# Dynamical description of quantum computing: Generic nonlocality of quantum noise

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We develop a dynamical non-Markovian description of quantum computing in the weak-coupling limit, in the lowest-order approximation. We show that the long-range memory of the quantum reservoir (such as the $1/t^4$ one exhibited by electromagnetic vacuum) produces a strong interrelation between the structure of noise and the quantum algorithm, implying nonlocal attacks of noise. This shows that the implicit assumption of quantum error correction theory—independence of noise and self-dynamics—fails in long time regimes. We also use our approach to present *pure* decoherence and decoherence accompanied by dissipation in terms of the spectral density of the reservoir. The so-called *dynamical decoupling* method is discussed in this context. Finally, we propose a *minimal decoherence model*, in which the only source of decoherence is vacuum. We optimize the fidelity of quantum-information processing under the trade-off between the speed of the gate and the strength of decoherence.

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## I. INTRODUCTION

In spite of many remarkable results on decoherence in open quantum systems [1] the problem still remains open, involving some conceptual and interpretational difficulties in the description of the dynamical quantum effect. Recently the problem has become crucial in connection with the idea of quantum computers [2,3]. The latter stimulated a huge theoretical and experimental effort to control evolution of quantum systems. This progress allows one to hope that theoretical obstacles to building a quantum computer (QC) can be overcome. To deal with unavoidable decoherence, the quantum error correcting codes have been designed [4,27], resulting in the theory of fault-tolerant (FT) quantum computation [6–9]. In this theory it is tacitly assumed that decoherence acts *independently* of the structure of the controlled self-evolution of the QC (the one including the quantum algorithm and the scheme of error correction). The self-evolution only *propagates* the errors. In fact, decoherence depends on the kind of dynamics of the system of interest, and this should be taken into account.

The main purpose of this paper is to provide a more complete general analysis of interrelations between decoherence and controlled evolution, basing on the well-established theory of open systems [10]. We develop the dynamical, Hamiltonian description of the decohering quantum computer, deriving and analyzing the non-Markovian master equation. We show that the long-range quantum memory in conjunction with self-dynamics of the quantum computer implies a highly nonlocal structure of noise. This has remarkable implications for the quantum error correction concept. The key idea of the latter is that the noise is local, so that the quantum information can be hidden in multiqubit entanglement. Once we do not couple the qubits through quantum gates, the noise is local indeed. However, to protect the information against the noise, we need to produce multiqubit entanglement. Due to memory, the reservoir will then ''see'' the evolution as highly nonlocal, and after some time it will act nonlocally itself. We argue that the long-range quantum memory is unavoidable for fundamental reasons: it is exhibited by interaction with vacuum, which can never be removed. It turns out that the memory is relevant even for the quantum optics regime, despite the fact that the deviation from exponential decay is unobservable.

In Refs. [7,9] it was noted that the FT method may not work, if time or space correlations do not decay exponentially. What we show here is that correlations in time cause the noise to follow the evolution of the system that is to protect against the noise. It follows that the FT scheme should be revisited to take into account the memory effect.

One should mention that the long quantum memory is exhibited also by phonon environment that characterized stronger coupling than electromagnetic vacuum. This means that within the program of solid-state quantum computation the memory effects can become practically relevant.

Other programs to avoid decoherence use *dynamical decoupling* [11], and *decoherence-free subspaces* [12]. We discuss the first approach in the context of the real physical environment and conclude that it is hard to find conditions for which the method could be useful. The decoherent-free subspaces approach bases on the paradigm opposite to the one of quantum error correction—it exploits symmetries of collective interaction with environment. Even though our approach is fully general, in this paper we will discuss the independent interaction case.

We also introduce a *minimal decoherence model* of evolution, within the framework of the non-Markovian approach. The motivation behind the model is to try to take into account only fundamental obstacles for building the quantum computer. That is, we remove any decoherence mechanism that could be, in principle, avoided at some level of technology. In the model the only environment is the vacuum, and there is no internal, natural self-evolution (the only evolution of the system is constituted by quantum gates). It follows that *there is* some room to optimize the process of quantum computation within this model. Subsequently, we obtain time-energy trade-off due to that fact that to perform a given gate one needs a constant amount of ''action'' that can be split in different ways into time and energy. Higher energy implies a quick gate, but simultaneously, enhances decoher-

 

ence. As a result it is possible to maintain high fidelity at the expense of poorer scaling. For a sequence of $n$ gates, the physical time needed to maintain high fidelity scales as $n^{3/2}$.

This paper is organized as follows. In Sec. II we make some preliminary remarks. Then we develop the Hamiltonian approach in Sec. III. We obtain the transition map (called here the error map). Then we pass to the Markovian limit for the time-independent Hamiltonian and indicate that in the non-Markovian stage the reservoir recognizes the structure of levels of the systems. We discuss possible types of spectral densities causing decoherence and the problem of high-frequency cutoffs. Subsequently we discuss the dynamical decoupling method. In Sec. IV we present two examples—the QC in a memoryless reservoir, and the QC driven by kicked dynamics. We discuss the error path interference effect. In Sec. IV C memory causing by interaction with vacuum is derived. Then we pass to the main result of the paper (Sec. V), deriving and discussing the master equation for decohering QC. We relate the results to the quantum error correction concept. In Sec. VI we postulate the *minimal- decoherence model*, and provide a formula for fidelity within the model. An example of fidelity for single-qubit rotation is presented and the time-energy trade-off is discussed.

## II. PRELIMINARIES

### A. Fidelity in the interaction picture

We consider the general case of the QC interacting with environment $E$. The initial state of the QC is given by $\psi_{t_0}$. The dynamics of the QC without decoherence is given by unitary evolution of the quantum algorithm

$$\psi_{t_0}\rightarrow\psi_t=U(t,t_0)\psi_{t_0}. \tag{1}$$

We will call the algorithm the total controlled evolution, i.e., the computation algorithm itself plus the error correction procedures. Due to interaction with environment, the initial state will evolve into the mixture,

$$\psi_{t_0}\rightarrow\varrho_t=\tilde{\Lambda}(t,t_0)\varrho_{t_0}, \tag{2}$$

with $\varrho_{t_0}=\psi_{t_0}$. The fidelity of decohered computation is given by

$$F_t=\langle\psi_t|\varrho_t|\psi_t\rangle. \tag{3}$$

It is convenient to use the interaction picture. Putting

$$\Lambda(t,t_0)=\hat{U}(t_0,t)\tilde{\Lambda}(t,t_0) \tag{4}$$

we obtain

$$F_t=\langle\psi_{t_0}|\Lambda(t,t_0)(|\psi_{t_0}\rangle\langle\psi_{t_0}|)|\psi_{t_0}\rangle. \tag{5}$$

Here we used the notation

$$\hat{U}(\varrho)=U\varrho U^{\dagger}, \tag{6}$$

where $U$ is unitary operation.

Note that in the interaction picture the quantum algorithm does not compute, but only ''interacts'' with errors, causing them to spread more and more. The map $\Lambda(t,t_0)$ describes the net effect of decoherence. As we will see, in the FT model the decoherence is ''decoupled'' from the algorithm, so that the total decoherence amounts to the faults caused by environment, which are then propagated by the algorithm. In the Hamiltonian approach, the interaction with environment will be ''entangled'' with algorithm, due to quantum memory, and no such simple description will apply.

### B. General evolution

Through the paper we will deal with the equation

$$\frac{d}{dt}\sigma_t=\mathcal{L}_t\sigma_t+L\sigma_t, \tag{7}$$

where $\mathcal{L}_t$ is some time-dependent operator (it will describe self-evolution), while $L$ is time-independent operator (interaction), $\sigma_t$ is vector. Let the free evolution of $\sigma$ be given by the operator $\Gamma(t,t_0)$,

$$\Gamma(t,t_0)\sigma_{t_0}=\sigma_t. \tag{8}$$

The operator satisfies

$$\frac{d}{dt}\Gamma(t,t_0)=\mathcal{L}_t\Gamma(t,t_0). \tag{9}$$

The total evolution denoted by $\tilde{\Lambda}(t,t_0)$ satisfies

$$\frac{d}{dt}\tilde{\Lambda}(t,t_0)=\mathcal{L}_t\tilde{\Lambda}(t,t_0)+L\tilde{\Lambda}(t,t_0). \tag{10}$$

Then for the evolution in interaction picture given by $\Lambda(t,t_0)=\Gamma(t_0,t)\tilde{\Lambda}(t,t_0)$ we have the following formal expansion

$$\Lambda(t,t_0)=\mathbb{I}+\sum_{m=1}^{\infty}\int_{t_0}^{t}dt_m\cdots\int_{t_0}^{t_2}dt_1 \times\Gamma(t_0,t_m)L\Gamma(t_m,t_{m-1})L\cdots L\Gamma(t_1,t_0). \tag{11}$$

Full evolution is given by

$$\tilde{\Lambda}(t,t_0)=\Gamma(t,t_0)+\sum_{m=1}^{\infty}\int_{t_0}^{t}dt_m\cdots\int_{t_0}^{t_2}dt_1 \times\Gamma(t,t_m)L\Gamma(t_m,t_{m-1})L\cdots L\Gamma(t_1,t_0). \tag{12}$$

## III. EVOLUTION OF THE QC IN THE SECOND-ORDER APPROXIMATION

In this section we derive reduced dynamics of the QC interacting with environment $R$. Consequently, in Eq. (7) the operator $\mathcal{L}_t$ will be sum of free Hamiltonians of the QC and $R$, the operator $L$—the interaction Hamiltonian, and $\sigma_t$—the wave function of the total $QC+R$ system.

### A. Non-Markovian Born approximation

The Hamiltonian of the total system is of the form

$$H = H_{QC} + H_R + \lambda H_{int}. \tag{13}$$

Here $H_{QC}$ is time-dependent Hamiltonian of the computer, $H_R$ is time-independent self-Hamiltonian of the environment, $\lambda$ is coupling constant, which is assumed to be small ($\lambda \gg 1$). In the following we will remove $\lambda$ from formulas by incorporating it into interaction Hamiltonian $H_{int}$. The latter is of the form

$$H_{int} = \sum_{\alpha} S_\alpha \otimes R_\alpha, \tag{14}$$

where $S_\alpha$ and $R_\alpha$ are self-adjoint operators. The resulting evolution of the total system is described by unitary transformation $U(t,s)$ such that

$$\hat{U}(t,s)\varrho(s) = \varrho(t). \tag{15}$$

We use the notation

$$\hat{H}(\varrho) = [H, \varrho]. \tag{16}$$

The self-evolution of QC is given by

$$U_C(t,s) = \mathcal{T}\exp\left[-i\int_s^t H_C(u)du\right], \tag{17}$$

where $\mathcal{T}$ is time-ordering operator. Finally, the free evolution of the total system is

$$\hat{U}_0(t,s) = \hat{U}_C(t,s) \otimes e^{-i\hat{H}_R(t-s)}. \tag{18}$$

The initial state of the total system is given by

$$\varrho(s) = \varrho_C(s) \otimes \omega_R, \tag{19}$$

where $\varrho_C$ is the initial state of quantum computer, while $\omega_R$ is stationary state of environment. Without loss of generality, one can assume that

$$\mathrm{Tr}\, R_\alpha \omega_R = 0. \tag{20}$$

Besides, the state $\omega_R$ commutes with dynamics of environment,

$$[\omega_R, H_R] = 0. \tag{21}$$

We consider the lowest-order approximation of the evolution, obtaining from Eq. (11)

$$\hat{U}(t,s) \simeq \hat{U}_0(t,s) - i\int_s^t du \hat{U}_0(t,u)\hat{H}_{int}\hat{U}_0(u,s) - \int_s^t du \int_s^u dw \hat{U}_0(t,u)\hat{H}_{int}\hat{U}_0(u,w)\hat{H}_{int}\hat{U}_0(w,s). \tag{22}$$

Then the evolution of the reduced density matrix reads as $\varrho_C(t) = \mathrm{Tr}_R[\varrho(t)]$ is given by

$$\varrho_C(t) = \hat{U}_C(t,s)\varrho_C(s) - \mathrm{Tr}_R\left\{\int_s^t du \int_s^u dw \hat{U}_0(t,u) \times \hat{H}_{int}\hat{U}_0(u,w)\hat{H}_{int}\hat{U}_0(w,s)\varrho_C(s)\otimes\omega_R\right\}. \tag{23}$$

Subsequently, we find that $\varrho_C(t)$ can be written in the following compact form:

$$\varrho_C(t) = \hat{U}_C(t,s)\{\varrho_C(s) - \tfrac{1}{2}[A(t,s)\varrho_C(s) - \varrho_C(s)A(t,s)] + \hat{\Phi}(t,s)\varrho_C(s) - i[h(t,s), \varrho_C(s)]\}. \tag{24}$$

The completely positive superoperator $\hat{\Phi}(t,s)$ is given by the following formula:

$$\hat{\Phi}(t,s)\varrho_C(s) = \sum_{\alpha\beta}\int_s^t du \int_s^t dw \langle R_\alpha R_\beta(u-w)\rangle_{\omega_R} \times S_\beta(u,s)\varrho_C(s)S_\alpha(w,s), \tag{25}$$

where $S_\alpha(u,s) = \hat{U}_C^{-1}(u,s)S_\alpha$, $R_\alpha(t) = e^{iH_R t}R_\alpha e^{-iH_R t}$ are the versions of $S_\alpha$ and $R_\alpha$ evolving in the Heisenberg picture according to free evolution; $\langle R_\alpha R_\beta(t)\rangle_{\omega_R} = \mathrm{Tr}(\omega_R R_\alpha R_\beta(t))$ is the autocorrelation function of the environment. The operator $A(t,s)$ is given by

$$A(t,s) = A^\dagger(t,s) = \hat{\Phi}^*(t,s)\mathbb{I}, \tag{26}$$

where $\hat{\Phi}^*$ is dual map to $\hat{\Phi}$. Explicitly,

$$A(t,s) = \sum_{\alpha\beta}\int_s^t du \int_s^t dw \langle R_\alpha R_\beta(u-w)\rangle_{\omega_R} S_\alpha(u,s)S_\beta(w,s). \tag{27}$$

Finally, $h(t,s)$ describes the Hamiltonian contribution to the dynamics due to the interaction with environment (Lamb shift, collective Lamb shift, etc.). In the following we put $h(s,t) \equiv 0$ by applying the renormalization procedure. We add to the Hamiltonian $H_{QC}(t)$ the appropriate counterterms that cancel the contribution $h(t,s)$ in a given order of perturbation calculus. Therefore, in the following, $H_{QC}(t)$ is the full physical Hamiltonian containing all relevant terms. In this way, when passing to Markovian approximation for the reservoir at the thermal equilibrium we obtain the Gibbs state corresponding to the full Hamiltonian, as it should be.

One can pass to the frequency domain, putting

$$\langle R_\alpha R_\beta(t)\rangle_{\omega_R} = \int_{-\infty}^{+\infty} R_{\alpha\beta}(\omega)e^{-i\omega t}d\omega, \tag{28}$$

$$Y_\alpha(\omega) = \int_s^t S_\alpha(u,s)e^{-i\omega u}du. \tag{29}$$

The function $R_{\alpha\beta}$ is called *spectral density*. We then can write $\hat{\Phi}$ as

$$\hat{\Phi}(t,s)\varrho_C(s)=\sum_{\alpha,\beta}\int_{-\infty}^{+\infty}d\omega R_{\alpha\beta}(\omega)Y_\beta(\omega)\varrho_C(s)Y_\alpha^\dagger(\omega), \tag{30}$$

where we do not write explicitly the dependence on $t$ and $s$ of the operators $Y_\alpha$. Denoting $\hat{\Phi}(t,0)=\hat{\Phi}_t$, the evolution in the interaction picture can be thus written (for $s=0$) as

$$\varrho_C(t)=\hat{U}_0(t,0)[\varrho_0-\tfrac{1}{2}\{\hat{\Phi}_t^*(\mathbb{I}),\varrho_0\}+\hat{\Phi}_t\varrho_0], \tag{31}$$

where $\varrho_0=\varrho_C(0)$. The above equation closely resembles the form of the generator of the Markovian semigroup [13,10]. In the next section we will exhibit the Markovian approximation of this formula.

One notes that the crucial element here is the transition map $\hat{\Phi}$ (we will call it *error map*). Operators $S_\alpha$ are errors that can occur during interaction with the reservoir (cf. [14]). In particular, the map contains a propagation of errors that is nothing but the evolution of $S_\alpha$ in the interaction picture. The overall error is proportional to $\|\hat{\Phi}\|$, where $\|\cdot\|$ is some norm (see [7]).

### B. Markovian limit for the time-independent Hamiltonian

Let us put $t=\tau/2$, $s=-\tau/2$, and

$$H_C=\sum_j \epsilon_j|j\rangle\langle j|, \tag{32}$$

where $\{|j\rangle\}$ is the orthonormal basis in the Hilbert space of QC. To find the meaning of the completely positive map $\Phi$, we will find the behavior of evolution for long time. We have

$$S_\alpha(u,s)=\sum_{\omega_k=\epsilon_j-\epsilon_j}S_\alpha(\omega_k)e^{i\omega_k u}e^{-i/2\omega_k\tau} \tag{33}$$

with

$$S_\alpha(\omega_k)=\sum_{j,j':\epsilon_{j'}-\epsilon_j=\omega_k}|j\rangle\langle j|S_\alpha|j'\rangle\langle j'|. \tag{34}$$

Then the formula (30) reads

$$\begin{aligned}\hat{\Phi}(s,t)\varrho_C(s)=&\Bigg[\pi\sum_{\alpha,\beta,\omega_k}\int d\omega R_{\alpha\beta}(\omega)\,\delta_\tau^{(2)}\\&\times(\omega-\omega_k)S_\beta(\omega_k)\varrho_C(s)S_\alpha^\dagger(\omega_k)\Bigg]\\&\times\tau+\pi^2\sum_{\alpha,\beta}\sum_{\omega_k\neq\omega_l}\int d\omega R_{\alpha\beta}(\omega)\,\delta_\tau^{(1)}\\&\times(\omega-\omega_k)\,\delta_\tau^{(1)}(\omega-\omega_l)S_\beta(\omega_k)\varrho_C(s)S_\alpha^\dagger(\omega_k)\\&\times\exp\left[-\frac{i}{2}(\omega_k-\omega_l)\tau\right].\end{aligned} \tag{35}$$

Here we have used two models of the Dirac $\delta$ function with width $1/\tau$,

$$\delta_\tau^{(1)}(x)=\frac{\sin(\tau x)}{\pi x},\quad \delta_\tau^{(2)}(x)=\frac{\sin^2(\tau x)}{\pi x^2\tau}. \tag{36}$$

Approximately we have $(\delta_\tau^{(1)})^2\simeq\pi\tau\delta_\tau^{(2)}$. The second, ''nonresonant'' term of Eq. (35) will vanish in the limit of large $\tau$, as the overlap between two $\delta$'s will decrease for large $\tau$. (This is an alternative form of a well-known rotating-wave approximation.) More precisely, one requires $\tau>\ll 1/\Delta\omega$, where

$$\Delta\omega=\min\{|\epsilon_j-\epsilon_{j'}|;\epsilon_j\neq\epsilon_{j'}\}.$$

Finally, we obtain

$$\hat{\Phi}(t,s)\varrho_C(s)=\tau\pi\sum_{\alpha,\beta,\omega_k}R_{\alpha\beta}(\omega_k)S_\beta(\omega_k)\varrho_C(s)S_\alpha^\dagger(\omega_k). \tag{37}$$

Note that the map depends on $t$ and $s$ only through the factor $\tau=t-s$. Thus $(1/\tau)\hat{\Phi}$ is exactly the transition map for the quantum dynamical semigroup in the weak coupling limit [10,15]

$$\dot{\varrho}_C=L\varrho_C \tag{38}$$

with

$$L\varrho_C=-i[H_C,\varrho_C]-\frac{1}{2\tau}\{\hat{\Phi}^*(\mathbb{I}),\varrho_C\}+\frac{1}{\tau}\hat{\Phi}\varrho_C. \tag{39}$$

One can see [16] that we can distinguish the initial, non-Markovian stage, when the reservoir is ''learning'' the structure of the Hamiltonian of the system. This requires time $\tau\gg 1/\Delta\omega$. Once the structure is recognized, we have the Markovian stage, during which the system is being relaxed towards the Gibbs state of thermal equilibrium $(1/Z)e^{-\beta H}$ determined by $H$.

Suppose now that we have a two-qubit system, where the components are coupled to one another; e.g., the self-evolution is to produce some two-qubit gate. Then, in the Markovian stage the system relaxes to the *compound* Gibbs state. Thus, the noise, after recognizing that the Hamiltonian is compound, becomes compound itself [17]. We will analyze it in more detail in Sec. V.

### C. Decoherence and dissipation in the Markovian regime

The properties of the decoherence in the Markovian limit can be read from the formula (37). We will now analyze the formula to get a first intuition about decoherence. First of all, one can single out *pure decoherence* that is not connected with energy exchange with environment. The populations of energy levels are kept, but the phases undergo randomization. The relevant term is

$$\pi\tau\sum_{\alpha,\beta} R_{\alpha,\beta}(0)S_\beta(0)\varrho_C S_\alpha^\dagger(0). \tag{40}$$

Indeed, $S_\alpha(0)$ commutes with $H_C$. Hence this term does not lead to transitions between energy levels. Thus the term $R_{\alpha\beta}(0)$ stands for the strength of pure dephasing. On the contrary, the other terms of the sum correspond to the decoherence accompanied by energy exchange, because the operators $S_\alpha(\omega_k)$ either do not exist (if $S_\alpha$ commute with $H_C$) or describe transitions between energy levels. Note that if the spectral density $R_{\alpha\beta}$ vanishes for $\omega=0$, there is no pure decoherence: the decoherence is always accompanied by dissipation. Assume, for example, that a system does not have self-Hamiltonian (e.g., spin of free electron). Then there is no dissipation. If in addition $R_{\alpha,\beta}(0)=0$ then there is *no* decoherence at all.

### D. Important examples of reservoirs

We note that an important characteristic of the interaction with environment is the shape of the spectral density. A general property of the heat bath at the temperature $T$ is the following:

$$R_{\alpha\beta}(-\omega)=e^{-\omega/kT}R_{\alpha\beta}(\omega). \tag{41}$$

In particular, at zero temperature, only non-negative frequencies are relevant. Let us now present three important examples of environments.

*(a) Linear coupling to bosonic field.* Operators $R_\alpha$ are given by

$$R_\alpha=a(\phi_\alpha)+a^\dagger(\phi_\alpha), \tag{42}$$

where $a^\dagger(\phi_\alpha)$, $a(\phi_\alpha)$ are creation and anihillation operators for fields $\phi_\alpha$, respectively. The following form of diagonal elements of $R_{\alpha\beta}$ can be obtained:

$$R_{\alpha\alpha}(\omega)=\int d\mathbf{k}\delta(\Omega(k)-\omega)(n(k)+1)|\phi_\alpha(\mathbf{k})|^2, \tag{43}$$

where $\Omega(k)$ is the energy of the boson [$\Omega(k)=k^2/2m$ or $\Omega=vk$], $n(k)=1/e^{\beta\Omega(k)}-1$. Typically $R_{\alpha\alpha}(\omega)$ grows like $|\omega|^d$ for small $\omega$ and then rapidly falls down for $|\omega|>\omega_c$ ($\omega_c$ is a cutoff frequency). For electromagnetic interactions in the dipole approximation we have $R_{\alpha,\alpha}(\omega)\approx\omega^3$ for small $\omega$, similarly for some cases of phonon interaction. The cutoff parameter $\omega_c$ is *model-dependent* (see, e.g., Ref. [18]) and characterizes the range of validity of model but does not mean that the frequencies $\omega>\omega_c$ are not influenced by noise. For example, the electromagnetic interaction can be described by the dipole approximation for $\omega/c=|\mathbf{k}|\ll 1/r_0$ where $r_0=e^2/m_ec^2$ is the classical electron radius [19]. For larger frequencies a different model involving nonbounded (scattering) states of electrons should be taken into account. The other cutoff for free electrons $\omega_c\simeq 2m_ec^2$ means only that for larger frequencies the pair production should be taken into account. In solid state $\omega_c$ is a Debye frequency. For higher frequencies the coupling to phonons makes no sense and we have to consider local interactions with the neighboring atoms or ions and electrons.

*(b) Interaction with a dilute gas.* We use the model of free bosonic or fermionic gas in the low-density approximation

$$R_\alpha=a^\dagger(\phi_\alpha)a(\phi_\alpha)-\langle a^\dagger(\phi_\alpha)a(\phi_\alpha)\rangle \tag{44}$$

The spectral density is then of the form

$$R_{\alpha\alpha}(\omega)=\int\int d\mathbf{k}d\mathbf{l}n(k)|\phi_\alpha(\mathbf{k})|^2|\phi_\alpha(\mathbf{l})|^2 \times\delta(\Omega(k)-\Omega(l)-\omega). \tag{45}$$

Typically $R_{\alpha\alpha}(0)>0$ and, again $R_{\alpha\alpha}(\omega)$, falls down for $|\omega|>\omega_c$ a model-dependent cutoff frequency.

*(c) Fluctuations of molecular field.* The influence of a local fluctuating field can be described by a classical noise $F_\alpha(t)$. For example, for a colored noise model we have

$$\langle F_\alpha F_\alpha(t)\rangle=De^{-|t|/\tau_c}. \tag{46}$$

Then the spectral density is Lorentzian

$$R_{\alpha\alpha}(\omega)=\frac{D}{\omega^2+\tau_c^{-2}}. \tag{47}$$

Such a density is relevant for the relaxation mechanism in pulsed nuclear magnetic resonance experiments.

### E. Discussion of the dynamical decoupling method

From Eq. (37) we see that one can gamble with decoherence by changing the spectrum of the Hamiltonian trying to choose frequencies $\omega_k$ for which the spectral density is small. In Ref. [11] the method of *dynamical decoupling* was exhibited. It bases on adding a periodic, rapidly alternating term to the self-Hamiltonian of the QC that averages out the interaction part, leaving some room for controlled evolution of the QC. The method sometimes called ''bang-bang'' control is supported by an elegant group-theoretic framework. Here we would like to determine what types of reservoirs allow us to apply such a method. Consider, as in [11], the decoherence in the QC with a rapidly oscillating term only, given by

$$H(t)=H\cos(\Omega t), \tag{48}$$

where $H$ is time independent, $\Omega$ is large frequency. We need to derive $Y_\alpha(\omega)$. Using the notation of Eq. (34) we have

$$S_\alpha(t,0)=\sum_{\omega_k} S_\alpha(\omega_k)\exp\left[i\frac{\omega_k}{\Omega}\sin(\Omega)t\right]. \tag{49}$$

Expanding the time-dependent term into Taylor series and applying the Fourier transform, we obtain

$$Y_\alpha(\omega)=\sum_{\omega_k} S_\alpha(\omega_k)\Big\{c_0\delta(\omega)c_1\frac{\omega_k}{\Omega}[\delta(\omega-\Omega) +\delta(\omega+\Omega)]+\cdots\Big\}, \tag{50}$$

where $\delta$ stands for $\delta_\tau^{(1)}$ [see Eq. (36)], $c_{0,1}$ are constants and we omitted higher harmonics. Putting $Y_\alpha$ into formula (30) we see that to obtain low decoherence, we need $R_{\alpha\beta}$ (i) to be small for $\omega \approx 0$ and (ii) to have cutoff frequency satisfying $\omega_c \leqslant \Omega$. Thus one would need bell-shaped spectral density. The bosonic field reservoirs satisfy (i), but, as we have already mentioned, do not have physical cutoffs. On the other hand, collisonal or the colored noise have cutoffs, yet allow for *pure decoherence due to nonzero* $R_{\alpha\beta}(0)$. The conditions for dynamical decoupling can in principle be met in the QED cavity. Then, however, the system loses its fundamental simplicity: the quantum computer must now include modes of cavity, while the reservoir becomes the atoms building the cavity.

There is another aspect of the discussed method that has not been investigated so far. Namely, the time-dependent Hamiltonian is obtained by use of external fields that can be described classically (e.g., coherent light of the laser beam). However, a rapidly alternating field will get entangled with the controlled system (this is called quantum back reaction), so it must be treated quantum mechanically. The resulting disturbance was evaluated to be weaker than the ''regular'' (say collisional) decoherence [20]. However, it seems that for the bang-bang control, the considered effect can become relevant.

An interesting version of the dynamical decoupling method was considered in Ref. [21]. Namely, the starting point was to modulate the coupling constant in the interaction picture rather than the self-Hamiltonian. As a result, in the Schrödinger picture, the self-Hamiltonian was of the form $\omega H_C \sin(\omega t)$. Thus, in addition to modulation, the level differences of the Hamiltonian were rescaled. Then the above analysis does not apply and condition (i) is not necessary. Earlier, in [28], the decoupling was achieved by adding ''kicks'' to the dynamics. Again, only (ii) is then needed. However, one can show that in the cases where the above method works, more elementary strategies can protect the system against the reservoir. Indeed, one can simply perform fast gates, instead of fast controlling kicks.

## IV. EXAMPLES

We will present here two examples. The first one will serve to introduce a *fault map* describing the action of noise with propagation excluded. In the second one we consider the kicked dynamics of algorithm, and illustrate the difference between quantum memory due to the interference of error paths and classical memory due to their probabilistic mixing .

### A. Memoryless reservoir: Error map and fault map

The memoryless reservoir has the white-noise spectral density. Its autocorrelation function is $R_{\alpha\beta}(s-t)=R^0_{\alpha\beta}\delta(s-t)$. Let us, for simplicity, assume independent interaction $R_{\alpha\beta}(\omega)=R_\alpha \delta_{\alpha\beta}$. The resulting error map is given by

$$\Phi_t(\varrho)=\sum_\alpha R^0_\alpha \int_s^t du S_\alpha(u,s)\varrho S_\alpha(u,s). \tag{51}$$

Suppose that $S_\alpha$ are one-qubit operators. According to the notation of Ref. [7] we will call them *faults*—these are errors caused by the reservoir, and should be distinguished from the *errors* resulting from propagation of faults by algorithm [we will denote it by $U(t,s)$ in this section]. The propagation of the faults is described by the fact that the error map involves $S_\alpha$ in the Heisenberg picture. Note, that the overall error is caused solely by one-qubit faults and their propagation, averaged in time. Such evolution can be described by the following master equation:

$$\frac{d}{dt}\varrho_t = i[H_t,\varrho_t]+L\varrho_t \tag{52}$$

with

$$L=-\tfrac{1}{2}\{\hat{\Phi}^*(\mathbb{I}),\varrho\}+\hat{\Phi}(\varrho) \tag{53}$$

for $\hat{\Phi}(\varrho)=\Sigma_\alpha S_\alpha \varrho S_\alpha$. Indeed, let Eq. (7) $\sigma_t$ be the density matrix of $QC$, $\mathcal{L}_t=-i\hat{H}_t$, where $H_t$ is the Hamiltonian of algorithm and $L$, the above generator. Applying the formula (12) in first-order approximation we obtain

$$\begin{aligned}\varrho_t &= \hat{U}(t,t_0)\left[\varrho_{t_0}-\frac{1}{2}\{A,\varrho_{t_0}\}+\sum_\alpha \int_s^t du S_\alpha(u,s)\varrho S_\alpha(u,s)\right]\\ &= \hat{U}(t,t_0)\left[\varrho-\frac{1}{2}\{\Phi_t^*(\mathbb{I}),\varrho\}+\Phi_t(\varrho)\right].\end{aligned} \tag{54}$$

Even though the memoryless case is not especially interesting from our point of view, the very form of the master equation (52) is particularly useful in our context, as the faults themselves and the propagation are separated from one another. The propagation is described by the Hamiltonian term, while the faults are described by the map $\Phi$, which we will call *fault map*. We will derive such a master equation in the general case in the next section. Here, let us only mention that the ''fault part'' of the master equation is a sum of one-qubit operators. This means that environment acts *locally*, and the multiqubit errors (i.e., correlated errors) occur solely due to propagation.

### B. Decoherence under the kicked dynamics of QC: Error path interference effect

We will assume that the gates are performed quickly, so that they can be generated by the Hamiltonian with time dependence given by $\delta$ function. This will not work for the electrodynamic or phonon vacuum reservoir, as due to the behavior $\omega^3$, any rapid changes in the self-Hamiltonian cause large or infinite contribution. Thus in this case one needs to work with Gaussian pulses. On the other hand, the kicked dynamics can be used for Lorentzian spectral density.

Thus we divide computation time $t$ into $N$ pieces of length $\tau$. The Hamiltonian is given by

$$H(t)=\sum_{j=1}^N \delta(t-(j-1)\tau)h_j, \tag{55}$$

where $h_j$ generates the $j$th step of computation. The unitary evolution is given by

$$U(t)=\vec{\Pi}_{j=1}^{N}\theta(t-j\tau)U(j,j-1), \tag{56}$$

where $U(j,j-1)=e^{-ih_j\tau}$ is the $j$th step of computation (the group of gates performed at time $j\tau$), $\theta$ is the step function. Consequently,

$$U_t=\begin{cases} U(1,0), & t\in\langle 0,\tau), \\ U(2,1)U(1,0), & t\in\langle \tau,2\tau), \\ \cdots & \\ U(N,N-1)U(N-1,N-2)\cdots & \\ \cdots U(1,0), & t\in\langle (N-1)\tau,N\tau). \end{cases} \tag{57}$$

Substituting such dynamics into the error map of Eq. (25), we obtain

$$\Phi^{H}=\sum_{\alpha}\sum_{j,k=1}^{N} r_{jk}^{\alpha}S_{\alpha}^{j}\varrho S_{\alpha}^{k}, \tag{58}$$

where $S_{\alpha}^{j}=U_j^{\dagger}S_{\alpha}U_j$. For simplicity we assumed that $R_{\alpha\beta}=R_{\alpha}\delta_{\alpha\beta}$. The coefficients $r_{jk}^{\alpha}$ are given by

$$r_{jk}^{\alpha}=\int_{(k-1)\tau}^{k\tau}du\int_{(j-1)\tau}^{j\tau}dv R_{\alpha\alpha}(u-v). \tag{59}$$

For the memoryless case we obtain the discrete counterpart of

$$\Phi^{Ph}=\tau\sum_{\alpha}R_{\alpha}^{0}\sum_{k=1}^{N}S_{\alpha}^{k}\varrho S_{\alpha}^{k}. \tag{60}$$

In both formulas we see propagation of error, this time in a discrete manner: $S_{\alpha}^{j}$ denotes the *error* resulting in propagation of *fault* $S_{\alpha}$ within the interval $(0,j\tau)$. The difference is that in Eq. (58) we have *interference* due to quantum memory. This effect of interference of error can remarkably modify the process of error propagation.

The difference is similar to that between separable and entangled mixed states. We can view the error map as the (subnormalized) density matrix and the Kraus operators as the pure components of the ensemble giving rise to the density matrix. In general, the noise process can then behave as classical-like or as quantum stochastic process with time correlations of the form

$$\Phi_{cl}\varrho=\sum_{j_1\cdots j_m}p(j_1\cdots j_m)S(t_{j_m})\cdots S(t_{j_1})\times\varrho S(t_{j_1})^{\dagger}\cdots S(t_{j_m})^{\dagger},$$

$$\Phi_{qu}\varrho=\sum_{j_1\cdots j_m,i_1,\ldots,i_n}C(j_1\cdots j_m|i_1\cdots i_m)\times S(t_{j_m})\cdots S(t_{j_1})\varrho S(t_{i_1})^{\dagger}\cdots S(t_{i_m})^{\dagger}, \tag{61}$$

respectively. In the first case we always have a mixture of the product of Kraus operators, in the second one we can have a map for which such representation may be impossible. Indeed, in Eq. (60) the errors occurring in different times do *not* interfere with each other, they are added as in classical probability calculus.

However, it seems that the crucial point is not the difference between quantum and classical memory, but its *range*. In Sec. V we will exhibit the remarkable connection between the range of the memory and the multiqubit faults that occur: the interference caused by long-range quantum memory will give rise to nonlocal noise.

### C. Memory of the electromagnetic vacuum reservoir

Here we will calculate and discuss the memory caused by vacuum in free space. This is an important point in our paper; as the vacuum is unavoidable, we cannot remove it. Of course we can put the QC into cavity, but then, we can consider the QC plus cavity as the total system; that is, again in free space. We then prefer the latter setup, as it is more fundamental.

Consider the two-level atom and electromagnetic field with dipole interaction,

$$H=\tfrac{1}{2}\omega_0\sigma_3+\sigma_1\otimes\mathbf{d}E. \tag{62}$$

The spectral density of reservoir in temperature $T$ is given by

$$R(\omega)=\frac{8\pi}{3}\mathbf{d}^2\omega^3\left(1+\frac{1}{e^{\hbar\omega/kT}-1}\right)\quad\text{for}\quad\omega>0,$$

$$R(\omega)=\frac{8\pi}{3}\mathbf{d}^2|\omega|^3\left(\frac{1}{e^{\hbar\omega/kT}-1}\right)\quad\text{for}\quad\omega\leqslant 0. \tag{63}$$

We must now calculate the autocorrelation function in zero temperature. Taking the inverse Fourier transform, we obtain

$$R(t)=\frac{1}{(t+i\epsilon)^4}. \tag{64}$$

We see that the memory scales as the power of $t$ rather than an exponential function. Then there is no characteristic time. This is due to the fact that Coulomb interaction has no characteristic range. Thus the range of vacuum memory is infinite. Nevertheless, in quantum optics there is a notion of time of memory, which is the inverse of $\omega_0$—the transition frequency for the atom—and one can work with the Markovian master equation, within times much longer than $1/\omega_0$.

This does not mean that there exists some characteristic time scale for the vacuum reservoir. Rather, in the interaction picture, the autocorrelation function acquires oscillating term $e^{i\omega_0 t}$ which effectively makes the memory much weaker. For optical transitions, the memory then has almost no observable consequences. In Ref. [22] the decoherence in the ion-trap quantum computer was analyzed. From the analysis it follows that non-Markovian effects are indeed negligible in comparison with other types of noise. However, in the con-

text of the FT scheme, under a pessimistic assumption that it is not able to correct non-Markovian errors, the latter could eventually dominate.

## V. QUANTUM LONG-RANGE MEMORY IMPLIES NONLOCAL STRUCTURE OF NOISE

### A. Master equation

As we have seen, to discuss the structure of noise without propagation effect, we need a master equation of the form

$$\dot{\varrho} = -i[H_{QC}(t), \varrho] + \hat{L}_t \varrho. \tag{65}$$

We will now derive such equation in second-order approximation. To this end, we differentiate Eq. (24), obtaining

$$\dot{\varrho}(t) = \hat{U}_C(t,s)\varrho(s) + \hat{U}_C(t,s)[-\tfrac{1}{2}\{\hat{\Phi}^* \mathbb{I}, \varrho(s)\} + \hat{\Phi}\varrho(s)] + \hat{U}_C(t,s)[-\tfrac{1}{2}\{\hat{\dot{\Phi}}\mathbb{I}, \varrho(s)\} + \hat{\dot{\Phi}}\varrho(s)]. \tag{66}$$

We take the first step of approximation by removing the second term. Differentiating $\hat{\Phi}(t,s)$ and putting $s=0$, we obtain

$$\dot{\varrho} = -i[H_{QC}(t), \varrho] - \tfrac{1}{2}\{\hat{\Upsilon}_t^* \mathbb{I}, \varrho\} + \hat{\Upsilon}_t \varrho, \tag{67}$$

where $\hat{\Upsilon}_t = \hat{\Upsilon}(t,0)$ is given by

$$\hat{\Upsilon}(t,s)(\cdot) = \sum_{\alpha\beta} [\Upsilon_{\alpha\beta}(\cdot) S_\alpha + S_\beta(\cdot)\Upsilon_{\alpha\beta}^\dagger] \tag{68}$$

with

$$\Upsilon_{\alpha\beta} = \int_s^t du R_{\alpha\beta}(u-t) S_\beta(u,t). \tag{69}$$

On the right-hand side of the equation we have density matrices evolving according to the self-Hamiltonian of the system. The second step of approximation is then to replace them with the ones subjected to full evolution. (Better justification of such an integral differential master equation can be obtained within the cumulant expansion method, see, e.g., [23].)

Now, the decoherence is essentially characterized by the map $\hat{\Upsilon}_t$ which we will call ''fault map'' in analogy to the error map $\hat{\Phi}_t$. The latter one characterizes error, i.e., the net effect of the ''attacks'' of noise (faults) and their propagation. The present map $\hat{\Upsilon}_t$ describes solely the action of noise. The propagation will be due to the Hamiltonian term in the above master equation.

Let us now discuss how the form of $\Upsilon_t$ depends on memory. The crucial term is, of course, $\Upsilon_{\alpha\beta}$. In the case of no memory (and, for simplicity, independent decoherence) we obtain $\Upsilon_{\alpha\beta} = \delta_{\alpha\beta} S_\alpha$, which is compatible with Sec. IV A. Assuming that each $\alpha$ denotes a different qubit, we obtain that decoherence is local, and multiqubit errors are solely due to propagation. We can then write the generator in the form of

$$L_t = \sum_\alpha L_t^\alpha, \tag{70}$$

where $L_t^\alpha$ operates only on the $\alpha$th qubit. If we admit exponentially decaying memory (e.g., colored noise) with characteristic time $\tau_R$ comparable with the time of performing a single computational step, but short in comparison with time between gates, there will be some time for the reservoir to learn something about the structure of two-qubit Hamiltonians, so that the noise will be a sum of two-qubit superoperators,

$$L_t = \sum_{\alpha\beta} L_t^{\alpha\beta}. \tag{71}$$

In both cases decoherence is local. The possible faults are, roughly speaking, symbolized by $L_\alpha^t, L_{\alpha\beta}^t$ and they couple at most two qubits (in general, they involve so many qubits, as the used gates do).

Suppose now that the reservoir has long-range memory. Still, in each step of computation, there are two-qubit gates. The environment learns about their structure a bit. The effect *accumulates* due to memory, so that after a long time, the noise is strongly nonlocal: the generator $L_t$ involves multiqubit operators, as it keeps the record of all the history of self-evolution (i.e., algorithm). The decoherence becomes as ''entangled'' as the quantum algorithm is entangling. Note that the effect does not depend on the time of performing gates. For short gates, the reservoir has a short time to distinguish the temporary levels (driving the gate), but they are easily distinguishable, as the energy differences are high. If instead the energy differences are low, then the time must be longer. This is the result of the time-energy uncertainty principle. Thus, the relevant parameter here is ''action,'' i.e., the product of the time of gate, and the strength of the applied Hamiltonian (energy). To perform the gate, the needed amount of action must be always of the order of 1. Thus the general structure of noise does not depend very much on the way the gates are performed. In Sec. VI we will see that the strength of the decoherence does depend on it, so that we have the mechanism of possible optimization.

### B. Quantum memory and error correction

Let us now discuss the implication of the above results on the quantum error correction (QEC) method [4–7]. The basic idea of the QEC method rests on two assumptions: (i) the environment acts locally; (ii) the environment acts independently of self-evolution of the system. If the above assumptions are satisfied, it is indeed possible to protect the quantum information against decoherence by encoding it into highly entangled multiqubit states. In the literature assumption (i) was made explicitly, while assumption (ii) was tacit. Thanks to assumption (i) such nonlocally encoded information will not be affected by decoherence (acting locally). Assumption (ii) ensures that the self-evolution will not modify the local structure of decoherence. As we have argued, the latter assumption is never satisfied. If there is short-range memory, the dependence on self-evolution is ir-

relevant, and the QEC method still applies. However, the interaction with vacuum, which is unavoidable, introduces long-range quantum memory, which causes the environment to be rather malevolent: it traces self-evolution, and the more entangled the latter is, the more nonlocal the faults become caused by noise. On the other hand, the self-evolution *must* be entangling, since it is supposed to hide the quantum information just into entangled states. Thus, we have something a bit analogous to the Lenz rule: the structure of the environment attack is proportional to the structure of the evolution that is to protect against it. It seems to be a rather pessimistic result, as it implies that the very idea of *active* protection against the environment is fundamentally inadequate, as far as *long* quantum computation is concerned.

However, the conclusion need not be so pessimistic. First of all, decoherence caused by vacuum is small in comparison with other sources of decoherence, e.g., dephasing due to collisions (pressure decoherence), colored noise or thermal noise. It seems that the QEC (and hence the FT method) can be used to fight with the latter sources of decoherence, within the time window in which the vacuum-memory effect can be neglected. Strictly speaking, other types of reservoirs also have long tails in memory, however, the dominant part decays exponentially. Thus the FT method could certainly be performed within the time regime satisfying the following conditions: (i) *the memory is exponentially decaying*, (ii) *the memory majorizes vacuum memory*. This regime may prove sufficient to perform some quantum computational tasks.

There is also another point to address [5]. The claim that lack of memory is sufficient for the FT method does not mean that it is in general impossible to improve the FT method against some kind of memory. As a matter of fact, the mechanism producing multiqubit faults we described has some features that may be used to defend the FT scheme. Namely, as seen from formula (69), the multiqubit fault at some place $\alpha$ arises from *backward* propagation (with decreasing strength) of the single-qubit fault $S_\alpha$. Now since the FT method is able to fight with propagation forward, it cannot be denied that it will be robust against faults correlated in such a way. However, it goes beyond the scope of this paper.

## VI. MINIMAL DECOHERENCE MODEL FOR QUANTUM CONTROLLED SYSTEMS

In this section we will postulate the *minimal decoherence model*. We then derive a formula for fidelity of quantum computation within the model. We shall consider a class of controlled open systems for which $H_C(t)\neq 0$ only during computation of subsequent gates. The spectral density $R_{\alpha\beta}$ satisfies

$$\int_{-\epsilon}^{\epsilon}\frac{R_{\alpha\beta}(\omega)}{\omega^2}d\omega<\infty \quad \text{for some } \epsilon>0. \tag{72}$$

This excludes reservoirs with any pure decoherence effects, such as gas or colored noise. Our motivation is that we would like to investigate only fundamental obstacles to build the quantum computer. The pure decoherence is not one such as in principle one could isolate the quantum computer from any influence of the above kind (even though it might require technology that will never be achieved). Since obtaining arbitrarily low temperature is also a matter of technology, we can work with the only environment being vacuum. We remove the time-independent part of the self-Hamiltonian, as it will only cause additional dissipation.

An example of the system could be $n$ well-separated spins-$\frac{1}{2}$ that interact with an electromagnetic field and possibly with phonons. The controlled Hamiltonian $H_C(t)$ can be realized by switching external magnetic fields and suitable interaction between spins. Another example is the atomic degenerate metastable level, with Stark or Zeeman splitting used for controlling. Of course it may be impossible to keep the possibility of controlling interactions (a cornerstone of a quantum computer) and remove the influence of pure decoherence. However, as said, we assume the most optimistic case, and keep only the obstacle that cannot be removed for fundamental reasons. By removing the time-independent self-Hamiltonian, we obtain that there is no decoherence on qubits that are not active. Thus decoherence will be caused by the vacuum solely due to gate operations that require the energy levels to split during the time of performing the operation. In the case of independent interactions of qubits with vacuum, this temporal difference of energy levels is used by the vacuum that forces the upper level to relax by spontaneous emission. In the case of collective interaction (Dicke limit) this is not the case, where one has decoherence-free subspaces [12].

### A. Fidelity in the minimal decoherence model

We put $\varrho_C(s)=|\psi\rangle\langle\psi|$ and $s=-\infty$, $t=+\infty$. The fidelity of computation is given by

$$F=1-\delta=\langle U_C(t,s)\psi|\varrho_C(t)|U_C(t,s)\psi\rangle. \tag{73}$$

We will denote $S_\alpha(t)=S_\alpha(-\infty,t)=\hat{U}_C^{-1}(t,-\infty)S_\alpha$. To avoid infinite contribution from the interaction, we will assume that the interaction is adiabatically switched off at infinity (the standard method in quantum field theory [24]). Thus the following operators are put into formula (30):

$$Y_\alpha(\omega)=\int_{-\infty}^{\infty}\hat{S}_\alpha(t)e^{-\epsilon|t|}e^{-i\omega t}dt. \tag{74}$$

Performing partial integration and then passing to the limit $\epsilon\to 0$ we obtain

$$Y_\alpha(\omega)=\frac{1}{i\omega}\int_{-\infty}^{\infty}dte^{-i\omega t}\left(\frac{d}{dt}S_\alpha(t)\right)=\frac{1}{\omega}X_\alpha(\omega). \tag{75}$$

We then obtain the following formula for error $\delta$:

$$\delta=\sum_{\alpha,\beta}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}R_{\alpha\beta}(\omega)\{\langle\psi|X_\alpha^\dagger(\omega)X_\beta(\omega)|\psi\rangle - \langle\psi|X_\alpha^\dagger(\omega)|\psi\rangle\langle\psi|X_\beta(\omega)|\psi\rangle\}. \tag{76}$$

### B. Quantum information processing in the minimal decoherence regime

We consider the simplest quantum gate: rotation of a single qubit by an angle $\alpha \in [0,2\pi]$, in the case of a two-level atom coupled to the electromagnetic field at zero temperature. The time dependent Hamiltonian will be of the form $H_C(t)=\frac{1}{2}f(t)\sigma_z$, with

$$\int_{-\infty}^{\infty} f(t)dt=\alpha, \qquad \lim_{t\to\pm\infty} f(t)=0. \tag{77}$$

The interaction Hamiltonian is $H_{int}=\sigma_x\otimes\phi$, and the spectral density, as discussed earlier, is

$$R(\omega)=\begin{cases} R_0\omega^3 & \text{for } \omega\geqslant 0, \\ 0 & \text{for } \omega<0. \end{cases} \tag{78}$$

The full unitary transformation is given by

$$U_C=U_C(\infty,-\infty)=e^{-(1/2)\alpha\sigma_z}. \tag{79}$$

The parameter $\alpha$ is the amount of ''action'' needed to perform the transformation. Roughly speaking, $\alpha$ is the product of energy pumped into the system during the performing gate and the time of operation. In the case of nonrectangular pulse shape, the time is given by the width of pulse. Typically, if the energy is low, then the accompanying dissipation is small; however, the time of the operation is long (while in quantum computation, one would prefer short-time gates). Adding more energy, we obtain short time, but simultaneously enhance decoherence, causing loss of fidelity. Let us now calculate the fidelity. We have

$$X(\omega)=F_-(\omega)\sigma^- + F_+(\omega)\sigma^+, \tag{80}$$

where

$$F_\pm=\int_{-\infty}^{\infty} dt e^{-i\omega t} f(t) e^{\pm i\phi(t)} \tag{81}$$

with $\phi(t)=\int_{-\infty}^{t} f(u)du$. From the formula (76) one easily finds that any rapid variation of $f(t)$ will result in large error. Indeed, the changes produce long tails of the Fourier transforms $F_\pm$, which, integrated with $R_0\omega$ for $\omega\geqslant 0$, give poor fidelity. In order to minimize this effect, we choose the Gaussian shape of the pulse

$$f(t)=\frac{\alpha}{\sqrt{2\pi}t_1}\exp\left[-\frac{1}{2}(t/t_1)^2\right], \tag{82}$$

where $t_1$ can be taken as the time of performing a single gate. Using the approximate formula $\phi(t)\simeq f(0)t+\alpha/2$ one finds

$$|F_\pm(\omega)|=\alpha\exp\left\{-\frac{1}{2}\left(\omega t_C \pm \alpha\frac{1}{\sqrt{2\pi}}\right)^2\right\}. \tag{83}$$

Consequently, the error can be estimated as follows:

$$\delta\simeq(\langle\psi|\sigma^+\sigma^-|\psi\rangle-|\langle\psi|\sigma^+|\psi\rangle|^2)\alpha^2 R_0 t_1^{-2}. \tag{84}$$

Taking the average over possible input states $\psi$ we get

$$\delta_1\sim R_0\alpha^2 t_1^{-2} \tag{85}$$

up to a constant factor.

We execute ''algorithm'' with action $A$ in $n$ steps (gates), i.e., $\alpha=A/n=O(1)$. It turns out that the formula (76) is additive with respect to the composition of gates. To see it consider $n$ Gaussian pulses of width $t_1$ separated with time $\tau=mt_1$, where $m\gg 1$. This is the smooth realization of kicked dynamics considered earlier. One finds

$$X_\alpha(\omega)=\sum_{j=1}^{n} e^{-i\omega_j\tau}\hat{U}^\dagger(j\tau,0)Y_\alpha^j(\omega), \tag{86}$$

where

$$Y_\alpha^j(\omega)=\int_{-\tau/2}^{\tau/2} e^{-i\omega t}\frac{d}{dt}\hat{U}^\dagger(t,j\tau)S_\alpha. \tag{87}$$

Now, we have, for example,

$$\langle\psi|X_\alpha^\dagger(\omega)X_\beta(\omega)|\psi\rangle = \sum_{j,j'} e^{-i\omega\tau(j'-j)}\langle U(j\tau,0)\psi|Y_\alpha^{j\,\dagger}(\omega)Y_\beta^j(\omega)|U(j\tau,0)\psi\rangle. \tag{88}$$

Taking the large separation between gates, we obtain that the terms involving $j\neq j'$ are averaged out by rapid phase rotation. The remaining term is nothing but the sum of single gate terms.

Consequently, for $n$ pulses with average error $\delta_1$ the total error is given by

$$\delta_n\sim n\delta_1. \tag{89}$$

Suppose now that we would like to keep the total error below some threshold $\epsilon$, then we obtain the following estimation for the time of computation $t_C=n(m+1)t_1$ (where $t_1$ is the time of single pulse):

$$t_C\geqslant\frac{1}{\sqrt{\epsilon}}R_0^{1/2}(m+1)n^{3/2}. \tag{90}$$

In conclusion, it is possible to keep high fidelity, at the expense of worse scaling of the physical time $t_C\sim n^{3/2}$ in comparison with ''algorithmic'' time $t_{alg}\sim n$.

Finally, one should mention that our result is not restricted to the simple one-qubit system. If one takes two qubits, and some two-qubit gate, the reasoning will be similar. If the system is $K$ qubit (still with, say, at most two-qubit gates), the only difference will be the need of scaling fidelity per one qubit as $1/K$ to get high fidelity of the total final $K$-qubit state. This can be achieved by further slowing down the gates. Finally, to keep high fidelity of the quantum computer in vacuum we need physical time to scale as $t_C\sim n^{3/2}\sqrt{K}$ or

$t_C \sim n\sqrt{v}$, where $v = nK$ is the complexity of the problem (or volume of the algorithm), i.e., the number of steps times the number of qubits needed to run the algorithm.

## VII. CONCLUDING REMARKS

We have developed a dynamical description of a decohering quantum computer. We have obtained dependence of the structure of decoherence on the quantum memory of the reservoir. The nonlocal structure of decoherence implied by vacuum memory suggests that long quantum computation at constant error rate is impossible. Instead, one should deal with short-time quantum computation by the use of low frequencies. This of course requires optimization of the kind we performed in the last section. However, that the scheme of FT quantum computing can be defended cannot be excluded. Indeed, the collective noise attacks due to memory are of a special type: they can be viewed as a result of the propagation backward in time of single-qubit errors.

Finally, we have designed the most optimistic *minimal decoherence model*, and provided a fidelity formula, in first-order approximation. Using it we have shown how the time-energy trade-off influences scaling of the physical time of computation.

In general, we have argued that in a description of quantum computing the Markovian approach fails, as far as linear coupling to a boson field is concerned. The effects are relatively small for coupling to electromagnetic vacuum. Our results can have practical meaning for the systems interacting with phonons, like, e.g., quantum dots (cf. [25]), where the coupling is much stronger. Then the memory effects can be of similar order as other sources of decoherence. There are also less fundamental but practical sources of memory such as the heating mechanism in ion traps [26].

Finally, we believe that our non-Markovian dynamical description of quantum computing will have some meaning for future practically useful implementations of the quantum computer involving large numbers of qubits and a relatively long time of computation [27].

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