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Mean of continuous variables observable via measurement on a single qubit

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It is shown that the mean value of any observable with a bounded spectrum can be uniquely determined from binary statistics of the measurement performed on a *single-qubit* ancilla coupled to a given system. The corresponding positive operator-valued measure fully encodes the observable structure. The method is generalized to the case of distant-laboratory paradigm and is considered in the context of entanglement detection with few local measurements. The results are also discussed in the context of quantum programming.

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One of the serious problems of quantum-information theory [1–4] is the fragility of quantum entanglement. The latter is a basic feature of some quantum cryptographic schemes, quantum teleportation, quantum dense coding, and quantum computing [5].

However, before using entanglement, one has to be sure that it is really present in the system (see Ref. [6] for some paradoxes). In particular, it is important to be able to detect it in the distant-laboratory scenario, where two observers are far apart and have restricted access to the composed system they share (see, for instance, Ref. [7]). There are many methods checking whether or not there is entanglement in the system (see Refs. [3,8]). However, they require prior state reconstruction, i.e., full knowledge about the density matrix of the system. Recently, a new paradigm was introduced [9]. Assume that we do not know the state of the system at all. Can we detect the presence of entanglement at all without state reconstruction then? Can we estimate the entanglement measure? The answers have been given in a series of papers [9–11]. In particular, for two-qubit entanglement can be detected both qualitatively [9] and quantitatively [10] without state reconstruction. If partial information about the state is provided, then entanglement can be detected [11] in the distant-laboratory paradigm with a minimal number of local measurements of mean values of product observables.

The problem is that any measurement of the latter usually requires the estimation of more than one parameter. Even determining the mean value of a *single* [sic] observable A usually requires the estimation of *many* parameters, namely, the probabilities of outcomes of von Neumann measurement (see the *problem* later). The problem is especially striking in the continuous variables case [12] where any von Neumann measurement can be only approximate due to finite number of outcomes of any real experiments. In this context, we address a quite general question: *Is there any way to associate the mean value of a given quantum observable A with the experimental estimation of a single parameter?*

Here, by the estimation of a single parameter we mean the estimation of the probability of some single outcome in the easiest way: counting detector clicks corresponding to the outcome and dividing the resulting number by the number of all runs of experiments. The simplest example is the spin-polarization measurement along a given axis: to get the probability of being "up," we count up events and divide them by the number of all ("up" and "down") results.

Surprisingly, the answer for the question above is positive for *any bounded observable, no matter whether it involves continuous variables or not*. The nature of the associated effect seems to be quite fundamental, and it has not been known in quantum measurement theory so far.

It can be explained as follows. If apart form our system we have a *single qubit* and can control the system-qubit interaction, then there exists a general quantum measurement (positive operator-valued measure, POVM) with two outcomes such that the mean value of A can be immediately reconstructed from the POVM statistics. Because binary POVM corresponds to the estimation of a single parameter (see previous discussion), it happens that in the presented scheme the estimation of the mean value of a single observable does correspond to a single parameter. The mechanism of this effect can be roughly summarized with the statement that the observable has been *encoded* into the interaction (represented by POVM) between the system and the qubit ancilla.

Note that to get the above POVM statistics, we need many runs of the experiment, i.e., we need many copies of our system and many qubit ancillas (each of them coupled to a single copy of the system), but this is always needed in quantum mechanics where mean values are measured.

In context of the results of Ref. [11], we also pose a similar issue in case of product observables measured by distant observers. It happens again that two binary POVMs are sufficient but with data analysis refined to get apart from the marginal statistics also one correlation probability (such as in Bell-type experiments).

Let us note that as a by-product of other investigations, we have provided a partial positive answer to the above question [16] earlier. The idea was to encode any spinlike observable A into some state $\varrho_A=\alpha I+\beta A$ (that can be viewed as a kind of program) of the auxiliary system. Then after interaction of the ancilla with our system in the given state ϱ, the value $\eta=\mathrm{Tr}(\varrho_A\varrho)$ was reconstructed with the help of an incomplete (binary) measurement giving finally the mean value $(\eta-\alpha)\beta^{-1}$ of the observable A in the state ϱ. However, the scheme required complex resources: for any d-level system, it needed $2d$-level ancillas. Moreover, as discussed subsequently, it cannot be applied for the continuous variables case [12], though local quantum operations and classical communication (LOCC) scheme for local (product) observables is possible [17]. Here, we provide a solution for both single system and bipartite (LOCC) system scenario. This unified approach has further advantages: (i) requires minimal ancilla—just single qubits and (ii) is applicable for the continuous variables case under the only assumption of boundness of the observable. We also provide a motivation: many-parameter estimation in the typical von Neumann measurement.

As we have already mentioned, the LOCC scheme we provide here is especially important for the local detection of unknown or partially unknown entanglement. In particular, it provides additional justification to approaches from Ref. [11].

It is worth mentioning that recently one developed ideas of quantum computing with quantum data structure [9,16], quantum programmable interferometric networks [16], and programmable quantum gates [18]. In this context, we address a natural (open) question: what observable can be implemented as a kind of quantum program, and, if so, how to do it optimally and how to quantify this process.

The paper is organized as follows. First we pose the problem with the standard von Neumann measurement and we show how to solve it by encoding a given observable into the binary POVM. Then, we provide a similar result for the product observable in the LOCC paradigm. Finally, we briefly discuss the result, especially in context of the recently considered computing with quantum data structure and related issues.

The problem. Consider an arbitrary quantum observable $A=\Sigma_i \lambda_i |\psi_i\rangle\langle\psi_i|$. If it has more than two different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, $n>2$, then the usual procedure to get the mean value of A in the given state ϱ,

$$\langle A\rangle_\varrho = \mathrm{Tr}(A\varrho), \tag{1}$$

requires von Neumann measurement with n outcomes corresponding to n eigenvectors of the observable A: ψ_1, $\psi_2, \ldots, \psi_n$. The measurement relies on the estimation of $n-1$ parameters that are probabilities of outcomes $p_1 = \langle\psi_1|\varrho|\psi_1\rangle$, $p_2=\langle\psi_2|\varrho|\psi_2\rangle, \ldots,$ $p_{n-1}=\langle\psi_{n-1}|\varrho|\psi_{n-1}\rangle$ (the last parameter p_n can be inferred from the normalization condition). Finally, we multiply the probabilities by eigenvalues and calculate the sum $\Sigma_{i=1}^n p_i\lambda_i$ that is equivalent just to the mean value $\langle A\rangle_\varrho$ we were looking for. Clearly, if n is greater than 2, we need an estimation of more than one parameter in the sense that (apart from counting runs of our experiment) we have to count clicks corresponding to *more than one* outcome. Moreover, if the observable corresponds to the continuous variables case ($n=\infty$ above), then there is no way to measure it directly and any indirect measurement must be approximate.

We shall see, however, that one can overcome these disadvantages under two assumptions: (i) boundness of the observable spectrum and (ii) additional resource: well-controlled interaction with a single quantum bit.

Solution for a single observable. Let us first assume that the observable has the spectrum bounded and its lower and upper bounds correspond to a_{min} and a_{max}, respectively. Let us define the non-negative number $a_- \equiv \max[0,-a_{min}]$. Then the following operator $D=a_- I+A$ is positive ($D \geqslant 0$), i.e., has no negative eigenvalue. Now, we define the new operator, i.e., the Hermitian operator $D'=D/(a_+)$ ($a_+ \equiv a_- + a_{\max}$]), such that it satisfies the property[1] $0\leqslant D'\leqslant 1$ which means that all its eigenvalues belong to the interval $[0,1]$. Consider now the following operators:

$$V_0=\sqrt{D'}=\sqrt{(a_- I+A)/a_+},$$

$$V_1=\sqrt{I-V_0^\dagger V_0}. \tag{2}$$

They satisfy the following condition: $\Sigma_{i=0}^1 V_i^\dagger V_i = I$ so they represent the so-called generalized quantum measurement and can be easily implemented on our system. It has two outcomes $i=0,1$ with probabilities $p_0=\mathrm{Tr}(V_0^\dagger V_0\varrho)$, $p_1 = \mathrm{Tr}(V_1^\dagger V_1\varrho)=1-p_0$. Note that only the *single* parameter p_0 describes this binary statistics. Now, it is elementary to see that because of the Hermicity of V_0 (which means that $V_0=V_0^\dagger$), one has $p_0=\mathrm{Tr}[(a_- I+A)\varrho]/a_+$ and, finally, because of $\mathrm{Tr}(A\varrho)=\langle A\rangle_\varrho$ this leads to the main conclusion

$$\langle A\rangle_\varrho = a_+ p_0 - a_- . \tag{3}$$

Thus, we have reproduced the mean value of the arbitrary observable A with a bounded spectrum with the help of a single parameter p_0 coming from the binary generalized quantum measurement (POVM).

It is remarkable that the above POVM can be performed on the system if we only have one-qubit ancilla (additional physical system) and can control the interaction between our system and the ancilla. In fact, this is all what a binary POVM requires ([13], see the Appendix of Ref. [7] for tutorial review). Indeed, we prepare our ancilla qubit in the pure state $|0\rangle$. Then, we subject our joint system (initially in a state $\varrho\otimes|0\rangle\langle 0|$) to unitary evolution U that leads to an interaction between our system and the ancilla (for definition of evolution U, realizing a given POVM, see Ref. [7]). Finally, we measure observable σ_z on our ancilla. If we get the result up (ancilla state unchanged i.e., remains in initial $|0\rangle$), this corresponds to the result $i=0$, and if we get the result down (ancilla state changed to $|1\rangle$) this corresponds to the result $i=1$; both occurring with probabilities p_0, p_1 defined above. This has a similarity to the scheme of a universal quantum estimator allowing one to detect nonlinear state functions [16], where one finally measures a single qubit to get the output of the measurement. There is a difference, however. Indeed, while there the mean value of the non-Hermitian unitary "shift" operator is estimated, here we have the *Hermitian* operator structure which is built in the POVM scheme in a more complex way. Still, it is interesting to perform a more detailed comparison of the two schemes.

Note that the above scheme allows one to detect the mean value of the non-Hermitian operator X defined by $\langle X\rangle_\varrho$ with the help of decomposing X into Hermitian and anti-Hermitian parts (cf. Ref. [14]) and detecting the corresponding observables with the help of two binary POVMs.

[1]We say that $A\geqslant B$, if for all Ψ one has $\langle\Psi|A-B|\Psi\rangle\geqslant 0$.

Product observables and the distant-laboratary paradigm. Suppose now that Alice and Bob are in the distant-laboratary paradigm, i.e., they are far apart and they share some bipartite quantum state ϱ_{AB}. This is similar to quantum teleportation process where they shared a single state (here, we allow ϱ to be mixed). In such a case, Alice and Bob are allowed to perform local operations (LO) and communicate classically (CC). Suppose now that they want to detect the mean value of some entanglement witness $W = \Sigma_{k=1}^{m} A_k \otimes B_k$ with its structure and number chosen properly (see Ref. [11]). Because of LOCC restrictions, this can be achieved only by the measurement of local measurements and exchange of information. Usually, it is done as in standard Bell inequalities (for similarity of entanglement witnesses formalism to Bell inequalities theory, see Ref. [15]): namely, this corresponds to local measurements of observables A_k, B_k (for each fixed k), but by keeping the record of results and finally establishing the mean value from joint statistics. However, there are more outcomes, in general, so again there is a question whether we can reduce the above scheme to binary experiments. The answer is ''yes,'' though the solution is not so simple as it was before. Suppose that Alice and Bob want to measure the mean value

$$\langle A \otimes B \rangle_{\varrho_{AB}} = \mathrm{Tr}(A \otimes B \varrho_{AB}) \tag{4}$$

of product observable $A \otimes B$ on the shared state ϱ_{AB}. Then they should perform local POVMs corresponding to local observables as defined in the preceding section, but they should use the data in a more sophisticated way. Let Alice POVM be $\{V_0, V_1\}$ (as before), while by Bob's POVM we denote $\{W_0, W_1\}$. They have pairs of possible local outcomes $i_A, i_B = 0,1$, where i_A (i_B) corresponds to the Alice (Bob) outcome. Then, performing measurements on their ancillas they should not only estimate parameters $p_0 = \mathrm{Tr}(W_0^\dagger W_0 \varrho_A)$, $q_0 = \mathrm{Tr}(W_0^\dagger W_0 \varrho_B)$ which correspond to normalized numbers of outcomes $i_A = 0$, $i_B = 0$, respectively. In addition, they should also communicate classically and count all cases when they get the results $i_A = i_B = 0$ correlated, i.e., coming from the copy of the state ϱ_{AB}. Normalizing the resulting number of the cases, i.e., dividing it by the number of all measurements by which they get the joint correlation probability

$$p_{00} = \mathrm{Tr}(V_0 \otimes W_0 \varrho_{AB}) \tag{5}$$

of getting the same outcome $i_A = i_B = 0$ on both sides from the same copy of the state.

The above process is equivalent to the estimation of mean values $\langle \sigma_z^{(A)} \rangle$, $\langle \sigma_z^{(B)} \rangle$, and $\langle \sigma_z^{(A)} \otimes \sigma_z^{B} \rangle$ on Alice and Bob local ancillas that were needed to implement the POVM. Thus, the process is virtually identical to what happens in the usual Bell-type inequality on two spin-$\frac{1}{2}$ particles where marginal and correlation probabilities are also determined. Summarizing, Alice and Bob need to determine probabilities p_0, q_0, and p_{00} of standard Pauli σ_z measurements on their ancillas. It is easy to see that from the probabilities they easily get the needed mean value as follows:

$$\langle A \otimes B \rangle_{\varrho_{AB}} = a_+ b_+ p_{00} + a_- b_- - [a_+ b_- p_0 + a_- b_+ q_0], \tag{6}$$

where $b_\pm$ are defined with respect to observable B in full analogy to $a_\pm$. Thus, again we have reduced the LOCC measurement of $A \otimes B$ to two binary POVMs with more careful data analysis, leading not only to binary marginal distributions (determined by probabilities p_0, q_0) but also to correlation probability p_{00}. Finally, let us note that the above reasoning can be generalized to the multipartite LOCC scheme. Then, only the proper hierarchy of correlation probabilities must be taken into account.

Applications to Bell inequalities. Note that, using the above formalism, any Bell inequality involving *arbitrary* bounded observables can be formally transformed to binary inequality that has formally ''Bell-like'' form: namely, it involves joint probabilities of binary events (such as p_0, q_0, p_{00}, and their multipartite analogs). However, the new Bell-like inequality, as it is, assumes validity of quantum mechanics (quantum interaction corresponding to local binary POVMs). As such, it does not represent the legitimate Bell inequality because it is *not independent of quantum formalism*. The question whether and when it is possible to overcome this difficulty will be considered elsewhere. It seems that the new inequalities might serve as an experimental test supporting (may be as a kind of preliminary stage) the fully detailed experimental tests of original Bell inequalities.

Discussion and conclusions. We have discussed the problem of whether measurement of a single observable with many eigenvalues can be restricted to the estimation of a single parameter. We have shown that it is always possible if (i) the observable is bounded, i.e., has upper and lower bounds on its spectrum and (ii) one has a well-controlled interaction with the single-qubit ancilla. We have constructed the corresponding POVM and pointed out that it can be achieved with only one additional quantum bit: namely, the estimated parameter corresponds to the probability of getting one outcome out of two that are possible in the measurement of Pauli operator σ_z on the single-qubit ancilla.

We have also considered the issue of detecting partially known entanglement with a minimal number of estimated parameters in context of Ref. [11]. In this case, it happens that the number of local observables involved in the measurement is equal to the corresponding binary POVMs that can supersede them. The result of POVMs, however, should be used in a more detailed way to get not only marginal (single parameter) binary statistics $\{p_0, 1-p_0\}$ ($\{q_0, 1-q_0\}$) on Alice (Bob) side, but also join correlation probability p_{00}. It can be generalized to multipartite systems and leads to the compression of usual Bell inequalities into ''binary Bell-like inequalities'' involving only joint probabilities of binary events.

Let us observe that in the continuous variables (CV) case, the measurement of a general observable is impossible—due to infinite number of outcomes, one can only measure some approximated (''digitized'') observable instead of the original one. The present binary POVM method seems to be *the only one* that provides the mean value of the observable itself rather than its approximation. The present result has some similarity to the recent interferometric method [16] where final estimation comes from the measurement of Pauli matrix σ_z. However, its fundamental difference can be seen easily when one realizes that the interferometric approach by no means can work for infinite-dimensional scenario called the CV case. The observable is there encoded by affine transformation $A \rightarrow \varrho_A = \alpha I + \beta A$, where I stands for identity operator. Hence, for CV, ϱ_A is no longer a quantum state (as the interferometric method of Ref. [16] requires) because it has no finite trace. There are no problems like that for the present method. Even the presence of a square root in formulas defining V_0, V_1 does not mean that the discussed difference is equivalent to that between probabilities and amplitudes in quantum theory. There is a deeper reason: in the present method, the observable A is encoded directly *in global dynamics* (ancilla-system interaction Hamiltonan that can be inferred from the POVM) rather than ''programmed'' into the ''static'' ancilla as it was proposed in Ref. [16].

There is, however, an important point that links the present approach with that of Ref. [16]. Let us recall that some kind of ''quantum programs'' that implement some physical observable in the physical system (ancilla) has been already presented in the previous approach [16]. Moreover, there is a general idea of quantum programming with quantum data structure [9,16] and a systematic way of quantum gates programming [18].

In the above context, an intriguing question arises naturally: is it possible to implement a given observable as a kind of program, and if so, what is the most optimal way to do it? From the present analysis, we already know that some CV observables cannot be implemented as a sort of program (state of the ancilla). But one can imagine the scenario where observable parameters are ''split'' into the programmable part and the one that is nonprogrammable but can be encoded into dynamics. In this context, one would need measures that would quantify both parts. It seems that by characterizing the second part, the entangling power concept can be important [19] as well as quantum gates programming [18] and gates cost [20]. Also, in the case of continuous variables, the concept of *both classical and quantum computability* of observable parameters will have to be taken into account.

Finally, it may be interesting to consider the application of the present result in context of Bell inequality tests for continuous variables systems [21].

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