



ELSEVIER

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

Discrete Applied Mathematics 129 (2003) 361–369

DISCRETE  
APPLIED  
MATHEMATICS

[www.elsevier.com/locate/dam](http://www.elsevier.com/locate/dam)

# The complexity of the $T$ -coloring problem for graphs with small degree

Krzysztof Giaro, Robert Janczewski<sup>\*,1</sup>, Michał Małafiejski

*Foundations of Informatics Department, Faculty of Electronics, Telecommunications and Informatics,  
Technical University of Gdańsk, ul. Narutowicza 11/12, Gdańsk, Poland*

Received 18 September 2000; received in revised form 10 July 2002; accepted 29 July 2002

---

## Abstract

In the paper we consider a generalized vertex coloring model, namely  $T$ -coloring. For a given finite set  $T$  of nonnegative integers including 0, a proper vertex coloring is called a  $T$ -coloring if the distance of the colors of adjacent vertices is not an element of  $T$ . This problem is a generalization of the classic vertex coloring and appeared as a model of the frequency assignment problem. We present new results concerning the complexity of  $T$ -coloring with the smallest span on graphs with small degree  $\Delta$ . We distinguish between the cases that appear to be polynomial or NP-complete. More specifically, we show that our problem is polynomial on graphs with  $\Delta \leq 2$  and in the case of  $k$ -regular graphs it becomes NP-hard even for every fixed  $T$  and every  $k > 3$ . Also, the case of graphs with  $\Delta = 3$  is under consideration. Our results are based on the complexity properties of the homomorphism of graphs.

© 2003 Published by Elsevier B.V.

*Keywords:* Vertex coloring;  $T$ -coloring;  $T$ -span; Homomorphism; NP-completeness

---

## 1. Introduction

We consider the  $T$ -coloring problem, as a generalized classical vertex coloring problem, which is one of the variants of the channel assignment problem in broadcast networks [8,16]. In this problem one wishes to assign to each transmitter  $x_i \in \{x_1, \dots, x_n\}$ , located in a region, a frequency  $f(x_i)$  avoiding interference between transmitters, i.e.

---

\* Corresponding author.

*E-mail addresses:* [giaro@eti.pg.gda.pl](mailto:giaro@eti.pg.gda.pl) (K. Giaro), [skalar@eti.pg.gda.pl](mailto:skalar@eti.pg.gda.pl) (R. Janczewski), [mima@eti.pg.gda.pl](mailto:mima@eti.pg.gda.pl) (M. Małafiejski).

<sup>1</sup> Supported by FNP.

two interfering transmitters (because of proximity, meteorological or other reasons) must be assigned frequencies so that the distance between them does not belong to the forbidden set  $T$  of nonnegative integers including 0. The most common objective is to minimize the span of a frequency band. For more about applications of this problem the reader is referred to [2,3,14,15].

Let  $G = (V, E)$  be a simple loopless graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . By  $\Delta(G)$  we mean the maximum degree  $\rho(v)$  over all vertices  $v$  of graph  $G$ , by  $\chi(G)$  and  $\omega(G)$  we denote the chromatic number and the clique number of graph  $G$ , respectively. Let  $G(W)$  denote the subgraph of graph  $G$  induced by  $W \subset V$ .

**Definition 1.** Let  $T$  be a finite set of nonnegative integers satisfying  $0 \in T$ . By a  $T$ -coloring of graph  $G$  we mean a vertex coloring  $c: V \rightarrow \mathbb{N}$  satisfying  $|c(v) - c(w)| \notin T$ , whenever  $\{v, w\} \in E$ . The  $T$ -span is defined as  $\text{sp}_T(G) = \min_c \text{sp}_T(G, c)$ , where  $\text{sp}_T(G, c) = \max c(V) - \min c(V)$  and  $c$  is a proper vertex  $T$ -coloring of graph  $G$ . A  $T$ -coloring  $c$  is said to be optimal if  $\text{sp}_T(G, c) = \text{sp}_T(G)$ .

Following [13] we introduce the notion of  $T$ -graphs.

**Definition 2.** For a given set  $T$ , we define an infinite  $T$ -graph  $G_T$ , with vertex set  $V(G_T) = \mathbb{N} \cup \{0\}$  and edge set  $E(G_T) = \{\{x, y\} : |x - y| \notin T\}$ . By  $G_T^{d+1}$  we mean the subgraph of  $G_T$  induced by  $\{0, \dots, d\}$ .

Given a graph  $G$ , set  $T$  and positive integer  $k$ , the problem of verifying the inequality  $\text{sp}_T(G) \leq k$  we call the  $T$ -SPAN PROBLEM. This differs from the  $T$ -COLORING PROBLEM, which requires an optimal  $T$ -coloring as its output. The notion of a  $T$ -coloring was introduced in [8]. The problem has been studied extensively (see [3,4,12,13–18]). The majority of results concern lower and upper bounds on  $\text{sp}_T(G)$ , see [3,11,17]. The first complexity result comes independently from [6,12], where the authors showed NP-completeness in the strong sense of the  $T$ -SPAN PROBLEM on complete graphs (so even a pseudopolynomial algorithm for the  $T$ -SPAN PROBLEM cannot exist unless  $P=NP$ ). We call the above problems FIXED  $T$ -SPAN PROBLEM and FIXED  $T$ -COLORING PROBLEM if set  $T$  is fixed. Furthermore, in [7] the authors have developed a linear algorithm for solving the FIXED  $T$ -COLORING PROBLEM on complete graphs (but exponential with respect to  $\max T$ ). So far, the problem on graphs with “small” degree has been still open. Therefore, in Sections 2 and 3 we deal with some new properties of homomorphisms and in Section 5 we show NP-completeness of the FIXED  $T$ -SPAN PROBLEM on subcubic graphs (i.e. with  $\Delta \leq 3$ ), and  $r$ -regular graphs (i.e. with all vertices of degree  $r$ ) with  $r \geq 3$ . In Section 4 we show a polynomial time algorithm for the  $T$ -COLORING PROBLEM on graphs with  $\Delta \leq 2$ .

## 2. Simple properties of graph homomorphisms

The idea of graph homomorphism is a generalization of vertex coloring. Moreover, it generalizes the  $T$ -coloring problem as well.



**Definition 3.** For two simple graphs  $G$  and  $H$  a *graph homomorphism* is a function  $h: V(G) \rightarrow V(H)$  such that  $\{h(v), h(w)\} \in E(H)$ , whenever,  $\{v, w\} \in E(G)$  for all  $v, w \in V(G)$ .

We write  $G \rightarrow H$  if there exists a homomorphism from  $G$  to  $H$ . Furthermore, if the homomorphism is onto, then it is called an *epimorphism*. In addition, if there exists  $h^{-1}$  and it is a homomorphism from  $H$  to  $G$ , then we call it an *isomorphism* and graphs  $G$  and  $H$  are said to be isomorphic, in symbols  $G \simeq H$ . We write  $H \tilde{\subset} G$  if  $H$  is isomorphic to any subgraph of  $G$ .

There is a straightforward equivalence between the properties of  $T$ -span and the existence of homomorphism from  $G$  to  $G_T^{d+1}$  (see [13]).

**Proposition 4.** Given a graph  $G$ , any set  $T$  and a nonnegative integer  $d$  we have  $\text{sp}_T(G) \leq d$  if and only if  $G \rightarrow G_T^{d+1}$ .

Let us note that if  $T = \{0\}$ , then the  $T$ -coloring problem reduces to the well-known vertex coloring problem, and moreover  $G_T^{d+1} \simeq K_{d+1}$ . Thus we get

**Corollary 5.** Given a graph  $G$  and a positive integer  $d$  we have  $\chi(G) \leq d$  if and only if  $G \rightarrow K_d$ .

The composition of graph homomorphisms is still a graph homomorphism. Moreover, an image of a complete graph under a homomorphism is a complete graph with the same number of vertices so

**Corollary 6.** If  $K_n \rightarrow G$  then  $K_n \tilde{\subset} G$ .

And

**Proposition 7.** If  $h: V(G) \rightarrow V(H)$  is a homomorphism then  $\psi(G) \leq \psi(H(h(V(G))))$ , where  $\psi$  is any of the functions from the list  $\{\chi, \omega, \text{sp}_T\}$ .

From the above is easy to see that if  $G \rightarrow H$  and  $H$  is bipartite, then graph  $G$  is bipartite. Concluding this section note an important upper bound proved in [17].

**Theorem 8** (Tesman [17]). For any given graph  $G$  and set  $T$  the following inequality holds

$$\text{sp}_T(G) \leq |T| \cdot (\chi(G) - 1).$$

Let us also recall that

**Theorem 9** (Brooks). If  $G$  is a connected graph that is neither a complete graph nor an odd cycle, then  $\chi(G) \leq \Delta(G)$ .

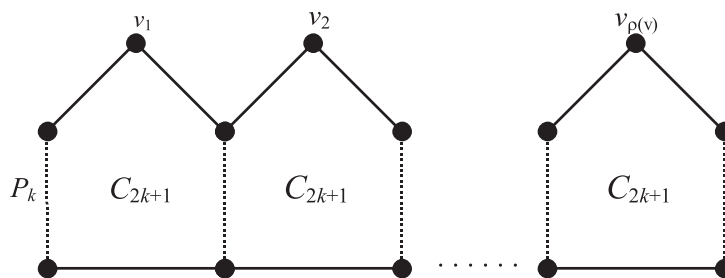


Fig. 1. Graph  $A_v^k$  replacing the vertex  $v$ .

### 3. Homomorphisms into odd cycles

The problem of graph homomorphism is considered in [1,5]. Let  $H$  be a fixed graph, the decision problem of the existence of a homomorphism from  $G$  to  $H$  will be denoted  $\text{HOM}(H)$ , where  $G$  is any graph from the specified family. The most important result comes from [9].

**Theorem 10** (Hell and Nešetřil [9]). *The problem  $\text{HOM}(H)$  on arbitrary graphs is polynomial, whenever  $H$  is bipartite, otherwise it is NP-complete.*

In this section we prove that the problem  $\text{HOM}(C_{2k+1})$  on subcubic graphs is NP-complete for every positive integer  $k \geq 2$ , in contrast to the problem  $\text{HOM}(C_3)$ , which is polynomial. Moreover, we prove analogous result for 3-regular graphs and NP-completeness of the problem  $\text{HOM}(C_{2k+1})$  on  $r$ -regular graphs, for every  $r \geq 4$  and  $k \geq 1$ .

We start with a general construction. Let  $G$  be an arbitrary graph and  $k$  be any positive integer greater than 1. We replace each vertex  $v \in V(G)$  of degree  $\rho(v)$  with the graph  $A_v^k$  shown in Fig. 1 (the dotted vertical lines in Fig. 1 mean path  $P_k$ ). We replace also every edge  $\{v, w\} \in E(G)$  with the edge  $\{v_i, w_j\}$  such that no two inserted edges are incident. Let  $G'_k$  be the graph constructed from  $G$  as above. It is easy to see that  $G'_k$  is always a subcubic graph.

**Theorem 11.** *The problem  $\text{HOM}(C_{2k+1})$ ,  $k \geq 2$  is NP-complete on subcubic graphs.*

**Proof.** By Theorem 10 it suffices to show  $G \rightarrow C_{2k+1}$  iff  $G'_k \rightarrow C_{2k+1}$ . First, observe that  $A_v^k \rightarrow C_{2k+1}$  and moreover for every homomorphism  $h_v: V(A_v^k) \rightarrow V(C_{2k+1})$  we have  $|h_v(\{v_1, \dots, v_{\rho(v)}\})| = 1$ . Otherwise, we have  $h_v(v_i) \neq h_v(v_{i+1})$  for some  $i \in \{1, \dots, \rho(v) - 1\}$ , hence  $h_v(v_i) = h_v(x)$ , where  $\{v_i, s\}, \{v_{i+1}, s\}, \{s, x\} \in E(A_v^k)$  and  $x \notin \{v_1, \dots, v_{\rho(v)}\}$ . Thus  $C_{2l-1}$  is subgraph of  $C_{2k+1}(h(V(A_v^k)))$  for some  $l < k$ , which is impossible. So, constructing a homomorphism  $g: V(G) \rightarrow V(C_{2k+1})$  from a homomorphism  $g': V(G'_k) \rightarrow V(C_{2k+1})$  is straightforward.

Conversely, let  $g: V(G) \rightarrow V(C_{2k+1})$  be a homomorphism, then we let  $g'(v_i) = g(v)$  and for  $w \in V(A_v^k) \setminus \{v_1, \dots, v_{\rho(v)}\}$   $g'(w) = \tau_v \circ h_v(w)$ , where  $h_v: V(A_v^k) \rightarrow V(C_{2k+1})$  is a



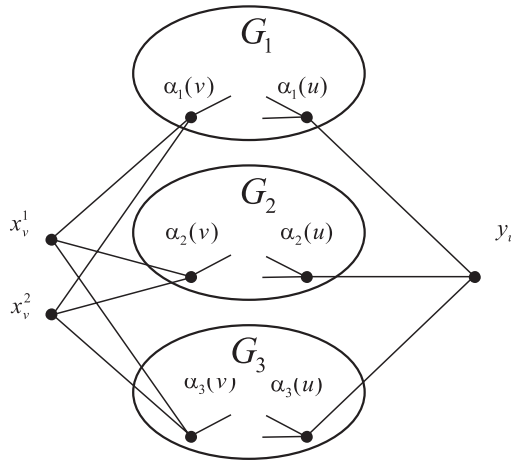


Fig. 2. A graph  $G'$ .

homomorphism and  $\tau_v$  is any automorphism of  $C_{2k+1}$  such that  $\tau_v(h_v(v_i)) = g(v)$ . One can check that  $g' : V(G'_k) \rightarrow V(C_{2k+1})$  is a homomorphism.  $\square$

**Theorem 12.** *The problem  $\text{HOM}(C_{2k+1})$ ,  $k \geq 2$  is NP-complete on 3-regular graphs.*

**Proof.** It suffices to show the equivalence  $G \rightarrow C_{2k+1}$  iff  $G' \rightarrow C_{2k+1}$  for any subcubic connected graph  $G$ , where  $k \geq 2$  and  $G'$  is a cubic graph defined as follows. Let  $\alpha_i$  be an isomorphism from graph  $G$  to its  $i$ th isomorphic copy  $G_i$ , for  $i = 1, 2, 3$ , which are vertex disjoint. Let  $V_j \subset V(G)$  be the set of vertices of degree  $j$ . We define  $V(G') = \bigcup_{i=1}^3 V(G_i) \cup \bigcup_{v \in V_1} \{x_v^1, x_v^2\} \cup \bigcup_{u \in V_2} \{y_u\}$  and  $E(G') = \bigcup_{i=1}^3 E(G_i) \cup \bigcup_{v \in V_1} \bigcup_{i=1}^3 \{\{x_v^1, \alpha_i(v)\}, \{x_v^2, \alpha_i(v)\}\} \cup \bigcup_{u \in V_2} \bigcup_{i=1}^3 \{\{y_u, \alpha_i(u)\}\}$  (see Fig. 2). Assuming that  $x_v^j$  and  $y_u$  are different vertices for  $j = 1, 2$  and  $v, u \in V(G)$ , it is obvious that  $G'$  is a cubic graph.

Now, suppose  $g : V(G) \rightarrow V(C_{2k+1})$  is a homomorphism. Let  $g' : V(G') \rightarrow V(C_{2k+1})$  be defined  $g'(w) = g(v)$  for  $w \in \{\alpha_1(v), \alpha_2(v), \alpha_3(v)\}$  and  $v \in V(G)$ ,  $g'(x_v^i) = g(z)$  for  $\{z, v\} \in E(G)$ ,  $g'(y_u) = g(z)$  for any  $z$  adjacent to  $u$ . Thus  $g'$  is a well-defined homomorphism. Conversely, if  $g'$  is a homomorphism from  $G'$  to  $C_{2k+1}$  then  $g = g' \circ \alpha_1$  is a homomorphism from  $G$  to  $C_{2k+1}$ .  $\square$

**Theorem 13.** *The problem  $\text{HOM}(C_{2k+1})$  is NP-complete on  $r$ -regular graphs for every fixed integer  $k \geq 1$  and  $r \geq 4$ .*

**Proof.** By induction on  $r \geq 4$ , consider  $r + 1$  isomorphic copies of any  $r$  regular graph. Using the analogous method as that in Theorem 12 we can show that the problem  $\text{HOM}(C_{2k+1})$  is NP-complete for any  $k \geq 2$  and for all  $r \geq 4$ . In [10] the author proved NP-completeness of edge 3-chromaticity of 3-regular graphs. Since line

graphs of 3-regular graphs are 4-regular, the problem of 3-chromaticity of 4-regular graphs is NP-complete. The construction from Theorem 12 is carried over to the case  $\text{Hom}(C_3)$  on  $r$ -regular graphs with  $r \geq 4$ .  $\square$

#### 4. Polynomial algorithm for cycles

We show a polynomial-time algorithm for graphs with  $\Delta \leq 2$ .

**Theorem 14.** *The  $T$ -COLORING PROBLEM on graphs with degree not exceeding 2 can be solved in time  $O(n|T|^2 \log|T|)$ .*

**Proof.** Bipartite graphs can be optimally colored with 1 and  $\min \mathbb{N} \setminus T + 1$ , thus all we need is considering odd cycles. Let  $T$  be any set and  $a$  be an arbitrary integer. We ask if  $\text{sp}_T(C_{2k+1}) \leq a - 1$ . By Theorem 8 we have  $\text{sp}_T(C_{2k+1}) \leq 2|T|$ . Thus using the standard bisection method we need only check  $1 + \log_2|T|$  inequalities to find  $\text{sp}_T(C_{2k+1})$ .

In the following, we sketch the idea of the algorithm. Let  $\text{TAB}(v_i)[1 \dots a]$  be a table of logical values associated with vertex  $v_i$  and defined as follows:  $\text{TAB}(v_i)[j] = \text{TRUE}$  if and only if there exists a  $T$ -coloring of path  $v_1, \dots, v_i$  using colors not greater than  $a$  such that  $v_1$  is colored with 1 and  $v_i$  is colored with  $j$ . So,  $\text{TAB}(v_1)$  has value TRUE only on its first position and  $\text{TAB}(v_{i+1})[y] = \text{TRUE}$  if and only if there exists  $z \in \{1, \dots, a\}$  such that  $|z - y| \notin T$  and  $\text{TAB}(v_i)[z] = \text{TRUE}$ . We see that there exists a  $T$ -coloring iff  $\text{TAB}(v_{2k+1})[j] = \text{TRUE}$  for some  $j - 1 \notin T$ , so constructing the  $T$ -coloring is straightforward. It is obvious that the complexity of the above algorithm is  $O(k|T|^2 \log|T|)$ .  $\square$

#### 5. Main results

Based on Theorem 11 we can prove the main result of this paper. Before doing this, we introduce the following notion.

**Definition 15.** For a given set  $T$ , by  $d_T$  we mean the number such that  $G_T^{d_T}$  is bipartite and  $G_T^{d_T+1}$  is not bipartite.

**Lemma 16.** *For any set  $T$  the following inequality holds:*

$$d_T \leq \text{sp}_T(K_3)$$

and, moreover,  $d_T$  can be determined in polynomial time.

**Proof.** Let us notice that  $\chi(G_T^{d_T+1}) = \chi(G_T^{d_T}) + 1 = 3$ . Thus from Corollary 5 it follows  $G_T^{d_T+1} \rightarrow K_3$ , hence by Proposition 7  $\text{sp}_T(G_T^{d_T+1}) \leq \text{sp}_T(K_3)$ . By Proposition 4  $\text{sp}_T(G_T^{d_T+1}) \leq d_T$ . Assuming  $\text{sp}_T(G_T^{d_T+1}) \leq d_T - 1$  we get at once  $G_T^{d_T+1} \rightarrow G_T^{d_T}$  but this contradicts the definition of  $d_T$ . So, we get  $d_T = \text{sp}_T(G_T^{d_T+1}) \leq \text{sp}_T(K_3)$ . By



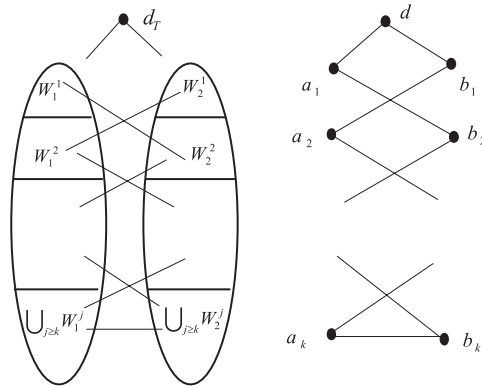


Fig. 3. A graph  $G_T^{d_T+1}$  (left) and a cycle  $C_{2k+1}$  (right).

Theorem 8  $sp_T(K_3) \leq 2|T|$ , hence using the bisection method we can determine the greatest  $d_T$  such that  $G_T^{d_T}$  is bipartite. This can be done in time  $O(|T|^2 \log|T|)$ .  $\square$

**Lemma 17.** Given any set  $T$ , we have  $d_T = sp_T(K_3)$  if and only if  $K_3 \tilde{C} G_T^{d_T+1}$ .

**Proof.** By Corollary 6  $K_3 \tilde{C} G_T^{d_T+1}$  is equivalent to  $K_3 \rightarrow G_T^{d_T+1}$ . Assume  $K_3 \tilde{C} G_T^{d_T+1}$ , then by Proposition 4  $sp_T(K_3) \leq d_T$ , hence from Lemma 16 it follows that  $d_T = sp_T(K_3)$ . The converse implication is straightforward by Proposition 4.  $\square$

Let us denote by  $C_T$  the shortest odd-length cycle in graph  $G_T^{d_T+1}$ .

**Lemma 18.** There exists a homomorphism  $h: V(G_T^{d_T+1}) \rightarrow V(C_T)$ .

**Proof.** We only have to construct a homomorphism on the vertices of the connected component of  $G_T^{d_T+1}$  containing vertex  $d_T$ , because the other components are bipartite. So let  $V_1$  and  $V_2$  be a bipartition of a bipartite graph obtained from this component by removing  $d_T$  and let  $W_i^j$ ,  $i = 1, 2$  and  $j \geq 1$ , be the vertex subset of  $V_i$  of distance  $j$  from vertex  $d_T$  in the graph  $G_T^{d_T+1}$ . Finally, let  $W_1^0 = W_2^0 = \{d_T\}$ . Let  $C_{2k+1} = (\{d, a_1, b_1, \dots, a_k, b_k\}, \{\{d, a_1\}, \{d, b_1\}, \{a_1, b_2\}, \{b_1, a_2\}, \dots, \{a_{k-1}, b_k\}, \{b_{k-1}, a_k\}, \{a_k, b_k\}\})$  be any cycle isomorphic to  $C_T$ . Let us define  $h(d_T) = d$ ,  $h(W_1^j) = \{a_j\}$  and  $h(W_2^j) = \{b_j\}$  for  $j = 1, \dots, k$  and  $h(W_i^j) = h(W_i^k)$  for  $j > k$ ,  $i = 1, 2$  (see Fig. 3).

The construction of  $h$  is correct because any vertex from  $W_i^j$ ,  $j > 0$ , can have neighbours only in the sets  $W_{3-i}^{j\pm 1}$  and  $W_{3-i}^j$ , and the latter case is impossible for  $j < k$ .  $\square$

**Lemma 19.** For any graph  $G$  the following equivalence holds:  $G \rightarrow G_T^{d_T+1}$  if and only if  $G \rightarrow C_T$ .

**Proof.** Let  $G \rightarrow G_T^{d_T+1}$ , hence from Lemma 18 it follows  $G \rightarrow C_T$ . Conversely, assume that  $G \rightarrow C_T$ . By definition  $C_T \tilde{C} G_T^{d_T+1}$ , thus we get  $G \rightarrow G_T^{d_T+1}$ .  $\square$

**Theorem 20.** *The T-SPAN PROBLEM can be solved in polynomial time on subcubic graphs for all sets  $T$  satisfying  $K_3 \tilde{C} G_T^{d_T+1}$ . The FIXED T-SPAN PROBLEM is NP-complete on cubic graphs for all sets  $T$  not satisfying  $K_3 \tilde{C} G_T^{d_T+1}$ .*

**Proof.** Let  $T$  be a fixed set and  $k$  be any positive integer. By Theorem 8 the case  $G = K_4$  is polynomial and can be solved in  $O(|T|^3)$  time (by Proposition 4 it reduces to the problem of finding the smallest  $d$  such that  $K_4 \tilde{C} G_T^d$ ; by Theorem 8  $K_4 \tilde{C} G_T^{3|T|+1}$  and the fact that 0 is a vertex of a maximal clique of  $G_T^d$ , it reduces to searching all the triples of vertices of  $G_T^{3|T|+1}$ ). For any subcubic graph  $G \neq K_4$  we ask if  $\text{sp}_T(G) \leq k$ .

Suppose that  $K_3 \tilde{C} G_T^{d_T+1}$ . Brooks' theorem implies  $G \rightarrow K_3$ , thus by Lemma 17 and Proposition 7  $\text{sp}_T(G) \leq d_T$ . According to Proposition 4 we have  $\text{sp}_T(G) < d_T$  iff  $G$  is bipartite, hence to solve T-SPAN PROBLEM for graph  $G$  we only need to check if  $G$  is bipartite ( $O(n+m)$  time) and if it is so then  $\text{sp}_T(G)$  equals the smallest positive integer not belonging to  $T$  (which we can find in  $O(|T|)$  time). Otherwise,  $\text{sp}_T(G) = d_T$ , computable in time  $O(|T|^2 \log|T|)$ .

Now assume that  $K_3$  is not isomorphic to any subgraph of  $G_T^{d_T+1}$  and let  $k = d_T$ . From Proposition 4 we have  $\text{sp}_T(G) \leq k$  iff  $G \rightarrow G_T^{d_T+1}$ . By Lemma 19 we get  $\text{sp}_T(G) \leq k$  iff  $G \rightarrow C_T$  and, moreover,  $C_T$  is an odd cycle of length greater than 4. By Theorem 12 the problem  $\text{HOM}(C_T)$  on cubic graphs is NP-complete and so is the FIXED T-SPAN PROBLEM.  $\square$

**Corollary 21.** *The T-SPAN PROBLEM is NP-complete in the strong sense on 3-regular graphs.*

**Proof.** By Theorem 20 and Lemma 17 it suffices to verify that for  $T = \{0, 2, 3\}$  we have  $d_T = 4 < \text{sp}_T(K_3) = 5$ .  $\square$

It is worth observing that if for some set  $T$  we put  $k = d_T$ , then by Lemma 19 for any graph  $G$  the question if  $\text{sp}_T(G) \leq k$  is equivalent to  $G \rightarrow C_T$ . So, if for every  $k \geq 1$  the problem  $\text{HOM}(C_{2k+1})$  is NP-complete on a class  $\mathcal{G}$ , then the FIXED T-SPAN PROBLEM on the class  $\mathcal{G}$  is NP-complete as well. Thus from Theorem 13 we have the following:

**Theorem 22.** *For every set  $T$  and integer  $r \geq 4$  the FIXED T-SPAN PROBLEM is NP-complete on  $r$ -regular graphs.*

**Corollary 23.** *The T-SPAN PROBLEM is NP-complete in the strong sense on  $r$ -regular graphs for any  $r \geq 3$ .*

Table 1 Now we sum up all the above results in the following table. Recall that the numbers appearing in the third column are polynomially computable functions of  $T$ .





Table 1

The complexity of the  $T$ -SPAN PROBLEM and  $T$ -COLORING PROBLEM on graphs with bounded degree

Graph	Problem	Property of $T$	Complexity	Reference
$\Delta \leq 2$	$T$ -COLORING PROBLEM	any	$O(n T ^2 \log T )$	Theorem 14
$\Delta \leq 3$	$T$ -COLORING PROBLEM	$\omega(G_T^{d_T+1}) \geq 3$	$O(n^2 +  T ^3)$	Theorem 20
3-regular	FIXED $T$ -SPAN PROBLEM	$\omega(G_T^{d_T+1}) \leq 2$	NPC	Theorem 20
$r$ -regular, $r \geq 4$	FIXED $T$ -SPAN PROBLEM	any	NPC	Theorem 22

### Acknowledgements

The authors are grateful to the two anonymous referees and prof. Kubale for valuable comments.

### References

- [1] M.O. Albertson, Generalized Colorings, Academic Press, New York, 1987, pp. 35–49.
- [2] M. Bellare, O. Goldreich, M. Sudan, Free bits, PCP and non-approximability towards tight results, Proceedings of the 36th IEEE Symposium on Foundations of Computer Science, Los Alamos, 1995, pp. 422–431.
- [3] M.B. Cozzens, F.S. Roberts,  $T$ -colorings of graphs and the channel assignment problem, Congr. Numer. 35 (1982) 191–208.
- [4] M.B. Cozzens, F.S. Roberts, Greedy algorithms for  $T$ -colorings of complete graphs and the meaningfulness of conclusions about them, J. Combin. Inform. System Sci. 16 (1991) 286–299.
- [5] A.M.H. Gerards, Homomorphisms of graphs into odd cycles, J. Graph Theory 12 (1988) 73–83.
- [6] A. Gräf, Distance graphs and the  $T$ -coloring problem, Discrete Math. 196 (1999) 153–166.
- [7] J.R. Griggs, D.D.-F. Liu, The channel assignment problem for mutually adjacent sites, J. Combin. Theory Ser. A 68 (1994) 169–183.
- [8] W.K. Hale, Frequency assignment: theory and applications, Proceedings IEEE 68 (1980) 1497–1514.
- [9] P. Hell, J. Nešetřil, On the complexity of  $H$ -coloring, J. Combin. Theory Ser. B 48 (1990) 92–110.
- [10] I. Holyer, The NP-completeness of edge-coloring, SIAM J. Comput. 10 (1981) 718–720.
- [11] R. Janczewski, A note on divisibility and  $T$ -span of graphs, Discrete Math. 234 (2001) 171–179.
- [12] K. Jansen, A rainbow about  $T$ -colorings for complete graphs, Discrete Math. 154 (1996) 129–139.
- [13] D.D.-F. Liu,  $T$ -colorings of graphs, Discrete Math. 101 (1992) 202–212.
- [14] D.D.-F. Liu,  $T$ -graphs and the channel assignment problem, Discrete Math. 161 (1996) 197–205.
- [15] A. Raychaudhuri, Further results on  $T$ -colorings and frequency assignment problem, Discrete Math. 7 (1994) 605–613.
- [16] F.S. Roberts,  $T$ -coloring of graphs: recent results and open problems, Discrete Math. 93 (1991) 229–245.
- [17] B. Tesman,  $T$ -colorings, list  $T$ -colorings and set  $T$ -colorings of graphs, Ph.D. Thesis, Department of Math. Rutgers University, New Brunswick, NJ, 1989.
- [18] B. Tesman, Applications of forbidden difference graphs to  $T$ -colorings, Congressus Numerantium 74 (1990), 15–24.

