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Simulation and Discretization of Random Field in the Slip-Line Method

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Abstract

The paper deals with the problem of bearing capacity of a strip footing based on the random subsoil. The solution has been obtained in a framework of the slip-line method. In order to include randomness of the soil medium, a modification of the method of characteristics is proposed. In such an approach, the stochastic finite difference method based on the Monte Carlo technique has been adopted. It enables including a spatial variability of soil properties into the analysis and to determine its influence on the variance of the ultimate collapse load. It is assumed that the soil medium is purely cohesive and only its cohesion can be considered as a random field. The simulation algorithm of multi-dimensional random field, based on the diagonalisation of the covariance matrix by a transformation using a lower-triangle matrix is described. The problems of the medium discretization and stabilization of the solution are discussed.

Key words: bearing capacity, method of characteristics, random field

1. Introduction

Studies on the bearing capacity problem started over sixty years ago. Many different methods utilizing the concept of perfect plasticity have been proposed. For frictional soils, the Coulomb criterion is widely used, whereas for ideally cohesive soils, it is the Tresca criterion which is primarily used for the yield condition. Combining respective criterion with the equations of equilibrium gives a set of differential equations of plastic equilibrium determining the basis of the slip-line method (Chen 1975). Together with the stress boundary conditions, this set of equations can be used to investigate the stresses in the soil beneath the footing.

The analytical closed form solution of the slip-line equations can be obtained only for special cases of soil and boundary conditions. For other cases, many approximate methods have been developed. One of the most effective, so called method of characteristics, is based on a finite difference approximation. Sokolovskii (1965) applied this method to different soil mechanics problems and obtained a number of interesting solutions of the bearing capacity problems, for which it is impossible to find closed form solutions. The method of characteristics is extensively described by Abbott (1966) and its application in geotechnical engineering by Szczepiński (1974).

In nature, soils intrinsically involve randomness and uncertainty. Its spatial variabilities are evident and can significantly influence the computing results. Considerable effort has been recently made to improve models of the soil properties by describing them as random fields. In such approach, the method of solution of bearing capacity problems should be reformulated into the stochastic one. So, method of characteristics should be suitably modified and stochastic finite difference method employed.

The stochastic approach to the bearing capacity of strip footing in a framework of the slip-line method, is proposed in the paper. It is assumed that the soil is ideally cohesive and behaves as the perfectly plastic material. Method of characteristics in conjunction with stochastic difference method is presented. The simulation algorithm proposed by Wilde (1981) is described and problems of soil medium discretization and stabilization of the solution are discussed.

2. Bearing Capacity Problem

The problem under consideration is the determination of the ultimate bearing capacity of a single, strip footing founded on the plane surface of a semi-infinite mass of soil that is assumed to be perfectly plastic material. This assumption was taken only for convenience and compatibility with another paper written by the author (Przewłócki, Dardzińska 2002), concerning probabilistic limit analysis. The footing and its equivalent static scheme are illustrated in Fig.1. It is further assumed that the load acting on the footing is normal and uniformly distributed and increases until penetration occurs as a result of a plastic flow in the soil. The investigation is limited to the bearing capacity of the strip footing on the horizontal bearing area. In practice, the footing is usually based at some depth h (Fig. 1a), which corresponds to a uniform load acting in the vicinity of the footing $q = \gamma h$ (Fig. 1b).

3. Method of Characteristics

The method of characteristics is an effective, numerical method for the solving of bearing capacity problems. In practice, in order to determine components of the stress tensor at every point of a soil medium (for a two-dimensional state of strains), it suffices to solve two equilibrium equations and the equation of the yield condition. Closed form solutions of the above defined problem of bearing capacity can be obtained using the method of characteristics.

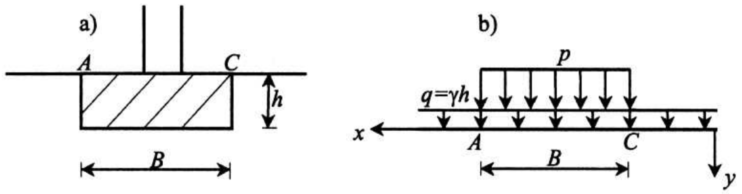

Fig. 1. Strip footing and its static scheme

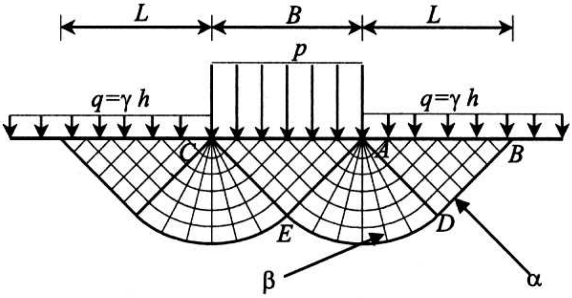

Fig. 2. Characteristics net for strip footing

There is a state of active pressure in a rigid wedge AEC which is formed directly under the footing (Fig. 2). A state of passive pressure (ABD) occurs in the region loaded by q. There is a fan of radiuses and arcs rolled from points A and C (edges of footing) between these two regions (ADE). In the limit analysis theory these three regions are qualified to different boundary problems that are known respectively as the Cauchy, characteristic with a singular point and the mix problems. According to Szczepiński (1974), the bearing capacity of the strip footing based on the ideally cohesive subsoil equals:

$$P = [2c\,(1 + \pi/2) + \gamma h]B \tag{1}$$

where: P – unit bearing capacity, c – cohesion of soil.

Assuming an associated flow rule, this is even a so-called exact solution, in the framework of the theory of limit analysis. Unfortunately, it is not possible to include the spatial variability of soil in Eq. (1), as the cohesion is constant and in the stochastic approach corresponds to random variable. For the cohesion considered as the random field, special derivations must be performed.

In the method of characteristics, two dependent variables θ and s, instead of three stress components σ_x, σ_y, τ_{xy} , are introduced. The variable θ denotes orientation of the major principal stress direction and the variable s represents a mean stress less overburden pressure and can be expressed as follow:

$$s = \frac{(\sigma_1 + \sigma_2)}{2} - \gamma y \tag{2}$$

According to the procedure described by Szczepiński (1974), some derivations for a case of random subsoil were performed by the author (Przewłócki 2001) and the following set of two quasi-linear, stochastic partial differential equations was obtained:

$$\left.\begin{aligned} &\frac{\partial s}{\partial x} + \frac{\partial c}{\partial x}\cos(2\theta) - 2c\sin(2\theta)\frac{\partial \theta}{\partial x} + \frac{\partial c}{\partial y}\sin(2\theta) + 2c\cos(2\theta)\frac{\partial \theta}{\partial y} = 0 \\ &\frac{\partial s}{\partial y} - \frac{\partial c}{\partial y}\cos(2\theta) + 2c\sin(2\theta)\frac{\partial \theta}{\partial y} + \frac{\partial c}{\partial x}\sin(2\theta) + 2c\cos(2\theta)\frac{\partial \theta}{\partial x} = 0 \end{aligned}\right\} \tag{3}$$

This set can be replaced by two sets of stochastic ordinary differential equations:

$$\left.\begin{aligned} &\alpha : \frac{dy}{dx} = \frac{\sin(2\theta) + 1}{\cos(2\theta)} = \tan\left(\theta + \frac{\pi}{4}\right) \\ &\beta : \frac{dy}{dx} = \frac{\sin(2\theta) - 1}{\cos(2\theta)} = \tan\left(\theta - \frac{\pi}{4}\right) \end{aligned}\right\} \tag{4}$$

and

$$\left.\begin{aligned} &\alpha : ds + 2cd\theta - \frac{\partial c}{\partial y}dx + \frac{\partial c}{\partial x}dy = 0 \\ &\beta : ds - 2cd\theta + \frac{\partial c}{\partial y}dx - \frac{\partial c}{\partial x}dy = 0 \end{aligned}\right\} \tag{5}$$

It is worth noting, that Kuznicov (1958) derived similar equations for the analogous boundary problem, but deterministically non-homogeneous medium.

Equations (4) describe two families of characteristics, i.e. lines denoted by α and β. For ideally cohesive soil, characteristics are inclined in each point of soil medium under the angle $\pi/4$ in the direction of the major principal stress. For the considered problem of strip footing, characteristics are shown in Fig. 2. It is worth noting that the equations of characteristics (4), derived for the stochastic soil medium are the same as in the deterministic case (Szczepiński 1974). However, they have a stochastic nature, as dependent variable θ appearing here is a random field. It is shown in the sequel that the expression defining this variable differs from a deterministic one. Contrary to the deterministic case, additional terms containing cohesion increments, i.e., partial derivatives of cohesion with respect to x and y, appear in the stochastic equations (5). These derivatives impose the necessity for the assumption of a differential random field.

4. Stochastic Finite Difference Method

Usually, in order to solve any boundary problem by the method of characteristics, equations (4) and (5) must be solved one after the other. First the direction ratios of characteristics are computed and then dependent variables θ and s, defined along characteristic, are calculated. Next, the components of the stress tensor and finally the bearing capacity are found.

Equations (4) and (5) can be effectively solved using the finite difference method. However, due to the randomness of the cohesion, considered as the random field $c(x,y)$, the dependent variables $s(x,y)$ and $\theta(x,y)$ are also random, thus the stochastic finite difference method should be used. In this method a number of points sufficiently close to each other is chosen. Knowing the position of two neighboring points (x_1, y_1) , (x_2, y_2) and attributing to them dependent variables s_1, θ_1 and s_2, θ_2 , the coordinates of neighboring point (x_M, y_M) and variables s_M , θ_M are sought. Replacing the derivatives and differentials occurring in (4) and (5) with finite differences, two sets of recurrent equations are obtained:

$$\left.\begin{aligned} x_M &= \frac{x_1 \tan\left(\theta_1 + \frac{\pi}{4}\right) - x_2 \tan\left(\theta_2 - \frac{\pi}{4}\right) + y_2 - y_1}{\tan\left(\theta_1 + \frac{\pi}{4}\right) - \tan\left(\theta_2 - \frac{\pi}{4}\right)} \\ y_M &= y_1 + (x_M - x_1)\tan\left(\theta_1 + \frac{\pi}{4}\right) \end{aligned}\right\} \tag{6}$$

and attributing to this point dependent variables:

$$\left.\begin{aligned} & s_M = \frac{s_1 + s_2}{2} + c_M(\theta_1 - \theta_2) + (c_M - c_1)\cot\left[2\left(\theta_1 + \frac{\pi}{4}\right)\right] + \\ & -(c_M - c_2)\cot\left[2\left(\theta_2 - \frac{\pi}{4}\right)\right] = 0 \\ & \theta_M = \theta_1 - \frac{s_M - s_1}{2c_M} + \frac{c_M - c_1}{c_M}\cot\left[2\left(\theta_1 + \frac{\pi}{4}\right)\right] = 0 \end{aligned}\right\} \tag{7}$$

It should be noted that the expressions (6), defining the coordinates of point M are the same as in the case of a deterministic soil medium. However, dependent variables θ_1 and θ_2 appearing here are random. The considerable differences for the stochastic and the deterministic soil media occur in expressions (7). In the case of a deterministic soil medium, only the first two terms of each expression (7) exist.

The calculations according to (6) and (7) should be performed for all pairs of neighboring points. Finally, it allows determination of states of stresses in the defined region. For the stochastic soil medium it is tantamount to realization of the random field. Only application of the Monte Carlo technique enables determination of the statistical characteristics of the bearing capacity.

The procedure based directly on the Monte Carlo technique is the simplest stochastic variant of the finite difference method. It is more effective the lesser the number of simulations of random field is needed to achieve the stabilized solution. This number depends on the assumed random variable generator, as well as the simulation method. Many random variable generators have been developed and usually are accessible in a form of utilizable computing programs. There are also several simulation methods that generate multi-dimensional random fields. The straightforward method of simulation of random field proposed Wilde (1981). This method, based on diagonalization of the covariance matrix by a transformation using a lower-triangle matrix, has been adopted in the present paper and is described in the following.

5. Random Field Simulation

Consider first the soil cohesion to be a one-dimensional random field $c(x)$ with normal distribution. Replacing it by a random series C_k, the n-dimensional random variable is obtained. It can be written using matrix notation in the following form:

$$\mathbf{C}^T = [C_1, C_2, ..., C_n] \tag{8}$$

where $\mathbf{C}$ is a random column matrix and the index T denotes transpose.

Such n-random variable is defined by its expected value:

$$E[\mathbf{C}^T] = \overline{\mathbf{C}}^T = [\overline{C}_1, \overline{C}_2, ..., \overline{C}_n] \tag{9}$$

And the covariance matrix:

$$\mathbf{K} = E\left[\left(\mathbf{C} - \overline{\mathbf{C}}\right)\left(\mathbf{C} - \overline{\mathbf{C}}\right)^T\right] = \begin{bmatrix} k_{11} & k_{12} \dots\dots\dots k_{1n} \\ k_{21} & k_{22} \dots\dots\dots k_{2n} \\ \dots\dots & \dots\dots\dots\dots \\ k_{31} & k_{32} \dots\dots\dots k_{3n} \end{bmatrix} \tag{10}$$

Let us assume that both expected value and the covariance matrix are known.

Without loosing generality one can introduce a new n-dimensional random variable $U_1, U_2, \dots, U_n$ defined by the relationship:

$$U_i = C_i - \overline{C}_i \tag{11}$$

where the bar denotes averaging.

The expected values of the variables U_i are equal to zero and the covariance matrix $\mathbf{K}$ remains unchanged.

A simple diagonalization algorithm of the covariance matrix can be obtained assuming linear transformation of independent n-dimensional random variables $V_1, V_2, \dots, V_n$ into dependent variables $U_1, U_2, \dots, U_n$, according to the formula:

$$\mathbf{V} = \mathbf{pU} \tag{12}$$

where $\mathbf{p}$ is the lower-triangular matrix having units on the diagonal:

$$\mathbf{p} = \begin{bmatrix} 1 & 0 & 0......0 \\ p_{21} & 1 & 0......0 \\ p_{31} & p_{32} & 1......0 \\ ...... & ...... & ...... \\ p_{n1} & p_{n2} & p_{n3}..1 \end{bmatrix} \tag{13}$$

The determinant of the matrix $\mathbf{p}$ is equal to one, thus the matrix is non-singular. As random variables $V_1, V_2, \ldots, V_n$ are independent, they are also uncorrelated. The covariance matrix of such variables, denoted by $\mathbf{d}$, is diagonal and can be presented in the form:

$$\mathbf{d} = \begin{bmatrix} d_1 & 0 & 0......0 \\ 0 & d_2 & 0......0 \\ ...... & ..... & .... \\ 0 & 0 & 0......d_n \end{bmatrix} \tag{14}$$

where d_{i} is the variance of the independent variable V_{i}.

The covariance matrix of n-dimensional random variable $\mathbf{U}$, given by the linear transformation (12), is defined as follows:

$$\mathbf{K} = \mathbf{pdp}^T \tag{15}$$

The matrix $\mathbf{d}$ can be found by multiplying expression (15) by inverse matrices:

$$\mathbf{d} = \mathbf{p}^{-1}\mathbf{K}\left(\mathbf{p}^{-1}\right)^T \tag{16}$$

The formulae defining elements of matrices $\mathbf{p}$ and $\mathbf{d}$ can easily be worked out by expanding equation (15). Substituting (10), (13) and (14) into it yields:

$$\begin{bmatrix} 1 & 0 & 0......0 \\ p_{21} & 1 & 0......0 \\ p_{31} & p_{32} & 1......0 \\ ...... & ...... & ...... \\ p_{n1} & p_{n2} & p_{n3}......1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0......0 \\ 0 & d_2 & 0......0 \\ 0 & 0 & d_3......0 \\ ...... & ...... & ...... \\ 0 & 0 & 0...d_n \end{bmatrix} \begin{bmatrix} 1 & p_{21} & p_{31}......p_{n1} \\ 0 & 1 & p_{32}......p_{n2} \\ 0 & 0 & 1......p_{n3} \\ ...... & ...... & ...... \\ 0 & 0 & 0...1 \end{bmatrix} = \\ = \begin{bmatrix} k_{11} & k_{12} & k_{13}......k_{1n} \\ k_{21} & k_{22} & k_{23}......k_{2n} \\ k_{31} & k_{32} & k_{33}......k_{3n} \\ ...... & ...... & ...... \\ k_{n1} & k_{n2} & k_{n3}......k_{nn} \end{bmatrix} \tag{17}$$

Performing matrix multiplications and comparing corresponding elements, the following expressions are obtained:

$$\begin{aligned}
&d_1 = k_{11}\\
&d_1 p_{21} = k_{12}, d_1 p_{31} = k_{13}, ..., d_1 p_{n1} = k_{1n}\\
&d_2 + d_1 p_{21}^2 = k_{22}\\
&d_2 p_{32} + d_1 p_{31} p_{21} = k_{23}, ..., d_2 p_{n2} + d_1 p_{n1} p_{21} = k_{2n}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{18}$$

It is easy to notice that from the expressions, other relationships can consequently be worked out. In general, unknown elements are defined as follows:

$$\begin{aligned}
&d_1 = k_{11}\\
&d_i = k_{ii} - \sum_{k=1}^{i-1} p_{ik}^2 d_k \quad \text{for} \quad i = 2, ..., n\\
&d_1 p_{j1} = k_{1j} \quad \text{for} \quad j = 2, ..., n\\
&d_i p_{ji} = k_{ij} - \sum_{k=1}^{i-1} p_{ik} p_{jk} d_k \quad \text{for} \quad j > i, i, j = 2, ..., n.
\end{aligned} \tag{19}$$

Knowing matrices **p** and **d**, the simulation algorithm for n-dimensional random variable **U** based on the linear relationship between random variables **U** and **V** (12), is obtained. The normal random number generator with expected value equal to zero and standard deviation equal to one can be presumed. It enables the obtaining of n statistically independent random numbers, which should be multiplied by standard deviation (square root of d). Thus, n-dimensional realization of the random variables V_1, V_2,..., V_n is obtained. Multiplying matrix **V** by the matrix **p**, the n-dimensional random variable in a form of the column matrix **U** is obtained. Adding to it the expected value according to (11), transforms this matrix into the realization of one-dimensional random field **C**.

The covariance matrix of random field (10) can be considered to be the discrete values of correlation function $K(x_1, x_2)$. For the stationary case it becomes a function of the separation distance. Let us consider two different correlation functions:

$$R(\tau) = \exp(-\lambda|\tau|) \tag{20a}$$

$$R(\tau) = (1 + \lambda|\tau|)\exp(-\lambda|\tau|) \tag{20b}$$

where: λ is the correlation decay coefficient and τ the distance between two points of soil medium.

Decay coefficient λ indicates the character of random field variability. For small values, a field is slow-speeded which indicates high correlation and for λ approaching infinity, there is lack of correlation and the random field is high-speeded. Some realizations of the one-dimensional random field, for both correlation functions (20) and two values of the decay coefficients, obtained by the described above simulation algorithm, are shown in Figs. 3 and 4.

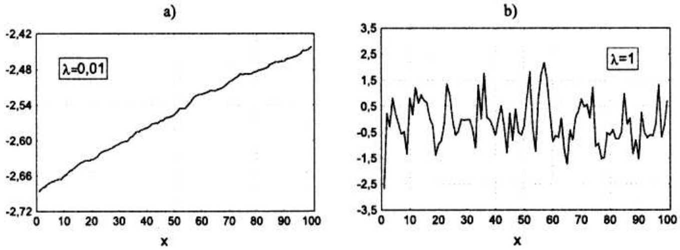

Fig. 3. Realization of the non-differentiable, one-dimensional random field

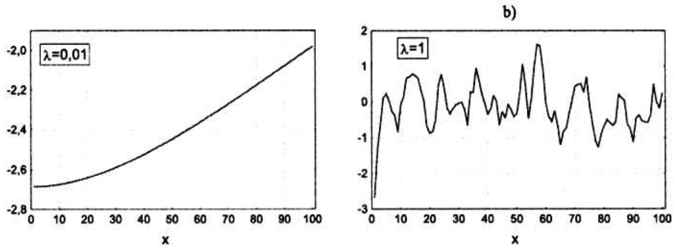

Fig. 4. Realization of the differentiable, one-dimensional random field

The correlation function defined by (20a) is non-differentiable and by (20b) is a differentiable one. According to (3), the random field of cohesion must be differentiable, thus the latter is considered in the following analysis. Comparing Fig. 3 and Fig. 4, it can be seen that the realization of the differentiable random fields is smoother. Smoother realizations can be obtained for random field of higher order of differentiability. For such fields, the stochastic difference method applied in the paper would be much more effective.

It is assumed in the present analysis, that the cohesion is an isotropic and homogeneous, two-dimensional random field. Thus the described simulation algorithm should be generalized for such a case.

So, let us now consider the two-dimensional random function $u(x, y)$. It can also be replaced by the random series U_{ij} defined at the points of intersections of lines of the rectangular net (Fig. 5). Two natural numbers are assigned to each point, corresponding to its coordinates: $x = j\,\Delta x$ and $y = i\,\Delta y$.

Assume that there are n points at the abscissa and m points at the ordinate. For the numbering shown in Fig. 5, random field $\mathbf{U}$ can be written in the form of the following matrix:

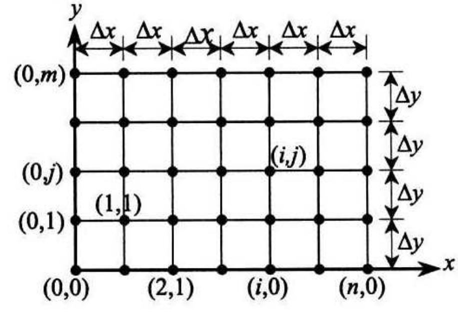

Fig. 5. Rectangular net of discretization

$$\mathbf{C} = \begin{bmatrix} C_{00} & C_{01} & \ldots\ldots & C_{0,n-1} \\ C_{10} & C_{11} & \ldots\ldots & C_{1,n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ C_{m-1,0} & C_{m-1,1} & \ldots\ldots & C_{m-1,n-1} \end{bmatrix} \tag{21}$$

It can be alternatively presented as the column matrix:

$$\mathbf{C}^T = [\mathbf{C}_0, \quad \mathbf{C}_1, \ldots, \quad \mathbf{C}_{m-1}] \tag{22}$$

where each element is also the column matrix:

$$\mathbf{C}_i^T = [C_{i0} C_{i1} \ldots C_{i,n-1}] \tag{23}$$

Similarly, the expected value $\overline{\mathbf{C}}$ and the random field $\mathbf{U}$ can be defined as follows:

$$\overline{\mathbf{C}}^T = \left[\overline{\mathbf{C}}_0, \overline{\mathbf{C}}_1, \ldots, \overline{\mathbf{C}}_{m-1}\right] \quad , \quad \overline{\mathbf{C}}_i^T = [\overline{C}_{i0} \overline{C}_{i1} \ldots \overline{C}_{i,n-1}] \tag{24}$$

$$\mathbf{U}^T = [\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_{m-1}] \quad , \quad \mathbf{U}_i^T = [U_{i0} U_{i1} \ldots U_{i,n-1}] \tag{25}$$

According to the notation introduced, the covariance matrix can be presented in the form of a block matrix:

$$K = \begin{bmatrix} \mathbf{K}_{0,0} & \mathbf{K}_{0,1} & \ldots\ldots & \mathbf{K}_{0,m-1} \\ \mathbf{K}_{1,0} & \mathbf{K}_{1,1} & \ldots\ldots & \mathbf{K}_{1,m-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \mathbf{K}_{m-1,0} & \mathbf{K}_{m-1,1} & \ldots\ldots & \mathbf{K}_{m-1,m-1} \end{bmatrix} \tag{26}$$

where elements $\mathbf{K}_{i,j}$ are defined as follows:

$$\mathbf{K}_{i,j} = E\left[\left(\mathbf{C}_i - \bar{\mathbf{C}}_i\right)\left(\mathbf{C}_j - \bar{\mathbf{C}}_j\right)^T\right] = E\left[\mathbf{U}_i \mathbf{U}_j^T\right] =$$
$$E\begin{bmatrix} U_{i0}U_{j0} & U_{i0}U_{j1} & \ldots\ldots & U_{i0}U_{j,n-1} \\ U_{i1}U_{j0} & U_{i1}U_{j1} & \ldots\ldots & U_{i1}U_{j,n-1} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ U_{i,n-1}U_{j0} & U_{i,n-1}U_{j1} & \ldots\ldots & U_{i,n-1}U_{j,n-1} \end{bmatrix} \tag{27}$$

The matrices on diagonal correspond to covariance matrices of particular rows.

Now, the proposed simulation algorithm can be utilized for the two-dimensional random field. However, in the case of a matrix, which elements are also matrices, the lower-triangular matrix $\mathbf{p}$ should also be defined in the form of a block matrix:

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{0} & \ldots\ldots & \mathbf{0} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \ldots\ldots & \mathbf{0} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \mathbf{p}_{m-1,0} & \mathbf{p}_{m-1,1} & \ldots\ldots & \mathbf{p}_{m-1,m-1} \end{bmatrix} \tag{28}$$

where $\mathbf{p}_{ii}$ is the lower-triangular matrix with units on the diagonal.

In such notation, matrix $\mathbf{d}$ also becomes a block one with diagonal matrices on the diagonal. Following the same steps as (15)-(19), the following expressions are finally obtained:

$$\begin{aligned} &\mathbf{p}_{00}\mathbf{d}_0\mathbf{p}_{00}^T = \mathbf{K}_{00} \\ &\mathbf{p}_{0,0}\mathbf{d}_0\mathbf{p}_{1,0}^T = \mathbf{K}_{0,1},\ \mathbf{p}_{0,0}\mathbf{d}_0\mathbf{p}_{2,0}^T = \mathbf{K}_{0,2},\ \ldots \\ &\mathbf{p}_{1,0}\mathbf{d}_0\mathbf{p}_{1,0}^T + \mathbf{p}_{1,1}\mathbf{d}_1\mathbf{p}_{1,1}^T = \mathbf{K}_{1,1} \\ &\mathbf{p}_{1,0}\mathbf{d}_0\mathbf{p}_{2,0}^T + \mathbf{p}_{1,1}\mathbf{d}_1\mathbf{p}_{2,1}^T = \mathbf{K}_{1,2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{29}$$

Matrices $\mathbf{p}_{0,0}$ and $\mathbf{d}_0$ can be determined from the first row of (28). Then consequently matrices $\mathbf{p}_{1,0}$ and $\mathbf{p}_{1,0}$ from the second row, then matrices $\mathbf{p}_{1,1}$ and $\mathbf{d}_1$ from the third and so on. In this way, analogical to (23) relationships can be worked out and the simulation algorithm defined.

6. Discretization of Soil Medium

In the formulation presented, the fundamental equations of the method of characteristics are given by expressions including finite differences. Such an approach closely depends on fixing the values of these differences, or in other words, with the assumption of a computing step, i.e., with the discretization of soil medium. The computing step is assumed in two stages. In the first stage, section AB in the Cauchy problem, is divided into N parts, usually of the same length Δx. This is tantamount to the assumptions $N-1$ characteristics from each family, in triangles ABD (Fig. 2). In the second stage, in the problem with a singular point, any decrement of the angle θ is divided into M parts of $\delta\theta$ each. This means that M characteristics of β family, in the fan ADE shown in Fig. 2, are assumed. So, for assumed decrements Δx and $\delta\theta$, the discretization is governed by two parameters M and N, giving together N-1 characteristics of family α and $2(N\text{-}1)+M$ characteristics of family β.

The exemplary net of characteristics for deterministically homogeneous soil medium, given values M, N and $q = 2$ kPa distributed at the section of length $L = 8$ m, is shown in Fig. 6. The calculations were performed for a weightless soil medium and cohesion equal to $c = 10$ kPa.

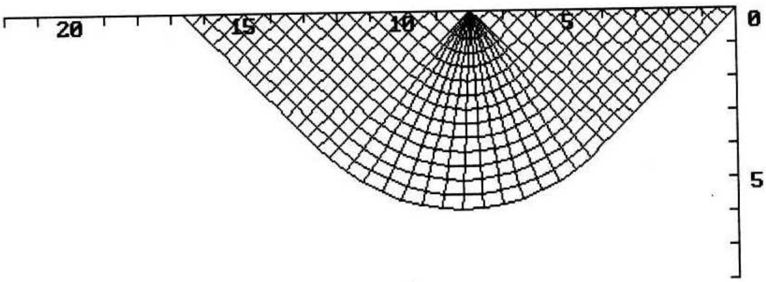

Fig. 6. Net of characteristics for $M = 15$, $N = 15$

The bearing capacity was computed for several pairs of parameters M and N and the results varied in range P = 53,416–53,516 kPa. Thus one can conclude that the influence of discretization for the deterministically homogeneous medium is inconsiderable.

Adequate discretization is of great importance in the case of the stochastic soil medium. In this case, the character of spatial variability of the random field describing a soil medium is very significant. For a high-speeded random field, the network of characteristics should be considerably thickened. For example, a net of characteristics for one realization of the isotropic and homogeneous random field, described by the correlation function (21) is shown in Fig. 7a. The calculations were performed for the following data: average cohesion $\bar{c}$ = 10 kPa, standard deviation σ_c = 2 kPa, correlation decay coefficient $\lambda = 1$, $M = 3$ and $N = 5$ and $q = 2$ kPa, $L = 8$ m, as before. A more detail description of the course of computing is given in another paper written by the author (Przewłócki 2001).

It can be seen in Fig. 7a that the characteristics of the same family intersect. This is an incorrect solution, having no physical meaning. The net of characteristics should be more dence, as shown in Fig. 7 b-d, for the same realization and different values of parameters M and N. It is seen in these figures that besides parameter M, the N also significantly influences the range B. For a sufficiently dense division, i.e., for high values of parameters M and N, the range B tends to the settled value (B = 8 m).

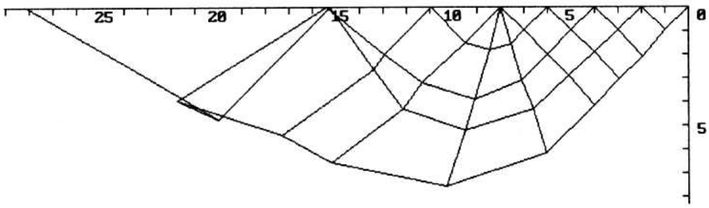

Fig. 7a. Net of characteristics for stochastic soil medium (M = 3, N = 5)

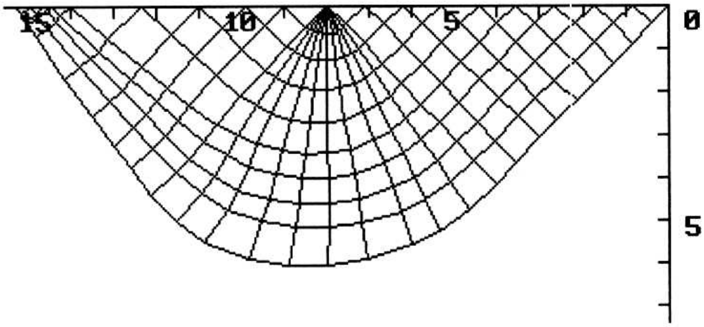

Fig. 7b. Net of characteristics for stochastic soil medium (M = 10, N = 10)

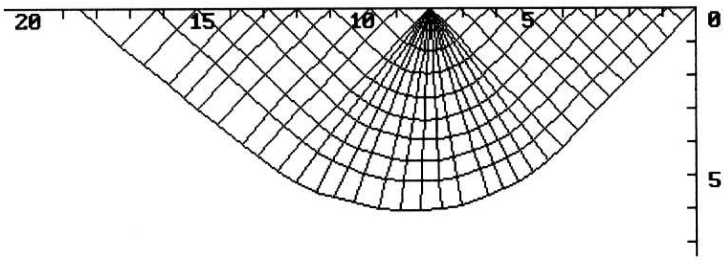

Fig. 7c. Net of characteristics for stochastic soil medium (M = 15, N = 10)

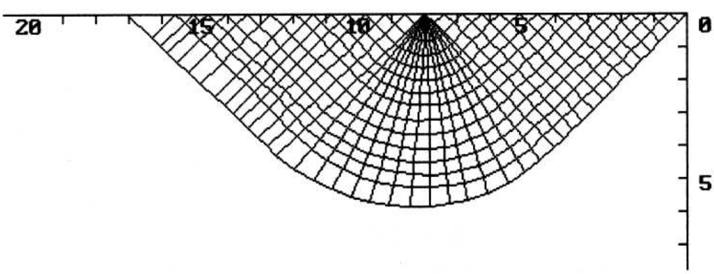

Fig. 7d. Net of characteristics for stochastic soil medium (M = 15, N = 15)

Parameters M and N should be fixed numerically, assuming for a given random field that their values for which the average intensity of load $\overline{p}$ and its range B will stabilize. The results of calculations B and p generally differ in succeeding realizations of a random field. The expected values and variances can be obtained using the Monte Carlo method.

It is worth noting that contrary to the deterministic case, the characteristics for the stochastic medium become broken lines, forming curvilinear triangles and a fan of irregular arcs and radiuses.

7. Stabilization of the Solution

The correlation decay coefficient determines the rate of random field variation, and for its small values the random field is slow-speeded. The rate of random field variation is extremely important in the case of the method of characteristics based on the stochastic finite difference method. It is necessary to assume a sufficiently small computing step for a high-speed random field. It makes calculation much more time-consuming.

The expected value and standard deviation of a unit bearing capacity versus the number of random field simulations, for $\lambda = 1$, $M = 3$ and $N = 5$ are shown in Fig. 8a and 8b, respectively. It is seen in these figures that good stabilization is achieved for 1000 random field simulations. Similar results were obtained for other values of decay coefficients λ.

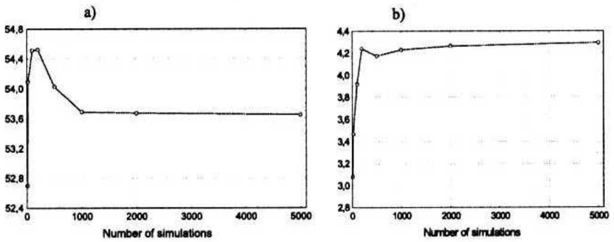

Fig. 8. Expected value and Standard deviation of unit bearing capacity versus a number of random field simulations

The expected value and standard deviation of a range of footing B versus the number of random field simulations, for $\lambda = 1$ are shown in Figs. 9a and 9b, respectively. Again, it is seen that quite good stabilization is achieved after 1000 random field simulations. However, it should be emphasized, that the width of footing B depends not only on soil properties, but also on the assumed range of the loading q, resulting from foundation depth. In the case considered range L was assumed to be equal to 8 m. The computations were performed for $M = 3$ and $N = 5$, i.e. for a "rough" discretization of the soil medium. For a fine mesh, the range of footing should approach $B = 8$ m.

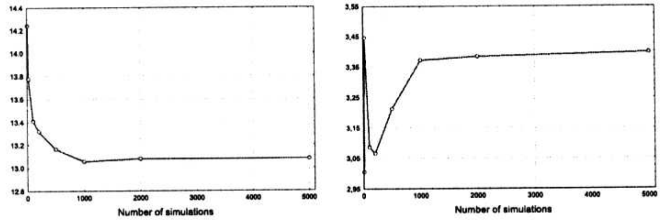

Fig. 9. Expected value and standard deviation of range of footing versus a number of random field simulations

8. Influence of Spatial Variability of Soil Properties on Bearing Capacity

Most essential is the influence of spatial correlation of cohesion on the standard deviation of the unit bearing capacity. The change of the standard deviation of the ultimate bearing capacity, being a product of the unit bearing capacity and the range of footing ($p \times B$), versus correlation decay coefficient is shown in Fig. 10. In this case, standard deviation of the ultimate bearing capacity decreases for diminishing correlation and tends to zero for its lack.

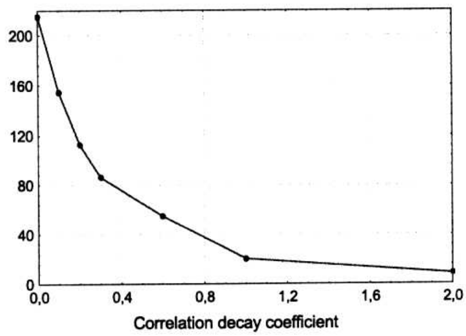

Fig. 10. Variation of standard deviation of the ultimate bearing capacity versus correlation decay coefficient λ

It should be noted that for real soils, the decay correlation coefficients are of the order of $\lambda = 1 - 2$ (Przewłócki 1998). This means a rather small correlation of soil properties. Thus the assumption of full correlation (only possible in limited analysis methods) is not reasonably acceptable. It confirms the necessity of application of the proposed methodology, i.e., the stochastic method of characteristics.

9. Conclusions

An extremely important influence on results of the stochastic analysis proposed in the paper is due to soil discretization and rate of the random field variation.

Soil discretization in the considered case of a bearing capacity of strip footing can be controlled by two parameters, describing the computing step in the Cauchy problem and a number of characteristics in the problem with a singular point. The latter one significantly influences the range of the footing. The "rough" discretization can lead to unrealistic results. The rate of random field variation is extremely important in the case of the method of characteristics based on the stochastic finite difference method. For quick-speed random fields, it is necessary to assume a sufficiently small computing step. Unfortunately, it makes calculation much more time-consuming. For slow-speed random fields, calculations become similar to those as in the deterministic case.

The effectiveness of the method of characteristics based on the stochastic finite difference method depends on the random variable generator and the simulation method assumed. The simulation algorithm, proposed by Wilde (1981), has been successfully applied in the analysis and good stabilization was achieved for 1000 random field simulations.

The method of characteristics based on the stochastic finite difference method enables the including of spatial variability of the soil medium, into the stochastic slip-line method. The results obtained give an additional measure to the bearing capacity and enable computation of the probability of failure of strip footing. The standard deviation of bearing capacity decreases with increasing values of the decay coefficient and tends to zero for lack of correlation.

References

Abbott M. B. (1966), *The Method of Characteristics*, American Elseviere, New York.

Chen W. (1975), *Limit Analysis and Soil Olasticity*,. Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York.

Kuznetzov A.I. (1958), The Problem of Torsion and Plane Strain of Non-homogeneous Plastic Bodies, *Archiwum Mechaniki Stosowanej*, No. 4, Vol. 10.

Przewłócki J. (1998), *Randomness in Choosen Problems of Soil Mechanics* (in Polish), Biblioteka Naukowa Hydrotechnika nr 25, Wydawnictwo IBW PAN.

Przewłócki J. (2001), *Stochastic Adaptation of the Method of Characteristics and its Application in Soil Mechanics* (in Polish), J. Kazimierz Szmidt (Ed.), Wydawnictwo IBW PAN.

Przewłócki J., Dardzińska I. (2002), Limit Analysis of a Strip Footing on Stochastic Subsoil, *Studia Geotechnica et Mechanica*, Vol. XXIII, No. 3–4.

Sokolovskii V. V. (1965), *Statics of Granular Media*, Pergamon Press, New York.

Szczepiński W. (1974), *Limit States and Kinematics of Granular Medium* (in Polish), Biblioteka Mechaniki Stosowanej, PWN Warszawa.

Wilde P. (1981), *Random Field Discretization in Engineering Computation* (in Polish), Biblioteka Naukowa Hydrotechnika No. 9, PWN, Warszawa-Poznań.