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States of light via reducible quantization

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Multi-photon and coherent states of light are formulated in terms of a reducible representation of canonical commutation relations. Standard properties of such states are recovered as certain limiting cases. The new formalism leads to field operators and not operator-valued distributions. The example of radiation fields produced by a classical current shows an automatic regularization of the infrared divergence.

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I. INTRODUCTION

The present paper is a continuation of [1] where a new approach to nonrelativistic quantum optics was proposed. The main idea of the new program of field quantization outlined in [1] is to treat the frequencies of electromagnetic oscillators as observables (or their eigenvalues) and not as parameters. The most important implication of this single modification is a new representation of canonical commutation relations (CCR) which naturally occurs for field operators. The field operators are in this representation indeed operators and not operator-valued distributions and the formalism is much less singular than the traditional one.

The results discussed in [1] were encouraging. However, before one will be able to say anything really conclusive about the role of the new quantization paradigm for the issue of infinities, one has to solve many intermediate problems. The list of open questions involves quantization of fermions, Poincaré covariant formalism, properties of multi-particle and coherent states, a full perturbative treatment of quantum electrodynamical S-matrix, and gauge invariance.

The goal of the present paper is to give an explicit Poincaré covariant formulation of free Maxwell fields in the new representation, investigate the structure of multi-photon and coherent states of light, and properties of radiation fields in the simplest exactly solvable case. Some of the results are only outlined, for details the readers are refered to the preprints [2–4]. Ref. [3] is devoted to free Dirac electrons and [4] gives a preliminary analysis of interactions in perturbation theory.

II. REDUCIBLE QUANTIZATION

The mode quantization of free electromagnetic fields reduces to an appropriate choice of the map $f(\boldsymbol{k}, s) \mapsto a(\boldsymbol{k}, s)$ which replaces wave functions by operators. The idea of reducible quantization is to take $a(\boldsymbol{k}, s)$ as an operator analogous to the operator one finds for a harmonic oscillator whose frequency is indefinite. In the nonrelativistic case there is only one parameter ω and one finds [1] the reducible representation of CCR $a(\omega) = |\omega\rangle\langle\omega| \otimes a$ where a comes from the irreducible representation $[a, a^\dagger] = 1$ [5]. The electromagnetic field involves the 3-momentum $\boldsymbol{k}$ and two polarizations. Let a_s be the annihilation operators corresponding to an irreducible representation of the CCR algebra $[a_s, a_{s'}^\dagger] = \delta_{ss'}1$. We accordingly define the *one-oscillator* indefinite-frequency annihilation operators [7]

$$a(\boldsymbol{k}, s) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a_s \tag{1}$$

satisfying the commutation relations characteristic of a reducible representation of the CCR algebra

$$[a(\boldsymbol{k}, s), a(\boldsymbol{k}', s')^\dagger] = \delta_{ss'}\delta_\Gamma(\boldsymbol{k}, \boldsymbol{k}') \underbrace{|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1}_{I_{\boldsymbol{k}}} \tag{2}$$

Multi-oscillator fields are defined in terms of the reducible representation which can be constructed from the single-oscillator one as follows. Let A be any operator acting in the single-oscillator Hilbert space. If we denote by $A^{(n)} = \underbrace{I \otimes \ldots \otimes I}_{n-1} \otimes A \otimes \underbrace{I \otimes \ldots \otimes I}_{N-n}$ the extension of A to the N-oscillator Hilbert space, then

$$\underline{a}(\boldsymbol{k},s)=\frac{1}{\sqrt{N}}\sum_{n=1}^{N}a(\boldsymbol{k},s)^{(n)}. \tag{3}$$

Although the definition (3) implies that the "oscillators" which form the electromagnetic field are themselves of a bosonic type, one should not confuse them with photons. Photons, in this formalism, are quasi-particles corresponding to excitations of the oscillators. The operator $\underline{a}(\boldsymbol{k},s)$ removes one excitation from the ensemble, i.e. annihilates one photon.

The factor $1/\sqrt{N}$ plays an important role for the thermodynamic limit $N\to\infty$. Its appearance can be given a physical interpretation if one considers a two-level system coupled to reducibly quantized vector potential. Since the coupling is linear, the interaction term becomes analogous to the one expressing a two-level system interacting with N indefinite-frequency harmonic oscillators. As is widely known, a dual situation, representing an oscillator coupled to N two-level systems (i.e. the Dicke model [8]), also involves the factor $1/\sqrt{N}$ in exactly the same place. Formally, the factor comes from the assumption that the density N/V of two-level systems is constant, and is essential for the thermodynamic limit (cf. [9]).

The reducible representation (3) satisfies

$$[\underline{a}(\boldsymbol{k},s),\underline{a}(\boldsymbol{k}',s')^\dagger]=\delta_{ss'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{I}_{\boldsymbol{k}}. \tag{4}$$

The operator $\underline{I}_{\boldsymbol{k}}$ at the right-hand-side of (4) is given by $\underline{I}_{\boldsymbol{k}}=\frac{1}{N}\sum_{n=1}^{N}I_{\boldsymbol{k}}^{(n)}$. Its presence will influence orthogonality properties of multi-photon states, as we shall see later. The electromagnetic field tensor and four-potential operators read

$$\underline{F}_{ab}(x)=\int d\Gamma(\boldsymbol{k})\Big(e_{ab}(\boldsymbol{k})\Big(\underline{a}(\boldsymbol{k},-)e^{-ik\cdot x}+\underline{a}(\boldsymbol{k},+)^\dagger e^{ik\cdot x}\Big)+\bar{e}_{ab}(\boldsymbol{k})\Big(\underline{a}(\boldsymbol{k},-)^\dagger e^{ik\cdot x}+\underline{a}(\boldsymbol{k},+)e^{-ik\cdot x}\Big)\Big), \tag{5}$$

$$\underline{A}_a(x)=i\int d\Gamma(\boldsymbol{k})\Big(m_a(\boldsymbol{k})\Big(\underline{a}(\boldsymbol{k},+)e^{-ik\cdot x}-\underline{a}(\boldsymbol{k},-)^\dagger e^{ik\cdot x}\Big)+\bar{m}_a(\boldsymbol{k})\Big(\underline{a}(\boldsymbol{k},-)e^{-ik\cdot x}-\underline{a}(\boldsymbol{k},+)^\dagger e^{ik\cdot x}\Big)\Big). \tag{6}$$

In the Penrose-Rindler spinor notation [10] one finds $k_a=\pi_A\bar{\pi}_{A'}$, $e_{ab}=\varepsilon_{A'B'}\pi_A\pi_B$, $m_a=\omega_A\bar{\pi}_{A'}$, $\bar{m}_a=\pi_A\bar{\omega}_{A'}$, where $\omega_A\pi^A=1$. It is well known that field "operators" of the standard theory are in fact operator-valued distributions. The reducibly quantized fields are more regular, as we shall see later.

III. ACTION OF THE POINCARÉ GROUP ON FIELD OPERATORS

Denote, respectively, by Λ and y the $SL(2,C)$ and 4-translation parts of a Poincaré transformation [6] (Λ,y). We are interested in finding the representation of the group in terms of unitary similarity transformations, i.e.

$$\underline{a}(\boldsymbol{k},\pm)\mapsto e^{\pm 2i\Theta(\Lambda,\boldsymbol{k})}e^{ik\cdot y}\underline{a}(\boldsymbol{\Lambda^{-1}k},\pm)=\underline{U}^\dagger_{\Lambda,y}\underline{a}(\boldsymbol{k},\pm)\underline{U}_{\Lambda,y} \tag{7}$$

where $\Theta(\Lambda,\boldsymbol{k})$ is the Wigner phase. It is sufficient to find an appropriate representation at the one-oscillator level. Indeed, assume we have found $U_{\Lambda,y}$ satisfying

$$e^{\pm 2i\Theta(\Lambda,\boldsymbol{k})}e^{ik\cdot y}a(\boldsymbol{\Lambda^{-1}k},\pm)=U^\dagger_{\Lambda,y}a(\boldsymbol{k},\pm)U_{\Lambda,y}. \tag{8}$$

Then $\underline{U}_{\Lambda,y}=\underbrace{U_{\Lambda,y}\otimes\ldots\otimes U_{\Lambda,y}}_{N}$. The definition of four momentum at a single-oscillator level reads

$$P_a=\int d\Gamma(\boldsymbol{k})k_a|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes\frac{1}{2}\sum_s\Big(a_s^\dagger a_s+a_s\,a_s^\dagger\Big). \tag{9}$$

One immediately verifies that

$$e^{iP\cdot x}a(\boldsymbol{k},s)e^{-iP\cdot x}=a(\boldsymbol{k},s)e^{-ix\cdot k},\quad e^{iP\cdot x}a(\boldsymbol{k},s)^\dagger e^{-iP\cdot x}=a(\boldsymbol{k},s)^\dagger e^{ix\cdot k} \tag{10}$$

implying $U_{\boldsymbol{1},y}=e^{iy\cdot P}$. Consequently, the generator of four-translations corresponding to $\underline{U}_{\boldsymbol{1},y}=e^{iy\cdot\underline{P}}$ is $\underline{P}_a=\sum_{n=1}^{N}P_a^{(n)}$ and

$$e^{i\underline{P}\cdot x}\underline{a}(\boldsymbol{k},s)^{\dagger}e^{-i\underline{P}\cdot x}=\underline{a}(\boldsymbol{k},s)^{\dagger}e^{ix\cdot k},\quad e^{i\underline{P}\cdot x}\underline{a}(\boldsymbol{k},s)e^{-i\underline{P}\cdot x}=\underline{a}(\boldsymbol{k},s)e^{-ix\cdot k}. \tag{11}$$

The x-dependence of field operators can be introduced via $\underline{F}_{ab}(x)=e^{i\underline{P}\cdot x}\underline{F}_{ab}e^{-i\underline{P}\cdot x}$. Defining

$$U_{\Lambda,0}=\exp\Big(\sum_s 2is\int d\Gamma(\boldsymbol{k})\Theta(\Lambda,\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes a_s^{\dagger}a_s\Big)\Big(\int d\Gamma(\boldsymbol{p})|\boldsymbol{p}\rangle\langle\boldsymbol{\Lambda^{-1}p}|\otimes 1\Big), \tag{12}$$

we find (7) with $y=0$. The transformations of the field tensor are finally

$$\underline{U}_{\Lambda,0}^{\dagger}\underline{F}_{ab}(x)\underline{U}_{\Lambda,0}=\Lambda_a{}^c\Lambda_b{}^d\underline{F}_{cd}(\Lambda^{-1}x),\quad \underline{U}_{\boldsymbol{1},y}^{\dagger}\underline{F}_{ab}(x)\underline{U}_{\boldsymbol{1},y}=\underline{F}_{ab}(x-y). \tag{13}$$

The zero-energy part of $\underline{P}$ can be removed by a unitary transformation leading to a *vacuum picture* dynamics (cf. [2]). We will describe this in more detail after having discussed the properties of states.

IV. ACTION OF THE POINCARÉ GROUP ON STATES

The one-oscillator Hilbert space consists of functions f satisfying $\sum_{n_+,n_-=0}^{\infty}\int d\Gamma(\boldsymbol{k})|f(\boldsymbol{k},n_+,n_-)|^2<\infty$. The operator $U_{\Lambda,y}$ introduced in the previous section acts on states of a single oscillator by

$$|f\rangle\mapsto U_{\Lambda,y}|f\rangle=U_{\boldsymbol{1},y}U_{\Lambda,0}|f\rangle=\sum_{n_\pm}\int d\Gamma(\boldsymbol{k})f(\boldsymbol{\Lambda^{-1}k},n_+,n_-)e^{2i(n_+-n_-)\Theta(\Lambda,\boldsymbol{k})}e^{ik\cdot y(n_++n_-+1)}|\boldsymbol{k},n_+,n_-\rangle. \tag{14}$$

The Poincaré transformation of an arbitrary multi-oscillator state $|\underline{f}\rangle$ is $\underline{U}_{\Lambda,y}|\underline{f}\rangle=\underbrace{U_{\Lambda,y}\otimes\ldots\otimes U_{\Lambda,y}}_{N}|\underline{f}\rangle$. The form (14) is very similar to the zero-mass spin-1 representation, the difference being in the multiplier n_++n_-+1.

In what follows we will work in a "vacuum picture", i.e. with unitary transformations

$$f(\boldsymbol{k},n_+,n_-)\mapsto V_{\Lambda,y}f(\boldsymbol{k},n_+,n_-)=e^{i(n_++n_-)k\cdot y}e^{2i(n_+-n_-)\Theta(\Lambda,\boldsymbol{k})}f(\boldsymbol{\Lambda^{-1}k},n_+,n_-). \tag{15}$$

The transition $U_{\Lambda,y}\mapsto V_{\Lambda,y}=W_y^{\dagger}U_{\Lambda,y}$ is performed by means of the unitary transformation that commutes with reducible creation and annihilation operators.

V. VACUUM AND COHERENT STATES

Vacuum states are all the states which are annihilated by all annihilation operators. At the one-oscillator level these are the states of the form $|O\rangle=\int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})|\boldsymbol{k},0,0\rangle$. Even in the vacuum picture the vacuum states are not Poincaré invariant since $V_{\Lambda,y}O(\boldsymbol{k})=O(\boldsymbol{\Lambda^{-1}k})$ which means they transform as a 4-translation-invariant scalar field. We will often meet the expression $Z(\boldsymbol{k})=|O(\boldsymbol{k})|^2$ describing the probability density of the "zero modes".

We define an N-oscillator vacuum state as a tensor product of N copies of single-oscillator vacua,

$$|\underline{O}\rangle=\underbrace{|O\rangle\otimes\ldots\otimes|O\rangle}_{N}. \tag{16}$$

Vacuum may be regarded as a Bose-Einstein condensate of the ensemble of harmonic oscillators at zero temperature. Such a vacuum is simultaneously a particular case of a coherent state with $\alpha(\boldsymbol{k},s)=0$.

An analogue of the standard coherent (or "semiclassical") state is at the 1-oscillator level

$$|O_\alpha\rangle=\sum_{n_+,n_-}\int d\Gamma(\boldsymbol{k})O_\alpha(\boldsymbol{k},n_+,n_-)|\boldsymbol{k},n_+,n_-\rangle \tag{17}$$

where $O_\alpha(\boldsymbol{k},n_+,n_-)=\frac{1}{\sqrt{n_+!n_-!}}O(\boldsymbol{k})\alpha(\boldsymbol{k},+)^{n_+}\alpha(\boldsymbol{k},-)^{n_-}e^{-\sum_\pm|\alpha(\boldsymbol{k},\pm)|^2/2}$. Under (14) the vacuum-picture coherent-state wave function transforms by $V_{\Lambda,y}O_\alpha=O_{T_{\Lambda,y}\alpha}$ where $\alpha\mapsto T_{\Lambda,y}\alpha$ is the usual spin-1 massless unitary representation. Coherent states are related to the vacuum state via the displacement operator

$$\underline{\mathcal{D}}(\beta) = e^{\underline{a}(\beta)^\dagger - \underline{a}(\beta)}, \quad \underline{\mathcal{D}}(\beta)|\underline{O}_\alpha\rangle = |\underline{O}_{\alpha+\beta}\rangle, \tag{18}$$

$$\underline{\mathcal{D}}(\beta)^\dagger \underline{a}(\boldsymbol{k},s)\underline{\mathcal{D}}(\beta) = \underline{a}(\boldsymbol{k},s) + \beta(\boldsymbol{k},s)\underline{I}_{\boldsymbol{k}}, \quad \underline{\mathcal{D}}(\beta)^\dagger \underline{I}_{\boldsymbol{k}}\underline{\mathcal{D}}(\beta) = \underline{I}_{\boldsymbol{k}}, \tag{19}$$

as follows $\underline{\mathcal{D}}(\alpha)|\underline{O}\rangle = |\underline{O}_\alpha\rangle = |O_{\alpha_N}\rangle \otimes \ldots \otimes |O_{\alpha_N}\rangle$. Here $|O_{\alpha_N}\rangle$ is the 1-oscillator coherent state with $\alpha_N(\boldsymbol{k},s) = \alpha(\boldsymbol{k},s)/\sqrt{N}$. The appearance of $1/\sqrt{N}$ is of crucial importance for the question of statistics of excitations of multi-oscillator coherent states. Coherent-state averages of field operators

$$\langle \underline{O}_\alpha|^- \underline{\hat{F}}_{ab}(x)|\underline{O}_\alpha\rangle = \int d\Gamma(\boldsymbol{k}) e_{ab}(\boldsymbol{k}) Z(\boldsymbol{k}) \Big(\alpha(\boldsymbol{k},-)e^{-ik\cdot x} + \overline{\alpha(\boldsymbol{k},+)}e^{ik\cdot x}\Big) \tag{20}$$

are equivalent to classical electromagnetic fields. Let us note that (20) involves a "renormalized amplitude" $Z(\boldsymbol{k})\alpha(\boldsymbol{k},s) = |O(\boldsymbol{k})|^2\alpha(\boldsymbol{k},s)$ and not just $\alpha(\boldsymbol{k},s)$. The latter property is very characteristic for the reducible representation and has implications for infrared and ultraviolet regularization.

VI. NORMALIZED MULTI-PHOTON STATES

Consider the vector $\underline{a}(f)^\dagger|\underline{O}\rangle$ where $\underline{a}(f) = \sum_s \int d\Gamma(\boldsymbol{k})\overline{f(\boldsymbol{k},s)}\underline{a}(\boldsymbol{k},s)$. The form (16) of the vacuum state implies

$$\langle \underline{O}|\underline{a}(f)\underline{a}(g)^\dagger|\underline{O}\rangle = \sum_s \int d\Gamma(\boldsymbol{k}) Z(\boldsymbol{k})\overline{f(\boldsymbol{k},s)}g(\boldsymbol{k},s) = \langle fO|gO\rangle =: \langle f|g\rangle_Z. \tag{21}$$

fO denotes the pointlike product $fO(\boldsymbol{k},s) = O(\boldsymbol{k})f(\boldsymbol{k},s)$. Thinking of bases in the Hilbert space one can take functions f_i satisfying $\langle f_i|f_j\rangle_Z = \delta_{ij}$. The next theorem explains in what sense the orthogonality of multi-photon wave packets can be characterized by the same condition as for 1-photon states, i.e. in terms of $\langle f|g\rangle_Z$. Denote by $\sum_\sigma$ the sum over all the permutations of the set $\{1,\ldots,m\}$.

Theorem 1. Consider the vacuum state (16). Then

$$\lim_{N\to\infty} \langle \underline{O}|\underline{a}(f_1)\ldots\underline{a}(f_m)\underline{a}(g_1)^\dagger\ldots\underline{a}(g_{m'})^\dagger|\underline{O}\rangle = \delta_{mm'}\sum_\sigma \langle f_1|g_{\sigma(1)}\rangle_Z \ldots \langle f_m|g_{\sigma(m)}\rangle_Z$$

$$= \delta_{mm'}\sum_\sigma \sum_{s_1\ldots s_m} \int d\Gamma(\boldsymbol{k}_1)\ldots d\Gamma(\boldsymbol{k}_m) Z(\boldsymbol{k}_1)\ldots Z(\boldsymbol{k}_m)\overline{f_1(\boldsymbol{k}_1,s_1)}\ldots\overline{f_m(\boldsymbol{k}_m,s_m)}g_{\sigma(1)}(\boldsymbol{k}_1,s_1)\ldots g_{\sigma(m)}(\boldsymbol{k}_m,s_m)$$

Proof: For $m \neq m'$ the scalar product is zero, which is an immediate consequence of the fact that vacuum is annihilated by all annihilation operators and the right-hand-side of CCR is in the center of the algebra. So assume $m = m'$. The scalar product of two general unnormalized multi-photon states is

$$\begin{aligned}
&\langle \underline{O}|\underline{a}(f_1)\ldots\underline{a}(f_m)\underline{a}(g_1)^\dagger\ldots\underline{a}(g_m)^\dagger|\underline{O}\rangle \\
&= \sum_\sigma \sum_{s_1\ldots s_m} \int d\Gamma(\boldsymbol{k}_1)\ldots d\Gamma(\boldsymbol{k}_m)\overline{f_1(\boldsymbol{k}_1,s_1)}\ldots\overline{f_m(\boldsymbol{k}_m,s_m)}g_{\sigma(1)}(\boldsymbol{k}_1,s_1)\ldots g_{\sigma(m)}(\boldsymbol{k}_m,s_m)\langle \underline{O}|\underline{I}_{\boldsymbol{k}_1}\ldots\underline{I}_{\boldsymbol{k}_m}|\underline{O}\rangle \\
&= \sum_\sigma \sum_{s_1\ldots s_m} \int d\Gamma(\boldsymbol{k}_1)\ldots d\Gamma(\boldsymbol{k}_m)\frac{1}{N^m}\overline{f_1(\boldsymbol{k}_1,s_1)}\ldots\overline{f_m(\boldsymbol{k}_m,s_m)}g_{\sigma(1)}(\boldsymbol{k}_1,s_1)\ldots g_{\sigma(m)}(\boldsymbol{k}_m,s_m) \\
&\times\langle \underline{O}|\Big(I_{\boldsymbol{k}_1}\otimes\ldots\otimes I + \ldots + I\otimes\ldots\otimes I_{\boldsymbol{k}_1}\Big)\ldots\Big(I_{\boldsymbol{k}_m}\otimes\ldots\otimes I + \ldots + I\otimes\ldots\otimes I_{\boldsymbol{k}_m}\Big)|\underline{O}\rangle
\end{aligned} \tag{22}$$

Further analysis of (22) can be simplified by the following notation:

$$1_{k_j} = I_{\boldsymbol{k}_j}\otimes\ldots\otimes I; \quad 2_{k_j} = I\otimes I_{\boldsymbol{k}_j}\otimes\ldots\otimes I; \quad \ldots \quad N_{k_j} = I\otimes\ldots\otimes I_{\boldsymbol{k}_j}$$

with $j = 1,\ldots,m$; the sums-integrals $\sum_{s_j}\int d\Gamma(\boldsymbol{k}_j)$ are denoted by $\sum_{k_j}$. Then (22) can be written as

$$\sum_\sigma \sum_{k_1\ldots k_m} \overline{f_1(k_1)}\ldots\overline{f_m(k_m)}g_{\sigma(1)}(k_1)\ldots g_{\sigma(m)}(k_m)\frac{1}{N^m}\sum_{A\ldots Z=1}^{N} \underbrace{\langle O|\ldots\langle O|}_{N} A_{k_1}\ldots Z_{k_m}\underbrace{|O\rangle\ldots|O\rangle}_{N} \tag{23}$$

Since m is fixed and we are interested in the limit $N\to\infty$ we can assume that $N > m$. Each element of the sum over $A_{k_1}\ldots Z_{k_m}$ in (23) can be associated with a unique point $(A,\ldots,Z)$ in an m-dimensional lattice embedded in a cube with edges of length N.

Of particular interest are those points of the cube, the coordinates of which are all different. Let us denote the subset of such points by C_0. For $(A,\dots,Z)\in C_0$

$$\langle O|\dots\langle O|A_{k_1}\dots Z_{k_m}|O\rangle\dots|O\rangle = Z(\boldsymbol{k}_1)\dots Z(\boldsymbol{k}_m) \tag{24}$$

no matter what N one considers and what are the numerical components in $(A,\dots,Z)$. (This makes sense only for $N\geq m$; otherwise C_0 would be empty). Therefore each element of C_0 produces an identical contribution (24) to (23). Let us denote the number of points in C_0 by N_0.

The sum (23) can be now written as

$$\sum_\sigma\sum_{k_1\dots k_m}\overline{f_1(k_1)}\dots\overline{f_m(k_m)}g_{\sigma(1)}(k_1)\dots g_{\sigma(m)}(k_m)\mathcal{P}_0 Z(\boldsymbol{k}_1)\dots Z(\boldsymbol{k}_m)+$$
$$\sum_\sigma\sum_{k_1\dots k_m}\overline{f_1(k_1)}\dots\overline{f_m(k_m)}g_{\sigma(1)}(k_1)\dots g_{\sigma(m)}(k_m)\frac{1}{N^m}\sum_{(A\dots Z)\notin C_0}\underbrace{\langle O|\dots\langle O|}_{N}A_{k_1}\dots Z_{k_m}\underbrace{|O\rangle\dots|O\rangle}_{N}.$$

The coefficient $\mathcal{P}_0=\frac{N_0}{N^m}$ represents a probability of C_0 in the cube. The elements of the remaining sum over $(A\dots Z)\notin C_0$ can be also grouped into classes according to the values of $\langle O|\dots\langle O|A_{k_1}\dots Z_{k_m}|O\rangle\dots|O\rangle$. There are $m-1$ such different classes, each class has its associated probability $\mathcal{P}_j$, $0<j\leq m-1$, which will appear in the sum in an analogous role as $\mathcal{P}_0$. The proof is completed by the observation that

$$\lim_{N\to\infty}\mathcal{P}_0=1;\quad \lim_{N\to\infty}\mathcal{P}_j=0,\quad 0<j. \tag{25}$$

Indeed, the probabilities are unchanged if one rescales the cube to $[0,1]^m$. The probabilities are computed by means of an m-dimensional uniformly distributed measure. $N\to\infty$ corresponds to the continuum limit, and in this limit the sets of points of which at least two coordinates are equal are of m-dimensional measure zero. ■

Remark: The coherent-state displacement operator is given by the usual power series in multiphoton states. Since in our representation the multiphoton states behave in the thermodynamic limit as those of the usual Fock one, one expects that statistics of excitations of a coherent state is, for $N\to\infty$, Poissonian. The proof that this is indeed the case follows the lines similar to those of Theorem 1 and can be found in [2].

VII. FIELD OPERATORS ARE INDEED OPERATORS

Acting with the vector potential operator on a vacuum we obtain the vector

$$|\underline{A}_a(x)\rangle=\underline{A}_a(x)|\underline{O}\rangle=\frac{1}{\sqrt{N}}\Big(|A_a(x)\rangle\underbrace{|O\rangle\dots|O\rangle}_{N-1}+\dots+\underbrace{|O\rangle\dots|O\rangle}_{N-1}|A_a(x)\rangle\Big) \tag{26}$$

where

$$|A_a(x)\rangle=-i\int d\Gamma(\boldsymbol{k})e^{ik\cdot x}O(\boldsymbol{k})|\boldsymbol{k}\rangle\Big(m_a(\boldsymbol{k})a_-^\dagger|0,0\rangle+\bar{m}_a(\boldsymbol{k})a_+^\dagger|0,0\rangle\Big). \tag{27}$$

The positive definite scalar product $\langle\underline{A}_a(y)|(-g^{ab})|\underline{A}_b(x)\rangle=\langle A_a(y)|(-g^{ab})|A_b(x)\rangle=2\int d\Gamma(\boldsymbol{k})e^{ik\cdot(x-y)}Z(\boldsymbol{k})$ shows that there is no ultraviolet divergence at $x=y$ since $\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})=1$. It is easy to understand that the same property will hold also for general states. To see this let us write the single-oscillator field operator as a function of the operator $\hat{k}_a=\int d\Gamma(\boldsymbol{k})k_a|\boldsymbol{k}\rangle\langle\boldsymbol{k}|$, i.e.

$$A_a(x)=im_a(\hat{\boldsymbol{k}})\big(e^{-i\hat{k}\cdot x}\otimes a_+-e^{i\hat{k}\cdot x}\otimes a_-^\dagger\big)+i\bar{m}_a(\hat{\boldsymbol{k}})\big(e^{-i\hat{k}\cdot x}\otimes a_--e^{i\hat{k}\cdot x}\otimes a_+^\dagger\big). \tag{28}$$

The operators $m_a(\hat{\boldsymbol{k}})$ and $\bar{m}_a(\hat{\boldsymbol{k}})$ are functions of the operator $\hat{\boldsymbol{k}}$ and are defined in the standard way via the spectral theorem. The remaining operators (a_j, $a_j^\dagger$, and $e^{\pm i\hat{k}\cdot x}$) are also well behaved. Particularly striking is the fact that the *distribution* $\int d\Gamma(\boldsymbol{k})e^{ik\cdot x}$ is replaced by the *unitary* operator $e^{i\hat{k}\cdot x}=\int d\Gamma(\boldsymbol{k})e^{ik\cdot x}|\boldsymbol{k}\rangle\langle\boldsymbol{k}|$. The latter property is at the very heart of various regularities encountered in the reducible formalism.

VIII. RADIATION FIELDS VIA S MATRIX IN REDUCIBLE REPRESENTATION

It is widely known [11,12] that in the canonical theory the scattering matrix corresponding to radiation fields produced by a classical transverse current is given, up to a phase, by a coherent-state displacement operator $e^{-i\int d^4y J(y)\cdot A_{\rm in}(y)}$. One of the consequences of such an approach is the Poissonian statistics of photons emitted by classical currents. An unwanted by-product of the construction is the infrared catastrophe.

Let us assume that we deal with a classical transverse current $J_a(x)$ whose Fourier transform is $\tilde{J}_a(k) = \int d^4x e^{ik\cdot x} J_a(x)$. Transversality means here that $\tilde{J}_a(|\boldsymbol{k}|,\boldsymbol{k}) = \bar{m}_a(\boldsymbol{k})\tilde{J}_{10'}(|\boldsymbol{k}|,\boldsymbol{k}) + m_a(\boldsymbol{k})\tilde{J}_{01'}(|\boldsymbol{k}|,\boldsymbol{k})$. An interaction term representing a classical current minimally coupled to the electromagnetic field takes the form $H_{\rm int} = \int d^3x J(x)\cdot \underline{A}(x)$ and the resulting scattering matrix is $S = e^{i\hat{\phi}} e^{-i\int d^4y J(y)\cdot \underline{A}_{\rm in}(y)}$ with some phase $\hat{\phi}$ which belongs to the center of CCR. Application of S to the incoming field produces $\underline{A}_{a\rm out}(x) = S^\dagger \underline{A}_{a\rm in}(x) S$ and

$$\underline{A}_{a\rm rad}(x) = i\int d\Gamma(\boldsymbol{k}) \underline{I}_{\boldsymbol{k}} \Big(m_a\big(e^{-ik\cdot x} j(\boldsymbol{k},+) - e^{ik\cdot x}\overline{j(\boldsymbol{k},-)}\big) + \bar{m}_a\big(e^{-ik\cdot x} j(\boldsymbol{k},-) - e^{ik\cdot x}\overline{j(\boldsymbol{k},+)}\big)\Big),$$

where $m_a = m_a(\boldsymbol{k})$, and

$$\underline{a}(\boldsymbol{k},s)_{\rm out} = \underline{a}(\boldsymbol{k},s)_{\rm in} + j(\boldsymbol{k},s)\underline{I}_{\boldsymbol{k}} = \underline{\mathcal{D}}(j)^\dagger \underline{a}(\boldsymbol{k},s)_{\rm in} \underline{\mathcal{D}}(j). \tag{29}$$

Comparison of (29) with (19) reveals that the S matrix is in the reducible theory proportional to the displacement operator constructed by means of the *reducible* representation, i.e. $S = e^{i\hat{\phi}}\underline{\mathcal{D}}(j)$ where $\hat{\phi}$ is in the center of CCR. The average number of photons and the average four-momentum read

$$\langle n\rangle = \langle \underline{O}_j|\underline{n}|\underline{O}_j\rangle = \sum_s \int d\Gamma(\boldsymbol{k}) Z(\boldsymbol{k}) |j(\boldsymbol{k},s)|^2, \quad \langle P_a\rangle = \langle \underline{O}_j|\underline{P}_a|\underline{O}_j\rangle = \sum_s \int d\Gamma(\boldsymbol{k}) k_a Z(\boldsymbol{k})|j(\boldsymbol{k},s)|^2. \tag{30}$$

A comparison with the infrared-divergent Fock result $\langle n\rangle_{\rm Fock} = \sum_s \int d\Gamma(\boldsymbol{k})|j(\boldsymbol{k},s)|^2$ shows that the reducible framework may regularize the divergence if $Z(\boldsymbol{k}) \to 0$ with $\boldsymbol{k}\to 0$. The point is that this is indeed the case if the wave function vanishes at the boundary of the set of momenta. For massive particles this means vanishing at infinity. For massless particles the boundary contains also the origin $\boldsymbol{k}=0$. This is a consequence of the fact that the cases $k=0$ and $k\neq 0$, $k^2=0$, correspond to representations of the Poincaré group induced from $SL(2,C)$ and $E(2)$, respectively.

It is quite remarkable that the ultraviolet cut-offs discussed in [1] appeared automatically due to the same property of the formalism: The nontrivial structure of the vacuum state. In the case of ultraviolet and vacuum divergences the regularization is a consequence of square integrability of $O(\boldsymbol{k})$.

IX. FINAL REMARKS

Let us end the paper with a few remarks on canonical commutation relations for renormalized (for simplicity scalar) fields. It is known that the CCR $[a(\boldsymbol{k}), a(\boldsymbol{k}')^\dagger] = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$ holds in the standard formalism only for free fields. The physical fields involve CCR in a form $[a(\boldsymbol{k}), a(\boldsymbol{k}')^\dagger] = Z\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$ where Z is related to a renormalization constant and the cut-off $|\boldsymbol{k}| < \infty$ is employed. Therefore Z can be treated as a non-zero constant only in a certain set of momenta asociated with the cut-off. Alternatively, one can incorporate the cut-off by turning Z into a function, i.e.

$$[a(\boldsymbol{k}), a(\boldsymbol{k}')^\dagger] = Z(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{31}$$

and make the theory nonlocal. The problem with such a modification is that the action of the Poincaré group $a(\boldsymbol{k}) \mapsto U^\dagger_{\Lambda,y} a(\boldsymbol{k}) U_{\Lambda,y} = a(\boldsymbol{\Lambda^{-1}k})$ will influence only the left-hand-side of (31) and thus the theory will not be Poincaré covariant. The reducible framework is very close to the modification (31) but with a 'quantized' $\hat{Z}(\boldsymbol{k}) = I_{\boldsymbol{k}}$ satisfying $U^\dagger_{\Lambda,y} I_{\boldsymbol{k}} U_{\Lambda,y} = I_{\boldsymbol{\Lambda^{-1}k}}$. It is striking that all the results we have found for $N\to\infty$ may be summarized by the following rule: In the limit $N\to\infty$ the products of the form $I_{\boldsymbol{k}_1}\dots I_{\boldsymbol{k}_m}$ are replaced by $Z(\boldsymbol{k}_1)\dots Z(\boldsymbol{k}_m) = |O(\boldsymbol{k}_1)|^2\dots|O(\boldsymbol{k}_m)|^2$. In effect the reducible representation may be regarded as a covariant implementation of the regularization (31). Modification of the RHS of CCR makes our formalism nonlocal but in an unusual sense (to compare with other nonlocal theories cf. [14–20]). An extension to a full quantum electrodynamics is a subject of ongoing study. Some preliminary results can be found in [4].

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[4] M. Czachor, "Reducible representations of CAR and CCR with possible applications to field quantization", hep-th/0304144.
[5] Consider a charged particle rotating in a constant magnetic field. Treating the field as a parameter we obtain the frequency of the oscillation proportional to the magnitude of the field. Replacing the field by a field operator we obtain an operator frequency. The constant-field approximation implies that momenta and coordinates of the particle commute with the operator cyclotron frequency. In a more realistic case the frequency will be replaced by a frequency operator that does not commute with momenta. The example is generic in the sense that the parameters one finds in various models of harmonic oscillations typically depend on physical quantities which, at a more fundamental level, are quantized. The construction we give in this Letter may be regarded as a toy model, a first step towards a more realistic description where the operator frequency is not in the center of the algebra of field operators.
[6] In this paper when we speak of the Poincaré group we mean the semidirect product of 4-translations and $SL(2, C)$, i.e. the universal covering space of the Poincaré group.
[7] The representation differs from the one introduced in [1]. The difference is that here *two* CCR operators are used to describe the polarization degree of freedom, as opposed to the one operator used in [1]. The reason for such a modification becomes clear when one introduces reducible (anti)commutation relations for *massive* particles. In the massless case there is no essential difference between the two approaches. That two CCR operators should be employed in the context of polarizations was stressed by J. Naudts (private communication), compare the discussion in [13].
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