathbf{u})$ converge – in some sense – to $f(\mathbf{u})$ as N tends to infinity. However, some other applications of the entropy concept are possible.

We apply the entropy in simulation of structural mechanics models (*cf.* [2]). Every structure is characterized by the compactness of the material – at least locally. Therefore, in the description of the structural geometry, the position parameter must be continuous in some domains. That is why the argument of random fields describing the structural properties is continuous at the first step of the modelling. Since our approach makes use of computers, the continuous parameters are discretized and a mesh of nodes arises. The scalar random field takes on the form of n-dimensional random variable, where n is the number of nodes. We shall regard the entropy given by formula (2) as a function of the number of nodes (this concept has been introduced in [3]).

Theoretically, for such n-variate integrals as (2), the Monte Carlo quadrature should ordinarily be used. This results from the independence of the sampling error of the dimensionality n.

However, for the second order fields it is possible to derive (for the normal distribution) an explicit formula for relation between second order moments and the entropy. Using the diagonalization procedure for a real covariance matrix, one obtains:

$$H(n) = \frac{1}{2} \ln((2\pi e)^n |\mathbf{K}|), \tag{3}$$

where $|\mathbf{K}|$ is the determinant of the covariance matrix $\mathbf{K}(n \times n)$. This formula will be used in our calculations.

2. Simulation of some scalar second order random fields

The complete description of these fields is given by the mean value functions and the covariance functions.

The following types of the covariance functions are considered (*cf.* [4]):

- Markov field

$$K(x_1,x_2)=\sigma^2\exp(-\beta|x_1-x_2|), \qquad \beta>0, \tag{4}$$

- Shinozuka field

$$K(x_1,x_2)=\sigma^2\exp(-\beta(x_1-x_2)^2), \qquad \beta>0, \tag{5}$$

- White noise field

$$K(x_1,x_2)=\begin{cases}\sigma^2 & x_1=x_2\\ 0 & x_1\neq x_2\end{cases}, \tag{6}$$

- Binary noise field

$$K(x_1,x_2)=\begin{cases}\sigma^2\left(1-\frac{|x_1-x_2|}{\beta}\right) & \beta>|x_1-x_2|\\ 0 & \beta\leq|x_1-x_2|\end{cases}, \tag{7}$$

- Wiener field

$$K(x_1,x_2)=\sigma^2\min(x_1,x_2), \tag{8}$$

where $x_1,x_2\in\mathscr{R}$.

The fields defined by Equations (4)–(7) are homogeneous since the covariance functions depend on the coordinate differences only. The field (8) is non-homogeneous.

In order to simulate the n-dimensional random variable with normal distribution the Cholesky decomposition of the covariance matrix $\mathbf{K}$ is applied:

$$\mathbf{K}=\mathbf{B}\mathbf{B}^T, \tag{9}$$

where $\mathbf{B}$ is a lower triangular matrix.

The following fact has been used here: if the coordinates of vector $\mathbf{U}$ are mutually independent and have the same normal distribution, then vector $\mathbf{V}=\mathbf{B}\mathbf{U}$ ($\det\mathbf{B}\neq 0$) has the normal distribution with the covariance matrix (9) (*cf.* [5]). A sample generated in this way is applied for calculation of the covariance matrix estimator.

In some cases the term "Cholesky decomposition" is used to the form $\mathbf{TDT}^T$, where $\mathbf{T}$ is a lower triangular matrix with units on the diagonal, and $\mathbf{D}$ is a diagonal matrix. In our case the form (9) is applied since the programming is more convenient. The decomposition $\mathbf{TDT}^T$ has some advantages for negative definite matrices. Since covariance matrices are always non-negative definite we control this property on the base of diagonal elements of $\mathbf{B}$ in decomposition (9).

3. Numerical example

Propagation of the entropy in a stochastically linear system is considered. Discussion is based on an elastic model of a clumped cantilever beam with a fixed length. A random field (*e.g.* a wind load) acts on the system (the load is illustrated in Figure 1). At the output the response of the beam is observed. In this case the response is the end displacement of the cantilever.

The displacement is an effect of the action of random forces. This relation is governed by some inner parameters of the beam, which are assumed here as deterministic. The assumption of small displacements is accepted such that the linear theory is used.

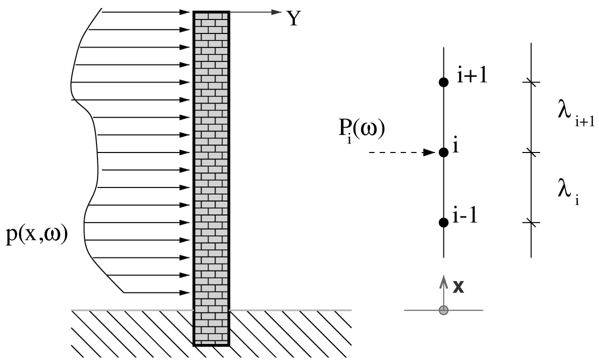

Figure 1. The cantilever beam model and the scheme for field discretization

Random field of the load $p(x,\omega)$, $\omega \in \Omega$ (the space of elementary events), $x \in \mathscr{R}$, is discretized by the mid-point method (*cf.* Figure 1):

$$P_i(\omega) = \frac{\lambda_{i+1}+\lambda_i}{2} p(x,\omega)\big|_{x=x_i}.$$

From this approach the formulae for the first two statistical moments follow:

$$\bar{P}_i = \frac{\lambda_{i+1}+\lambda_i}{2} E(p(x,\omega)\big|_{x=x_i}),$$

$$K_{ij} = \frac{(\lambda_{i+1}+\lambda_i)(\lambda_{j+1}+\lambda_j)}{4} K_p(x_i,x_j), \qquad i,j=1,\ldots,n.$$

Function $K_p(x_i,x_j)$ is the covariance function defined by one of the formulae from (4) to (8). The variances of the fields with discretized parameter are assumed as constant for fields with a variable number of nodes. From this assumption the relation between variance σ^2 of the field with continuous parameter and variance σ_{ii}^{2discr} of the field with discrete parameter follows:

$$\sigma^2 = \frac{4}{(\lambda_{i+1}+\lambda_i)^2} \sigma_{ii}^{2discr}.$$

Positions of the nodes are described by coordinates x_i. Increasing the number of nodes leads to decreasing the distance between the nodes, since the length of the beam is fixed. The distances between the nodes are equal.

The variance of the displacement at the beam top (the output of the system) is calculated from the formula:

$$\sigma_Y^2 = \mathbf{c}\mathbf{K}\mathbf{c}^T,$$

where $\mathbf{c}$ is the flexibility row vector of the system and $\mathbf{K}$ is the estimator of the load covariance matrix.

In order to get a constant mean displacement of the beam, the mean load vector is reduced, depending on the number of nodes. From properties of the covariance matrix it follows that a change in the mean value vector does not influence the covariance matrix, thus does not change the entropy. The entropies at the input and the output are calculated from formula (3).

4. Results and conclusions

Numerical results presented in Figures 2–5 are expressed in the coordinate system: the number of nodes – the entropy. Numbers given in parentheses correspond

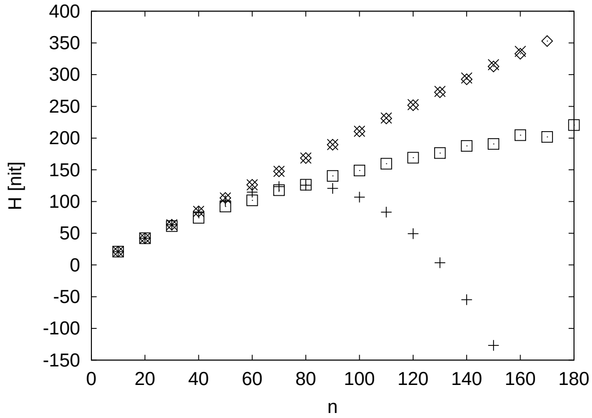

Figure 2. The entropy of four homogeneous fields at the input of the beam; ◇ – Markov (4.0,26.0), + – Shinozuka (4.0,26.0), ⊡ – binary noise (4.0,0.5), × – white noise (4.0)

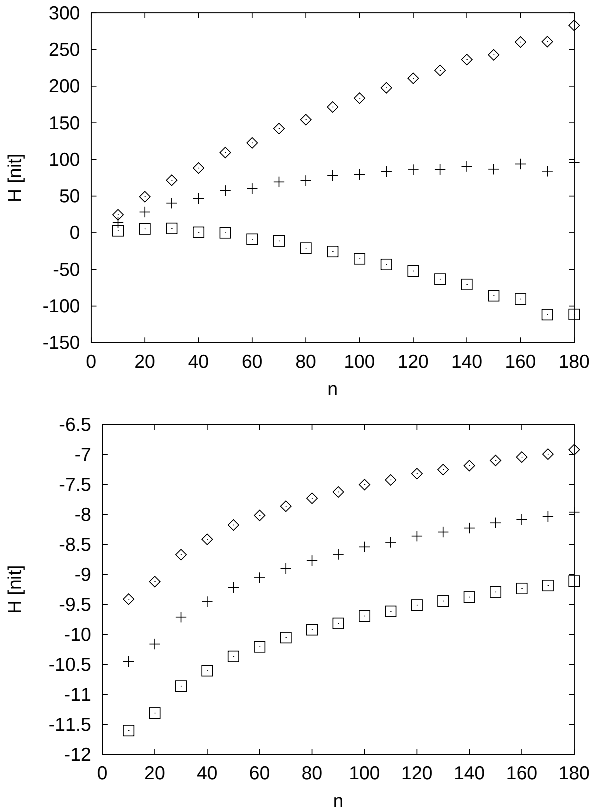

Figure 3. The entropy of binary noise at the input (top) and at the output of the cantilever beam (bottom); ◇ – binary noise (8.0,0.5), + – binary noise (1.0,0.5), ⊡ – binary noise (0.1,0.5)

to: the variance of a field with discretized parameter (the first number) and the correlation function parameter (the second number). It should be noticed that the meaning of the correlation function parameter for the binary noise field is quite

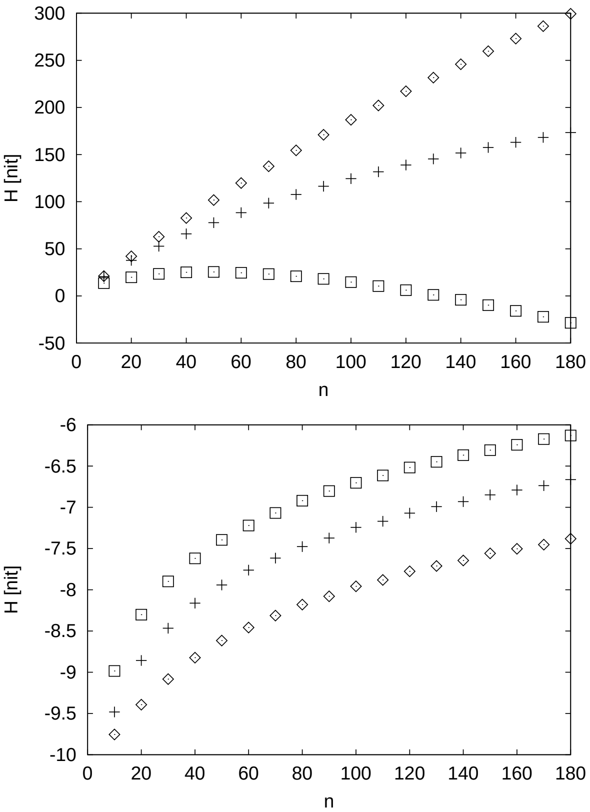

Figure 4. The entropy of Markov field at the input (top) and at the output of the cantilever beam (bottom); ◇ – Markov (4.0,5.0), + – Markov (4.0,1.0), ⊡ – Markov (4.0,0.1)

different from the meaning of the corresponding parameters for Markov and Shinozuka fields.

- The most unexpected result is a great difference in the entropy at the input for Markov and Shinozuka fields (Figure 2). However, it should be stressed that field (4) is nondifferentiable (in the mean square sense) while field (5) is differentiable. The main reason for this discrepancy in the entropy functions is the difference in the shape of correlation functions in the neighbourhood of zero. For sufficiently small distances between the nodes the covariances of Shinozuka field are high and the determinant tends to zero. Therefore the Shinozuka field is not a good model of fully random phenomena, although it is very useful in some analytical models.
- Making a comparison of the functions at the input and at the output (Figure 3) one can state that the entropy is an effective tool in evaluation of quality of the discretization of the random field. At the output one can observe an asymptotic behaviour of the graphs – the variance of the beam displacement becomes stabilized. This observation is valid for all homogeneous fields considered here.
- In agreement with our intuition, if the variances at the input and at the output increase then the entropy increases as well. However, some unexpected phe-

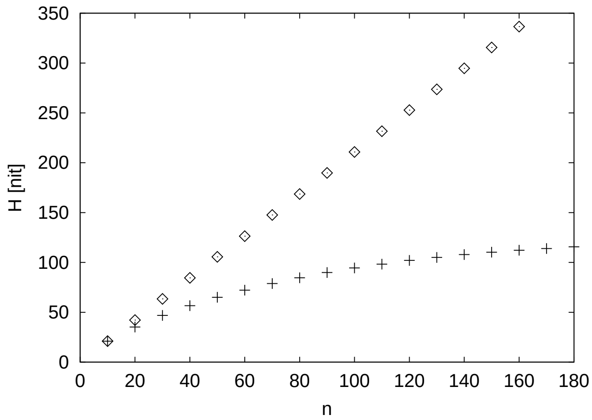

Figure 5. The comparison of entropies of white noise and Wiener fields at the input of the cantilever beam; ◇ – white noise (4.0), + – Wiener (4.0)

nomenon has been observed in the case of changing the correlation parameter. For the Markov field the "inversion" phenomenon (under a constant variance) appears (Figure 4). This means that even in a linear system it is not true that greater randomness at the input leads to greater randomness at the output. This observation concerns all fields with correlation function characterized by a parameter of decay.

- A comparison of the entropy functions for the white noise field and Wiener field (Figure 5) shows the effect of reduction of this measure of randomness. It follows (among other things) from continuity of the realizations of the Wiener field.

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