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Numerical Integration of a Coupled Korteweg-de Vries System

A. A. HALIM
Technical University of Gdansk
ul. G. Narutowicza 11/12, 80-952 Gdansk, Poland

S. P. KSHEVETSKII
Kaliningrad State University
Kaliningrad, Russia

S. B. LEBLE
Technical University of Gdansk
ul. G. Narutowicza 11/12, 80-952 Gdansk, Poland
leble@mif.pg.gda.pl

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Abstract—We introduce a numerical method for general coupled Korteweg-de Vries systems. The scheme is valid for solving Cauchy problems for an arbitrary number of equations with arbitrary constant coefficients. The numerical scheme takes its legality by proving its stability and convergence, which gives the conditions and the appropriate choice of the grid sizes. The method is applied to the Hirota-Satsuma (HS) system and compared with its known explicit solution investigating the influence of initial conditions and grid sizes on accuracy. We also illustrate the method to show the effects of constants with a transition to nonintegrable cases. © 2003 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Coupled Korteweg-de Vries (cKdV) system equations form a class of important nonlinear evolution systems. Its importance comes (physically) from the wide application field it covers and (mathematically) from including both (weak) nonlinearity and third-order derivatives (weak dispersion). It describes the interactions of long waves with different dispersion relations. Namely, it is connected with most types of long waves with weak dispersion ($\omega(k) \to 0$, $k \to 0$), e.g., internal, acoustic, and planetary waves in geophysical hydrodynamics.

It was introduced by Maxworthy *et al.* [1] in studying the nonlinear atmosphere Rossby waves. Hirota and Satsuma [2] give single- and two-soliton solutions to some version of the system. Dodd and Fordy [3] found an L-A pair for Hirota-Satsuma equations. Leble derived the cKdV system for different hydrodynamical systems with explicit expressions for the nonlinear and dispersion constants [4]. He also developed the approach to the cKdV integration. Leble and Kshevetskii [4,5] used the system in investigation of nonlinear internal gravity waves. Perelomova [6]

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used it in description of interaction of acoustic waves with opposite directions of propagation in liquids with bubbles.

Others deal with integrability of the system [3] from a Lax pair point of view, Leble [7] in Walquist-Estabrook theory. Foursov [8] described a new method for constructing an integrable system of differential equations that reduced to cKdV equations. Oevel [9] considers an integrable system of cKdV and found an infinite hierarchy of commuting symmetries and conservation laws in involution. Zharkov [10] obtained a new class of integrable KdV-like systems. Gurses and Karasu [11] found infinitely many coupled systems of KdV type equations which are integrable. They also give recursion operators. In studying the Painlevé test classification of the system, Karasu [12] found new KdV systems that are completely integrable in the sense of WTC paper. He was looking for the integrable subclass of KdV systems given by Svinolupov [13]. The latter has introduced a class of integrable multicomponent KdV equations associated with Jordan algebras. Weiss [14] derived the associated "modified" equations for the HS system, and from these the Lax pair is also derived. Kupershmidt [15] showed that a dispersive system describing a vector multiplet interacting with the KdV field is a member of a bi-Hamiltonian integrable hierarchy.

The significant achievement in numerical solution of the single KdV equation starts from the famous paper of Zabusky and Kruskal [16]. It develops the idea of soliton solutions set for the integrable equations and enlighten the problems of effective integration scheme elaborating. The paper launched numerous investigations and inventions in this field. Perhaps the last publication that develops applications of recent theoretical achievements in numerical integration schemes is based on the notion of isospectral deformations [18]. Recently a multisymplectic twelve points scheme was produced [17]. This scheme is equivalent to the multisymplectic Preissmann scheme and is applied to solitary waves over a long time interval.

Zhu [19] had produced a difference scheme for the periodic initial-boundary problem of the coupled KdV (Hirota-Satsuma case) system. He used the inner product of the discrete function to obtain a scheme keeping two conserved quantities. His scheme is a nonlinear algebraic system for which a catch-ran iterative method is designed to solve it.

The coupled KdV system representing most possible physical applications (related to the weak nonlinear dispersion) to be considered in this work takes the following general form:

$$(\theta_n)_t + c_n (\theta_n)_x + \sum_{k,m} g_{mkn}\, \theta_k\, (\theta_m)_x + d_n (\theta_n)_{xxx} = 0, \qquad n, m, k = 1, 2, 3, \ldots, N, \tag{1}$$

where $\theta_n(x,t)$ is the amplitude of the wave mode as a function of space x and time t, respectively. The constants c_n are the linear velocities and g_{mkn}, e_n are the nonlinear and dispersion coefficients.

In the present work, we introduce a numerical tool for solving a coupled KdV system which is a development of the two-step three-time levels as Lax-Wendroff scheme [5,20]. Proving the theorem about stability and convergence of the scheme gives the opportunity to use it for different applications like Cauchy problems for arbitrary number of equations and a wide class of initial conditions $\theta_n(x,0)$. We consider in our problem an infinite domain while the initial condition goes quickly enough to zero following the relation

$$\int (1 + |x|)|\theta(x,0)|\, dx = 0 < \infty,$$

keeping in mind the choice of a smooth and integrable function. As an important corollary of the theorem, one obtains conditions that have to be taken into account in choosing grid sizes. This numerical method is checked by applying it to the HS system, for which a good number of explicit solutions exist [21]. We examine also the effects of equations coefficients and conditions of the problem on the solution.

In Section 2, we introduce the difference scheme for an arbitrary number of coupled KdV equations. We investigate stability and prove the convergence giving the condition has to be taken into account in choosing the grid sizes and how they are related. In Section 3, we analyze the HS system with a two-parameters one-soliton explicit solution. The numerical method is applied to the HS system and compared with the explicit solution. We analyzed the effects of the two parameters and initial condition on the form of the resulting solitons as well as on accuracy and show the results in figures. We also produced (numerically) a multisoliton solution for the HS system and used the conservation law to estimate the expected number of solitons which agreed that we already obtained. Proving stability and convergence besides testing the results for the HS system allows us in Section 4 to use the scheme for the general cKdV system. Hence, we illustrate by plots the results of applying the scheme to slightly nonintegrable cKdV systems and others for a system with nonsmooth initial conditions.

2. THE NUMERICAL METHOD

2.1. The Difference Scheme

For the cKdV system (1), we introduce a numerical (finite-difference) method of solution, a scheme which is two-steps three-time levels similar to the Lax-Wendroff one [5,20]. The usual Lax-Wendroff is modified such that the order of the first derivative becomes of order $O(\triangle x^4)$. The approximation of the nonlinear terms is changed in such a manner that the integral of θ^2 is a conserved one. The approach gives a solution that can be considered as some generalized solution, in the sense of Schwartz distribution theory, where the dispersion constants vanish. This scheme is suitable to nonlinear equations and is valid for n equations with arbitrary coefficients. The scheme can be simply derived beginning from Taylor series expansion as

$$(\theta_n)_i^{j+1} = (\theta_n)_i^j + \Delta t\left((\theta_n)_t\right)_i^j + O\left[(\Delta t)^2\right], \tag{2}$$

where i and j are used to locate a point in the discrete domain and Δt is the time step, while the subscript t means time derivative. Substituting for $(\theta_n)_t$ in (2) using system (1) to obtain

$$(\theta_n)_i^{j+1} = (\theta_n)_i^j - \Delta t\left(c_n(\theta_n)_x + \sum_{k,m} g_{mkn}\theta_k(\theta_m)_x + d_n(\theta_n)_{xxx}\right)_i^j + O\left[(\Delta t)^2\right], \tag{3}$$

the difference scheme is elaborated applying the Lax idea for a half-time step and leapfrog method to the remaining half-time step. In both steps, $(\theta_n)_x$ and $(\theta_n)_{xxx}$ are replaced by fourth-order $O(\Delta x^4)$ and second-order accurate $O(\Delta x^2)$ central difference expressions. Hence, (3) gives the following difference scheme:

$$\begin{aligned}&\frac{\left((\theta_n)_i^{j+1/2} - (\theta_n)_i^j\right)}{\tau/2} + c_n\frac{\left((\theta_n)_{i+1}^j - (\theta_n)_{i-1}^j\right)}{2h} + \sum_{k,m} g_{mkn}\frac{(\theta_k)_i^j\left((\theta_m)_{i+1}^j - (\theta_m)_{i-1}^j\right)}{2h}\\ &+ e_n\frac{\left((\theta_n)_{i+2}^j - 2(\theta_n)_{i+1}^j + 2(\theta_n)_{i-1}^j - (\theta_n)_{i-2}^j\right)}{2h^3} = 0, \qquad e_n = \left(d_n - \frac{c_n h^2}{6}\right),\end{aligned} \tag{4a}$$

where n, m, k are the modes numbers; i and j are discrete space and time, respectively. The time step Δt is replaced for simplicity by τ while h denotes spatial step. Equation (4a) is accompanied with a discrete equation for the intermediate layer as

$$\begin{aligned}&\frac{\left((\theta_n)_i^{j+1} - (\theta_n)_i^j\right)}{\tau} + c_n\frac{\left((\theta_n)_{i+1}^{j+1/2} - (\theta_n)_{i-1}^{j+1/2}\right)}{2h}\\ &+ \sum_{k,m} g_{mkn}\frac{(\theta_k)_i^{j+1/2}\left((\theta_m)_{i+1}^{j+1/2} - (\theta_m)_{i-1}^{j+1/2}\right)}{2h}\\ &+ e_n\frac{\left((\theta_n)_{i+2}^{j+1/2} - 2(\theta_n)_{i+1}^{j+1/2} + 2(\theta_n)_{i-1}^{j+1/2} - (\theta_n)_{i-2}^{j+1/2}\right)}{2h^3} = 0.\end{aligned} \tag{4b}$$

2.2. Stability and Convergence Analysis

For simplicity of the analysis, we start by considering one equation of the system and give the details of stability and convergence. Then we apply the idea to the general cKdV system because it is rather close to that for one KdV equation but more bulky.

2.2.1. Stability analysis for KdV scheme

Consider one KdV equation of system (1),

$$\theta_t + c\theta_x + g\,\theta\theta_x + d\theta_{xxx} = 0. \tag{5}$$

Note again that the investigation we perform can be generalized for the case of any finite number of modes. Considering the numerical scheme applied for equation (5),

$$\frac{\left(\theta_i^{j+1} - \theta_i^j\right)}{\tau} + c\frac{\left(\theta_{i+1}^j - \theta_{i-1}^j\right)}{2h} + g\theta_i^j\frac{\left(\theta_{i+1}^j - \theta_{i-1}^j\right)}{2h} + e\frac{\left(\theta_{i+2}^j - 2\theta_{i+1}^j + 2\theta_{i-1}^j - \theta_{i-2}^j\right)}{2h^3} = 0. \tag{6}$$

Let us select a suitable norm. For this, multiply equation (5) by θ and integrate to yield

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\theta^2\,dx = 0 \qquad \text{or} \qquad \int_{-\infty}^{\infty}\theta^2\,dx = \text{const},$$

and hence, by definition of the L_2 norm, $(\|\theta\|_2)^2 = \int_{-\infty}^{\infty}\theta^2\,dx$, it may be written as $(\|\theta\|_2)^2 =$ const, i.e., the norm $\|\theta\|_2$ is conserved and the equation can be treated in the L_2 norm.

Now we will prove stability with respect to small perturbations (because we consider nonlinear equations) of initial conditions. Strictly speaking, it is the boundedness of the discrete solution in terms of small perturbation of the initial data. So let us consider the differential

$$d\theta_i^{j+1} = \sum_r \left(\frac{\partial\,\theta_i^{j+1}}{\partial\,\theta_r^j}\right) d\theta_r^j, \qquad r = \ldots, i-1, i, i+1, \ldots, \tag{7}$$

for equation (5) denoting

$$T_{i,r}^{j+1} = \left\{\frac{\partial\,\theta_i^{j+1}}{\partial\,\theta_r^j}\right\}, \qquad d\theta_r^j = \left\{\begin{matrix} d\theta_{i-2}^j \\ d\theta_{i-1}^j \\ d\theta_i^j \\ d\theta_{i+1}^j \\ d\theta_{i+2}^j \end{matrix}\right\}$$

and use $\|d\theta^j\| = \left(\sum_r (d\theta_r^j)^2\,h\right)^{1/2}$. Rewrite also relation (7) in the matrix form

$$d\theta_i^{j+1} = T_{i,r}^{j+1}\,d\theta_r^j = T^{j+1}T^j\,d\theta^{j-1} = \prod_r T^r\,d\theta^0,$$

where $d\theta^0$ is a small perturbation of initial data and the subscripts are omitted for simplicity.

Stability requires the boundedness of the product $\prod_r T^r$ in a sense that the norm $\|\prod_r T^r\|$ is bounded by some constant, i.e., $\|\prod_r T^r\| \le C$. Here, C is a constant, and the matrix norm is a spectral norm. For this, the sufficient condition is $\|T^r\| < e^{a\tau}$ where a is a constant, that is independent of τ. The case $\|T^r\| < e^{a(\tau,h)*\tau}$ is a sufficient condition of stability also, but only if $|a(\tau,h)| \le \text{const} < \infty$. If $|a(\tau,h)| < \text{const}$, including τ, $h \to 0$, for some dependence $\tau = f(h)$, then we can talk about conditional stability. Namely, this kind of stability will be established below.

To calculate T^{τ}, rewrite scheme (6) in the form $\theta_i^{j+1} = \theta_i^{j+1}(\theta_{i+2}^j, \theta_{i+1}^j, \theta_i^j, \theta_{i-1}^j, \theta_{i-2}^j)$. So,

$$\begin{aligned}\left(T^{j+1}\right)_{ir} = \delta_{i,r} - \left(\frac{c\tau}{2h}\right)\left[\delta_{i+1,r} - \delta_{i-1,r}\right] - \left(\frac{g\tau}{2h}\right)\left[\theta_i^j\left(\delta_{i+1,r} - \delta_{i-1,r}\right) + \delta_{i,r}\left(\theta_{i+1}^j - \theta_{i-1}^j\right)\right] \\ - \left(\frac{e\tau}{2h^3}\right)\left[\delta_{i+2,r} - 2\delta_{i+1,r} + 2\delta_{i-1,r} - \delta_{i-2,r}\right].\end{aligned} \tag{8}$$

Rewriting (8) in terms of the identity (E), symmetric (S), and antisymmetric (A) matrices $T^{j+1} = E + S^{j+1} + A^{j+1}$,

$$\begin{aligned}\left\{S^{j+1}\right\}_{i,r} &= -\frac{g\tau}{4h}\left(\left(\theta_i^j - \theta_{i+1}^j\right)\delta_{i+1,r} - \left(\theta_i^j - \theta_{i-1}^j\right)\delta_{i-1,r} + 2\delta_{i,r}\left[\theta_{i+1}^j - \theta_{i-1}^j\right]\right) \\ \left\{A^{j+1}\right\}_{i,r} &= -\frac{c\tau}{2h}\left[\delta_{i+1,r} - \delta_{i-1,r}\right] - \frac{g\tau}{4h}\left(\left(\theta_i^j + \theta_{i+1}^j\right)\delta_{i+1,r} - \left(\theta_i^j + \theta_{i-1}^j\right)\delta_{i-1,r}\right) \\ &\quad - \frac{e\tau}{2h^3}\left[\delta_{i+2,r} - 2\delta_{i+1,r} + 2\delta_{i-1,r} - \delta_{i-2,r}\right],\end{aligned}$$

$$\|S^{j+1}\| \le |g|\tau \max_i\left(\left|\theta_{x,i}^j\right|, \left|\theta_{x,i}'^j\right|\right), \qquad \theta_{x,i}^j = \frac{\left[\theta_{i+1}^j - \theta_i^j\right]}{(h)}, \qquad \text{and} \qquad \theta_{x,i}'^j = \frac{\left[\theta_{i+1}^j - \theta_{i-1}^j\right]}{(2h)},$$

$$\|A^{j+1}\| \le \frac{|g|\tau}{h}\max_i\left|\theta_i^j\right| + \frac{|c|\tau}{h} + \frac{3|e|\tau}{h^3}.$$

one arrives at

$$\begin{aligned}\|T^{j+1}\|^2 &= \left\|\left(T^{j+1}\right)^* T^{j+1}\right\| = \left\|\left(E - A^{j+1} + S^{j+1}\right)\left(E + A^{j+1} + S^{j+1}\right)\right\| \\ &\le 1 + 2\|S^{j+1}\| + \left(\|A^{j+1}\| + \|S^{j+1}\|\right)^2 \\ &\le 1 + 2|g|\tau \max_i\left|\theta_{x,i}^j\right| + \tau^2\left(|g|\max_i\left|\theta_{x,i}^j\right| + \frac{|g|}{h}\max_i\left|\theta_i^j\right| + \frac{|c|}{h} + \frac{3|e|}{h^3}\right)^2 \\ &\le e^{a\tau}, \quad \text{where } a = 2|g|\max\left|\theta_{x,i}^j\right| + \tau\left(|g|\max_i\left|\theta_{x,i}^j\right| + \frac{|g|}{h}\max_i\left|\theta_i^j\right| + \frac{|c|}{h} + \frac{3|e|}{h^3}\right)^2,\end{aligned}$$

which is a necessary condition of stability. The scheme is stable if $a \le \text{constant} \le \infty$ in spite of τ, $h \to 0$. This is a conditional stability of the scheme. It means that it is required for stability that $\tau \to 0$ is faster than $h \to 0$, or

$$\tau \le (\text{constant}). \qquad h^6, \text{ constant} < \infty. \tag{9}$$

Therefore, for small enough τ, we can simplify the expression for a

$$a = 2|g|\max\left|\theta_{x,i}^j\right| + \tau\left(\frac{3e}{h^3}\right)^2.$$

In practical calculations, the time step τ should be chosen so that it would satisfy $\tau(3e/h^3)^2 * t_0 = O(1)$, where t_0 is the time of simulation $(0 \le t \le t_0)$. In the future, when we shall suggest some better numerical scheme, we will essentially use our observation that stability depends only on the dispersion terms. And now we will try to accomplish our short investigation of scheme (6) by strongly proving the numerical scheme convergence.

2.2.2. The proof of the KdV scheme convergence

Now we prove that a solution of equation (6) converges to a solution of (5), if the exact solution is a continuously-differentiable one. Let us denote by $\theta(x,t)$ a solution of equation (5).

We substitute $\theta_i^j = \theta(x_i, t_j) + v_i^j$ into (6); v_i^j is an error between the difference solution θ_i^j and the exact solution $\theta(x_i, t_j)$. We obtain the equation for v_i^j,

$$\begin{aligned}
&\frac{\left(v_i^{j+1} - v_i^j\right)}{\tau} + c\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h} + g\theta(x_i, t_j)\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h} + gv_i^j\frac{(\theta(x_{i+1}, t_j) - \theta(x_{i-1}, t_j))}{2h}\\
&\qquad + gv_i^j\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h} + e\frac{\left(v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j\right)}{2h^3}\\
&= -\left(\frac{(\theta(x_i, t_{j+1}) - \theta(x_i, t_j))}{\tau} + c\frac{(\theta(x_{i+1}, t_j) - \theta(x_{i-1}, t_j))}{2h} + g\theta(x_i, t_j)\frac{(\theta(x_{i+1}, t_j) - \theta(x_{i-1}, t_j))}{2h}\right.\\
&\qquad \left. + e\frac{(\theta(x_{i+2}, t_j) - 2\theta(x_{i+1}, t_j) + 2\theta(x_{i-1}, t_j) - \theta(x_{i-2}, t_j))}{2h^3}\right).
\end{aligned}$$

Let us take into account that

$$\begin{aligned}
v_i^j - \tau\left(c\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h} + g\theta(x_i, t_j)\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h} + gv_i^j\frac{(\theta(x_{i+1}, t_j) - \theta(x_{i-1}, t_j))}{2h}\right.\\
\left. + e\frac{\left(v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j\right)}{2}h^3\right) = \sum_k \left(T^{j+1}\right)_{ik} v_k^j.
\end{aligned}$$

Using the operator T^{j+1} introduced above, this equation may be rewritten in the form

$$\begin{aligned}
&\frac{\left(v_i^{j+1} - \sum_k \left(T^{j+1}\right)_{ik} v_k^j\right)}{\tau} + gv_i^j\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h}\\
&= -\left(\frac{(\theta(x_i, t_{j+1}) - \theta(x_i, t_j))}{\tau} + c\frac{(\theta(x_{i+1}, t_j) - \theta(x_{i-1}, t_j))}{2h}\right.\\
&\qquad + g\theta(x_i, t_j)\frac{(\theta(x_{i+1}, t_j) - \theta(x_{i-1}, t_j))}{2h}\\
&\qquad \left. + e\frac{(\theta(x_{i+2}, t_j) - 2\theta(x_{i+1}, t_j) + 2\theta(x_{i-1}, t_j) - \theta(x_{i-2}, t_j))}{2h^3}\right).
\end{aligned}$$

The right part of this relation is a quantity of order $O(\tau + h^2)$. So, we can write

$$\frac{\left(v_i^{j+1} - \sum_k \left(T^{j+1}\right)_{ik} v_k^j\right)}{\tau} + gv_i^j\frac{\left(v_{i+1}^j - v_{i-1}^j\right)}{2h} = O\left(\tau + h^2\right)$$

or

$$v_i^{j+1} = \sum_k \left(T^{j+1}\right)_{ik} v_k^j - \tau f_i^j, \qquad f_i^j = gv_i^j\frac{v_{i+1}^j - v_{i-1}^j}{2h} - O\left(\tau + h^2\right). \tag{10}$$

We finally arrive at the inequality that compares the norms

$$\left\|f^j\right\| \le \frac{|g|}{h^{3/2}}\left\|v^j\right\|^2 + O\left(\tau + h^2\right).$$

To explain how this estimate was obtained, follow the expressions

$$\begin{aligned}
\left\|f^j\right\| &= \left(\sum_i \left(f_i^j\right)^2 h\right)^{1/2} \le |g|\left(\sum_i \left(v_i^j\frac{v_{i+1}^j - v_{i-1}^j}{2h}\right)^2 h\right)^{1/2} + O\left(\tau + h^2\right)\\
&\le |g|\left(\sum_i \left(v_i^j\right)^2 h * \sum_i \left(v_i^j\right)^2 h\right)^{1/2} \frac{1}{h^{3/2}} + O\left(\tau + h^2\right).
\end{aligned} \tag{11}$$

Using the Schwartz inequality $\|AB\| \leq \|A\|\|B\|$, formulas (10) could be transformed as

$$\begin{aligned}
\left\|v^{j+1}\right\| &\leq \left\|T^{j+1}\right\| \left\|v^{j}\right\| + \tau \left\|f^{j}\right\| \leq \left\|T^{j+1}\right\| \left\|T^{j}\right\| \left\|v^{j-1}\right\| + \tau \left(\left\|T^{j+1}\right\| \left\|f^{j-1}\right\| + \left\|f^{j}\right\|\right) \\
&\leq \left\|T^{j+1}\right\| \left\|T^{j}\right\| \left\|T^{j-1}\right\| \left\|v^{j-2}\right\| \\
&\quad + \tau \left(\left\|T^{j+1}\right\| \left\|T^{j}\right\| \left\|f^{j-2}\right\| + \left\|T^{j+1}\right\| \left\|f^{j-1}\right\| + \left\|f^{j}\right\|\right) \\
&\leq e^{a\tau j} \left\|v^{0}\right\| + \tau \left(e^{a\tau(j-1)} \left\|f^{0}\right\| + e^{a\tau(j-2)} \left\|f^{1}\right\| + \cdots \left\|f^{j}\right\|\right) \qquad (12) \\
&\leq e^{a\tau j} \left\|v^{0}\right\| + M \max_{k \leq j} \left(\left\|f^{k}\right\|\right) \\
&\leq e^{a\tau j} \left\|v^{0}\right\| + M \left(\frac{|g|}{h^{3/2}} \left\|v^{j+1}\right\|^{2} + O\left(\tau + h^{2}\right)\right), \qquad M = \tau \frac{e^{a\tau j} - 1}{e^{a\tau} - 1}.
\end{aligned}$$

To derive (12), we have used the iteration of the first of formula (we substituted the formula into itself, but for index less than 1) and using (11). Then we have utilized the formula for a sum of geometric series. Further, the inequality obtained in (12) has a solution

$$\left\|v^{j+1}\right\| \leq \frac{\left(1 - \sqrt{1 - 4M(|g|/h^{3/2})\left(e^{a\tau}\left\|v^{0}\right\| + MO(\tau + h^{2})\right)}\right)}{\left(2M|g|/h^{3/2}\right)}.$$

If we take into account (9), and use $\|v^0\| = 0$, we obtain

$$\begin{aligned}
\left\|v^{j+1}\right\| &\leq \frac{\left(1 - \sqrt{1 - (4M|g|/h^{3/2})MO\left(\tau + h^{2}\right)}\right)}{\left(2M|g|/h^{3/2}\right)} \\
&\approx \frac{\left(1 - \left(1 - (2M|g|/h^{3/2})MO\left(\tau + h^{2}\right)\right)\right)}{\left(2M|g|/h^{3/2}\right)} = M * O\left(\tau + h^{2}\right).
\end{aligned}$$

The constant M is bounded, in spite of $j \to \infty$, because of $j\tau < \infty$. Therefore, the convergence is proved.

2.2.3. The coupled KdV scheme

The numerical scheme for the system of the cKdV (4.1),(4.2) is also conditionally stable and convergent one. The proof for this scheme is close to that given before, but a bit bulky. We deal with a vector

$$U = \{\, \theta_1 \quad \theta_2 \quad \cdots \quad \theta_N \,\}^t$$

as a dependent variable instead of the simple variable θ in the case of one KdV. For this vector case, the norm used has the form

$$\|U\| = \left(\sum_{l=1}^{N} \sum_{i} \left|\theta_{l,i}\right|^{2} h\right)^{1/2}.$$

The conditions connecting time and space steps for this scheme also look similar, but with different constants

$$\max_{n}(|e_n|) * \frac{81 * \tau^3}{4 * h^{12}} * t_0 = O(1).$$

3. CHECKING THE NUMERICAL METHOD

The numerical method is tested by applying it to the Hirota-Satsuma system. Namely, the two parameters one-soliton explicit solution is used [21].

3.1. Analytic Solution (Explicit Formula) of Hirota-Satsuma System

Darboux transformation (DT) that accounts for a deep reduction for this specific HS case of cKdV is used [21] to obtain explicit solutions to the HS system. The Lax representation of the HS equations is based on the matrix 2×2 spectral problem of the second order. For this problem, the deep reduction scheme [21] is applied (with the help of the conserved bilinear-forms) and supports the constrains on the potential while the iterated DT are performed. The iterated DT in determinant form and the covariance of the bilinear forms with respect to DT under restrictions gives N soliton solutions of the HS system. We use the system HS to check that the scheme has the form

$$(\theta_1)_t - 0.25\,(\theta_1)_{3x} - 1.5\,(\theta_1)_x\,(\theta_1) + 3\,(\theta_2)_x\,(\theta_2) = 0, \tag{13a}$$

$$(\theta_2)_t + 0.5\,(\theta_2)_{3x} + 1.5\,(\theta_2)_x\,(\theta_1) = 0. \tag{13b}$$

This system has a two-parameters one soliton solution

$$\begin{aligned} \theta_1 &= \frac{-2m^2\left(-1 + d^2 + 2d\,\sin(\lambda_1) * \sinh(\lambda_2)\right)}{\left(d\cos(\lambda_1) + \cosh(\lambda_2)\right)^2}, \\ \theta_2 &= \frac{\left(2 + 2d^2\right)^{0.5} m^2}{\left(d\cos(\lambda_1) + \cosh(\lambda_2)\right)}, \\ \lambda_1 &= 0.5m^3 t + mx \qquad \text{and} \qquad \lambda_2 = 0.5m^3 t - mx, \end{aligned} \tag{14}$$

with real constants m, d. For small $|d|$, this solution is a smooth function, but for $|d| > 1$ poles appear. The following figures show some choice of m and d to show the effect of these two

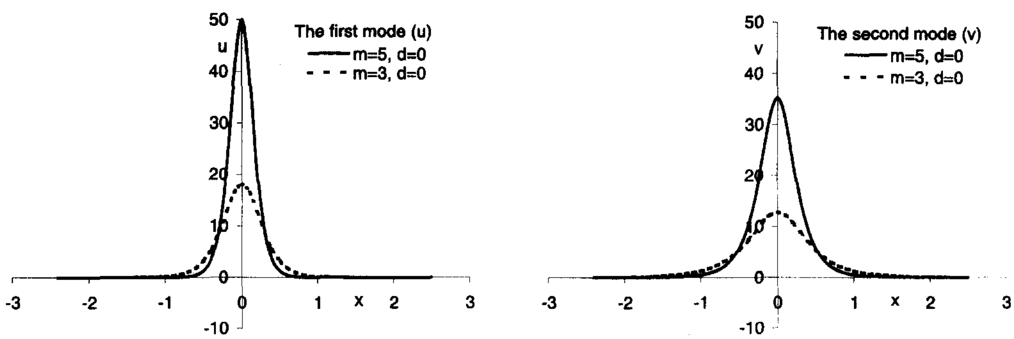


Figure 1. For a constant d, the amplitude is proportional to m while the wave width is inversely proportional to m.

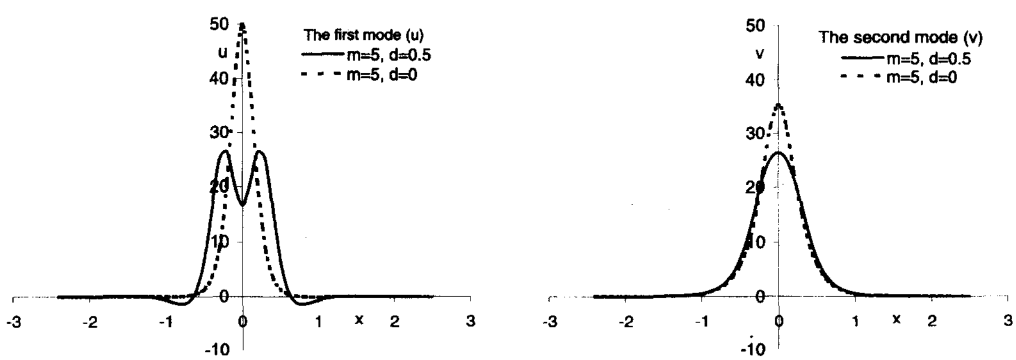


Figure 2. For the same m, d affects soliton shape, namely the first mode, while the amplitude of the second mode is inversely proportional to d.

parameters on the solution. Figure 1 shows that, for constant d, the amplitude is proportional to m while the wave width is inversely proportional to it. Figure 2 shows that, for the given m, d affects soliton shape, namely the first mode, while the amplitude of the second is inversely proportional to d.

3.2. Calculations by Numerical Scheme and Comparison Results

HS system (13) is solved numerically using scheme (4) with initial condition from formula (14) ($t = 0$), and the results are compared with the explicit solution. It is found that, keeping the restriction on the choice of τ and h and the relation between them, the initial wave modes amplitude affects the accuracy of the results. Also, the error decreases as the mesh is refined. Namely, smaller amplitude (of order one) gives better results as shown below in Figure 3. It gives the percentage error calculated as follows:

$$\%\ \text{Error} = \frac{|\text{Explicit solution} - \text{Numerical solution}|}{\text{Initial amplitude}}.$$

We relate the error to the initial amplitude to show the physical significance of the error.

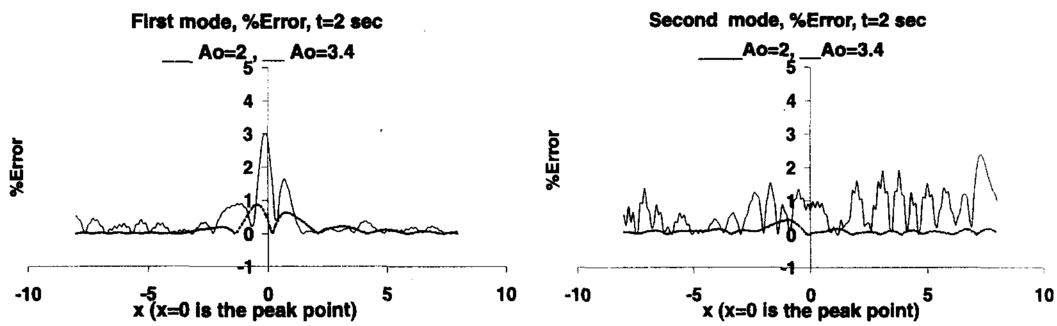


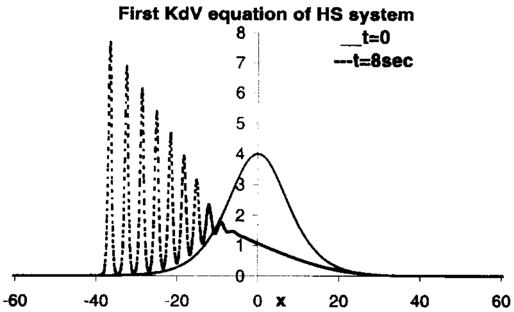
Figure 3. % error is proportional to the amplitude (A).

The plots show that the error is proportional to the amplitude (A), where as shown in the figure, the maximum relative error in the case ($A = 2$) is 1% while in the case ($A = 3.4$) is 3%. The reason may be due to the higher velocity in the larger one, and hence, more interactions impact. It also shows that the error increases near the peak points. The reason of these oscillations in plot appearance is that the numerical and analytical plots intersect over the space domain.

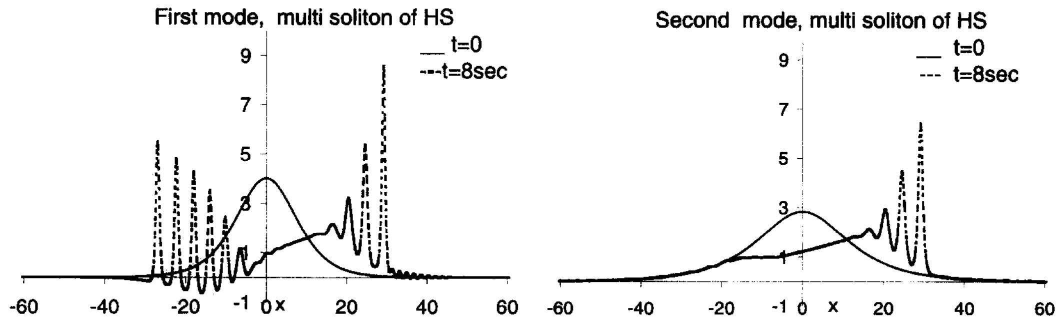
4. APPLYING THE SCHEME TO DIFFERENT APPLICATIONS

Stability analysis and checking performed on the scheme in the general cKdV equations give the ability of using this scheme to solve other problems for which analytical solutions have not been found. We first consider the multisoliton solution decay of the localized initial condition for the single KdV equation of the HS system (Figure 4a) and then for the complete HS system (Figure 4b). In both, we use the initial condition from formula (14) but with ten times the width and twice the amplitude.

The second mode affects (interacts) the first one, which results in the right direction soliton-like "tail" as shown in Figure 4b. We estimate, using the conservation low (derived below), the



(a) The multisoliton decaying of the isolated (first) KdV of HS system (13a).



(b) The multisoliton decaying of the complete HS system (13).

Figure 4.

expected number of solitons that are already obtained in the numerical solution as

$$\theta_1 \times (13.1) - 2\theta_2 \times (13.2)$$

$$\Rightarrow \frac{d}{dt}\left[0.5\theta_1^2 - \theta_2^2\right] + \frac{d}{dx}\left[-0.5\theta_1^3 - 0.25\theta_1\theta_{1xx} + 0.125\theta_{1x}^2 - \theta_2\theta_{2xx} + 0.5\theta_{2x}^2\right] = 0,$$

$$\theta_1 \text{ and } \theta_2 \to 0, \qquad \text{as } x \to \pm\infty,$$

$$\int_{\infty}^{-\infty} \left(0.5\theta_1^2 - \theta_2^2\right)\, dx = \text{const.}$$

Next we go to the solution of a nonintegrable HS system. The integrable HS system (13a,b) may be shifted to a "slightly" nonintegrable one by a small change of the dispersion constant of the first equation to have the new nonintegrable HS system

$$(\theta_1)_t - 0.2\,(\theta_1)_{3x} - 1.5\,(\theta_1)_x\,(\theta_1) + 3\,(\theta_2)_x\,(\theta_2) = 0, \tag{15a}$$

$$(\theta_2)_t + 0.5\,(\theta_2)_{3x} + 1.5\,(\theta_2)_x\,(\theta_1) = 0. \tag{15b}$$

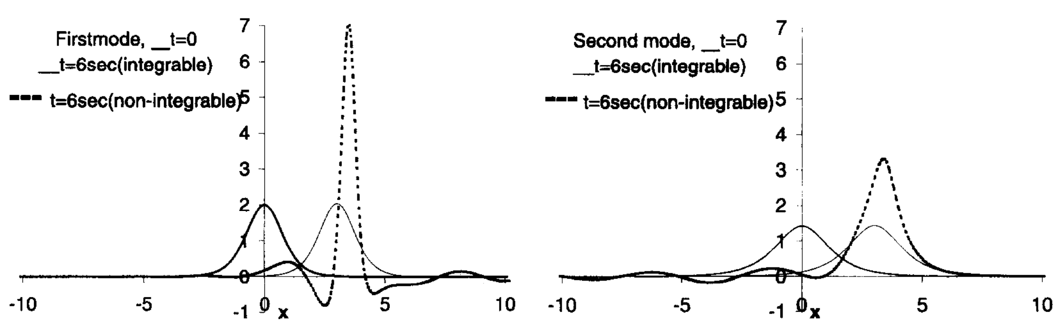


Figure 5. The numerical solution of integrable and slightly nonintegrable HS system.

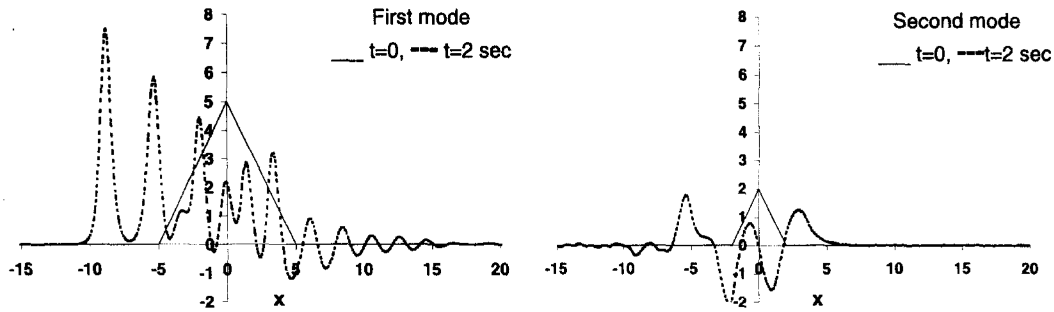


Figure 6. The numerical solution of nonsmooth initial condition for HS system (13).

Using our scheme with the initial condition from (14), we find that the scheme works satisfactorily (in the sense of convergence) even for the nonintegrable HS system as shown in Figure 5. The solution looks like a soliton one for small time.

Also, the solution using a nonsmooth initial condition for HS system (13) is shown in Figure 6.

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