

# Local properties of the solution set of the operator equation in Banach spaces in a neighbourhood of a bifurcation point

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**Abstract:** In this work we study the problem of the existence of bifurcation in the solution set of the equation  $F(x, \lambda) = 0$ , where  $F: X \times R^k \rightarrow Y$  is a  $C^2$ -smooth operator,  $X$  and  $Y$  are Banach spaces such that  $X \subset Y$ . Moreover, there is given a scalar product  $\langle \cdot, \cdot \rangle: Y \times Y \rightarrow R^1$  that is continuous with respect to the norms in  $X$  and  $Y$ . We show that under some conditions there is bifurcation at a point  $(0, \lambda_0) \in X \times R^k$  and we describe the solution set of the studied equation in a small neighbourhood of this point.

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## 1 Introduction

Let  $X$  and  $Y$  be real Banach spaces and  $F: X \times R^k \rightarrow Y$  be a continuous map. Suppose that the equation

$$F(x, \lambda) = 0, \tag{1}$$

where  $x \in X$  and  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in R^k$ , possesses the trivial family of solutions

$$\Lambda = \{(0, \lambda) \in X \times R^k : \lambda \in R^k\}.$$

A point  $(x, \lambda)$  such that  $F(x, \lambda) = 0$  and  $x \neq 0$  is called a nontrivial solution of (1). Bifurcation theory is concerned in part with the existence of nontrivial solutions of (1)

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in a small neighbourhood of  $\Lambda$ . A point  $(0, \lambda_0) \in \Lambda$  is called a bifurcation point of (1) if every neighbourhood of  $(0, \lambda_0)$  contains a nontrivial solution of (1).

Methods of bifurcation theory are often applied in mathematical physics. Let us mention some applications to mechanics of elastic constructions and hydromechanics. In [3] the buckling of a thin elastic plate subject to arbitrary forces and stresses along its boundary is studied by the use of a perturbation theory and a variational method. In [6] to describe a deformation of the minimal interface of two fluids in a vertical tube in a gravitational field one applies a method based on the Crandall-Rabinowitz bifurcation theorem and representation theory. In [9] the buckling of a thin elastic rectangular plate simply supported on sides is studied numerically. In [14] the forms of equilibrium of a thin elastic circular plate lying on an elastic foundation and simply supported along its boundary are investigated via a finite-dimensional reduction and the Krasnosielski bifurcation theorem. Finally, in [16] the buckling of a homogeneous finite beam clamped at the edges to an elastic foundation is studied by a method of a key function due to Sapronov.

Assume that  $F$  is  $C^1$ -smooth. For every  $\lambda \in R^k$ , let  $F'_x(0, \lambda): X \rightarrow Y$  denote the Fréchet derivative of  $F$  with respect to  $x$  at  $(0, \lambda)$ . Let  $N(\lambda) = \ker F'_x(0, \lambda)$  and  $R(\lambda) = \text{im } F'_x(0, \lambda)$ . It is easily seen that if  $F'_x(0, \lambda_0): X \rightarrow Y$  is a Fredholm operator of index zero then a necessary condition for  $(0, \lambda_0)$  to be a bifurcation point of (1) is

$$\dim N(\lambda_0) > 0.$$

In this paper we investigate bifurcation at  $(0, \lambda_0)$  when  $X$  is a linear subspace of  $Y$ , there is given a scalar product  $\langle \cdot, \cdot \rangle: Y \times Y \rightarrow R^1$  that is continuous with respect to the norms in  $X$  and  $Y$ , and  $F$  is a  $C^p$ -smooth map ( $p \geq 2$ ) that satisfies the following conditions:

- (I<sub>1</sub>)  $F(0, \lambda) = 0$  for every  $\lambda \in R^k$ ,
- (I<sub>2</sub>)  $\dim N(\lambda_0) = 1$ ,
- (I<sub>3</sub>)  $N(\lambda_0) \perp R(\lambda_0)$ ,
- (I<sub>4</sub>)  $F'_x(0, \lambda_0): X \rightarrow Y$  is a Fredholm operator of index 0.

Our aim is to prove a theorem on bifurcation at  $(0, \lambda_0)$  and a local structure of a solution set of equation (1) in a neighbourhood of a bifurcation point. We apply a kind of finite-dimensional reduction of Liapunov-Schmidt type and the implicit function theorem. We are motivated by applications in mathematical physics [6], [14], [16] in which the problems under considerations (see above) are described by (1) with  $F$  that satisfies (I<sub>1</sub>)–(I<sub>4</sub>) and is a variational gradient. The main results of this work are Theorem 3.7 and its variational version: Conclusion 3.10. Theorem 3.7 is an analogue of the Crandall-Rabinowitz bifurcation theorem (see [17], [21]). However, our theorem is formulated in terms of a finite-dimensional reduction and in a variational case it seems to be easier to apply. Conclusion 3.10 is well adapted to a class of nonlinear problems of elasticity described by the von Kármán equations with one or a few parameters (see [4], [15], [16]) in the case when the linearization space is one-dimensional. An example is given in Section 4.

The paper is divided into four sections. In Section 2 we introduce some notions and we briefly sketch a scheme of finite-dimensional reduction. Section 3 is devoted to the study of bifurcation and local properties of the solution set of (1) near a bifurcation point. In Section 4 some applications of our results are indicated.

In practice it suffices to suppose that  $F$  is defined in a neighbourhood of  $(0, \lambda_0)$  in  $X \times R^k$ , but we want to omit inessential details.

## 2 Finite-dimensional reduction

In this section we describe a kind of a finite-dimensional reduction of the Liapunov-Schmidt type. The scheme we present is adapted from [21] ( see also [10], [11], [17], [20]).

From now on we assume that  $X \subset Y$  are real Banach spaces with a scalar product  $\langle \cdot, \cdot \rangle: Y \times Y \rightarrow R^1$  that is continuous with respect to the norms in  $X$  and  $Y$ . The norms in  $X$  and  $Y$  can be defined independently of the scalar product  $\langle \cdot, \cdot \rangle$ , and the norm in  $X$  does not have to be induced by the norm in  $Y$ . In particular,  $X$  and  $Y$  with  $\langle \cdot, \cdot \rangle$  may be Hilbert spaces. Let  $F: X \times R^k \rightarrow Y$  be a  $C^p$ -smooth map, where  $p \geq 1$ , satisfying conditions:  $(I_1)$ ,  $(I_3)$ ,  $(I_4)$  and  $(I'_2)$   $\dim N(\lambda_0) = n \neq 0$ .

The aim is to show that under the above assumptions the problem of bifurcation for equation (1) at the point  $(0, \lambda_0) \in X \times R^k$  is reducible to the problem of bifurcation for the equation  $\varphi(\xi, \lambda) = 0$  with a certain map  $\varphi: S \subset R^n \times R^k \rightarrow R^n$  at the point  $(0, \lambda_0) \in R^n \times R^k$ . The reader may find the proofs of the propositions given below in [13] and [15].

**Proposition 2.1.** For every  $\lambda \in R^k$  the following equality holds:

$$Y = R(\lambda) \oplus N(\lambda). \tag{2}$$

Let  $G: X \times R^n \times R^k \rightarrow Y$  be a map defined by

$$G(x, \xi, \lambda) = F(x, \lambda) + \sum_{i=1}^n (\xi_i - \langle x, e_i \rangle) e_i, \tag{3}$$

where  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$  and  $\{e_1, e_2, \dots, e_n\}$  is a fixed orthonormal base of  $N(\lambda_0)$ .

**Proposition 2.2.** The operator  $G'_x(0, 0, \lambda_0): X \rightarrow Y$  is an isomorphism.

It is easily seen that  $G$  is  $C^p$ -smooth. From the implicit function theorem it follows that there exist two open sets  $U \subset X$  and  $S \subset R^n \times R^k$  such that  $0 \in U$ ,  $(0, \lambda_0) \in S$  and the solution set of the equation

$$G(x, \xi, \lambda) = 0 \tag{4}$$

in  $U \times S$  is a graph of a certain  $C^p$ -smooth function  $x: S \rightarrow U$  such that  $x(0, \lambda_0) = 0$ . Moreover, it is obvious that  $x(0, \lambda) = 0$  for all  $(0, \lambda) \in S$ , because  $G(0, 0, \lambda) = 0$ . Let

$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n): S \rightarrow R^n$  be defined by coordinates as follows:

$$\varphi_i(\xi, \lambda) = \xi_i - \langle x(\xi, \lambda), e_i \rangle, \quad i = 1, \dots, n. \quad (5)$$

**Proposition 2.3.**  $(0, \lambda_0) \in \Lambda$  is a bifurcation point of equation (1) if and only if  $(0, \lambda_0) \in S$  is a bifurcation point of equation

$$\varphi(\xi, \lambda) = 0. \quad (6)$$

### 3 Theorem on bifurcation

In this section our main results are stated and proved.

Let  $F: X \times R^k \rightarrow Y$  be a  $C^p$ -smooth map,  $p \geq 2$ , satisfying conditions  $(I_1)$ – $(I_4)$  (see p. 562). Fix  $e \in N(\lambda_0)$  such that  $\langle e, e \rangle = 1$  and denote  $\lambda_0 = (\lambda_{01}, \lambda_{02}, \dots, \lambda_{0k})$ . We will describe the solution set of (1) in terms of the finite-dimensional reduction. Notice that now in the formulas of maps  $G$  and  $\varphi$  there are  $n = 1$  and  $e_1 = e$ . Differentiating the equality  $G(x(\xi, \lambda), \xi, \lambda) = 0$  with respect to  $\xi$  at  $(0, \lambda_0)$  we obtain

$$F'_x(0, \lambda_0)x'_\xi(0, \lambda_0) + (1 - \langle x'_\xi(0, \lambda_0), e \rangle)e = 0.$$

From this and  $(I_3)$  it follows that  $x'_\xi(0, \lambda_0) = e$ .

**Theorem 3.1.** There exist open sets  $V_0 \subset X$  and  $V \subset R^k$  such that  $(0, \lambda_0) \in V_0 \times V$  and for every  $(x, \lambda) \in V_0 \times V$  we have  $F(x, \lambda) = 0$  if and only if  $(\langle x, e \rangle, \lambda) \in S$  and  $x = x(\langle x, e \rangle, \lambda)$ .

**Proof 3.2.** Suppose contrary to our claim, that there are no open sets  $V_0 \subset X$  and  $V \subset R^k$  with the above properties. Then for every  $n \in N$  there exists  $(x_n, \lambda_n) \in X \times R^k$  such that  $\|x_n\|_X \leq \frac{1}{n}$ ,  $|\lambda_n - \lambda_0| \leq \frac{1}{n}$  and one of the following conditions is satisfied:

1.  $F(x_n, \lambda_n) = 0$  and  $(\langle x_n, e \rangle, \lambda_n) \notin S$ ,
2.  $F(x_n, \lambda_n) = 0$ ,  $(\langle x_n, e \rangle, \lambda_n) \in S$  and  $x_n \neq x(\langle x_n, e \rangle, \lambda_n)$ ,
3.  $F(x_n, \lambda_n) \neq 0$ ,  $(\langle x_n, e \rangle, \lambda_n) \in S$  and  $x_n = x(\langle x_n, e \rangle, \lambda_n)$ .

If  $(\langle x_n, e \rangle, \lambda_n) \in S$  and  $x_n = x(\langle x_n, e \rangle, \lambda_n)$  then  $F(x_n, \lambda_n) = F(x(\langle x_n, e \rangle, \lambda_n), \lambda_n) + (\langle x_n, e \rangle - \langle x(\langle x_n, e \rangle, \lambda_n), e \rangle)e = G(x(\langle x_n, e \rangle, \lambda_n), \langle x_n, e \rangle, \lambda_n) = 0$ .

Since  $x_n \rightarrow 0$  in  $X$ , there exists  $n_0 \in N$  such that  $x_n \in U$  for every  $n \geq n_0$ . If for some  $n \geq n_0$  we have  $F(x_n, \lambda_n) = 0$  and  $(\langle x_n, e \rangle, \lambda_n) \in S$  then  $0 = F(x_n, \lambda_n) + (\langle x_n, e \rangle - \langle x_n, e \rangle)e = G(x_n, \langle x_n, e \rangle, \lambda_n)$ , and so  $x_n = x(\langle x_n, e \rangle, \lambda_n)$ .

Since  $(\langle x_n, e \rangle, \lambda_n) \rightarrow (0, \lambda_0) \in S$  there exists  $n_1 \in N$  such that  $(\langle x_n, e \rangle, \lambda_n) \in S$  for every  $n \geq n_1$  — a contradiction.

The equality  $\langle G(x(\xi, \lambda), \xi, \lambda), e \rangle = 0$  implies

$$\varphi(\xi, \lambda) = -\langle F(x(\xi, \lambda), \lambda), e \rangle. \quad (7)$$

From (7) we obtain

$$\varphi'_\xi(\xi, \lambda) = -\langle F'_x(x(\xi, \lambda), \lambda)x'_\xi(\xi, \lambda), e \rangle,$$

and hence  $\varphi'_\xi(0, \lambda_0) = 0$ . Moreover, since  $\varphi(0, \lambda) = 0$  for every  $(0, \lambda) \in S$  we have  $\varphi_{\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m}}^{(m)}(0, \lambda_0) = 0$  for all  $i_1, i_2, \dots, i_m \in \{1, 2, \dots, k\}$  and  $m \in \mathbb{N}$ . In order to get our main result we have to assume that there is  $i \in \{1, 2, \dots, k\}$  such that  $\varphi''_{\xi \lambda_i}(0, \lambda_0) \neq 0$ . There is no loss of generality if we assume

$$(I_5) \quad \varphi''_{\xi \lambda_k}(0, \lambda_0) \neq 0.$$

From now on, if  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k-1}, \lambda_k) \in R^k$ ,  $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_{k-1}) \in R^{k-1}$  we will write  $\lambda = (\lambda', \lambda_k)$ .

**Proposition 3.3.** There exist open sets  $\Omega_0 \subset R^1 \times R^{k-1}$  and  $\Omega \subset R^1$  such that  $(0, \lambda'_0) \in \Omega_0$ ,  $\lambda_{0k} \in \Omega$  and there exists a  $C^p$ -smooth map  $f: \Omega_0 \rightarrow \Omega$  that satisfies the following conditions:

- (1)  $f(0, \lambda'_0) = \lambda_{0k}$ ,
- (2) for every  $(\xi, \lambda') \in \Omega_0$  and  $\lambda_k \in \Omega$  we have  $\varphi(\xi, \lambda', \lambda_k) = 0$  if and only if  $\xi = 0$  or  $\lambda_k = f(\xi, \lambda')$ .

**Proof 3.4.** Let  $\psi: S \rightarrow R^1$  be a function defined by

$$\psi(\xi, \lambda) = \int_0^1 \varphi'_\xi(t\xi, \lambda) dt. \tag{8}$$

Observe that we have

$$\varphi(\xi, \lambda) = \xi \psi(\xi, \lambda). \tag{9}$$

Hence  $\varphi(\xi, \lambda) = 0$  only if  $\xi = 0$  or  $\psi(\xi, \lambda) = 0$ . From (8) and  $(I_5)$  it follows that  $\psi(0, \lambda_0) = \varphi'_\xi(0, \lambda_0) = 0$  and  $\psi'_{\lambda_k}(0, \lambda_0) = \varphi''_{\xi \lambda_k}(0, \lambda_0) \neq 0$ . Applying the implicit function theorem we get the desired claim.

Let  $B_r(\lambda'_0)$  denote a ball in  $R^{k-1}$  of radius  $r$  centered at  $\lambda'_0$ , and  $B_\delta(0)$  a ball in  $X$  of radius  $\delta$  centered at 0.

**Theorem 3.5.** Let  $f: \Omega_0 \rightarrow \Omega$  be a function of Proposition 3.3 and  $r > 0$  be a number such that  $(-r, r) \times B_r(\lambda'_0) \subset \Omega_0$ . There exist open sets  $\tilde{V}_0 \subset X$  and  $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$  such that  $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$  and for every  $(x, \lambda) \in \tilde{V}_0 \times \tilde{V}$  we have  $F(x, \lambda) = 0$  if and only if  $x = 0$  or there exists  $\xi \in (-r, r)$  such that  $\lambda_k = f(\xi, \lambda')$  and  $x = x(\xi, \lambda', f(\xi, \lambda'))$ .

**Proof 3.6.** There exists  $\delta \in (0, r)$  such that for every  $x \in X$  if  $\|x\|_X < \delta$  then  $|\langle x, e \rangle| < r$ . Let  $\tilde{V}_0 = V_0 \cap B_\delta(0)$  and  $\tilde{V} = V \cap (B_r(\lambda'_0) \times \Omega)$ , where  $V_0 \subset X$  and  $V \subset R^k$  are open sets of Theorem 3.1. Take  $(x, \lambda) \in \tilde{V}_0 \times \tilde{V}$ .

( $\Rightarrow$ ) By Theorem 3.1, if  $F(x, \lambda) = 0$  then  $(\langle x, e \rangle, \lambda) \in S$  and  $x = x(\langle x, e \rangle, \lambda)$ , which gives  $\varphi(\langle x, e \rangle, \lambda) = 0$ . From Proposition 3.3 it follows that  $\langle x, e \rangle = 0$  or  $\lambda_k = f(\langle x, e \rangle, \lambda')$ . If  $\langle x, e \rangle = 0$  then  $x = x(0, \lambda) = 0$ . If  $\lambda_k = f(\langle x, e \rangle, \lambda')$  then  $x = x(\langle x, e \rangle, \lambda', f(\langle x, e \rangle, \lambda'))$ .

( $\Leftarrow$ ) Assume now that  $x = 0$  or there exists  $\xi \in (-r, r)$  such that  $\lambda_k = f(\xi, \lambda')$  and  $x = x(\xi, \lambda', f(\xi, \lambda'))$ . In the first case,  $F(x, \lambda) = F(0, \lambda) = 0$ . In the second case, by Proposition 3.3, we have  $\varphi(\xi, \lambda) = 0$ , and hence  $F(x, \lambda) = F(x, \lambda) + \varphi(\xi, \lambda)e = F(x, \lambda) + (\xi - \langle x(\xi, \lambda'), e \rangle)e = F(x, \lambda) + (\xi - \langle x, e \rangle)e = G(x, \xi, \lambda) = 0$ .

We are now in a position to prove our main result.

**Theorem 3.7.** Under assumptions  $(I_1)$ – $(I_5)$ , the solution set of equation (1) in a certain neighbourhood of  $(0, \lambda_0) \in \Lambda$  is the union of two sets:  $\Lambda$  and  $\Xi$ . The set  $\Xi$  is given by

$$\Xi = \{(\hat{x}(\xi, \lambda'), \lambda', f(\xi, \lambda')) : |\xi| < r, |\lambda' - \lambda'_0| < r\},$$

where  $\hat{x}$  and  $f$  are  $C^p$ -smooth functions such that  $\hat{x}(0, \lambda'_0) = 0$ ,  $f(0, \lambda'_0) = \lambda_{0k}$ ,  $\hat{x}'_\xi(0, \lambda'_0) = e$ ,  $f'_\xi(0, \lambda'_0) = -\frac{1}{2} \frac{\varphi''_{\xi\xi}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$ ,  $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$  and  $f'_{\lambda_s}(0, \lambda'_0) = -\frac{\varphi''_{\xi\lambda_s}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$  for every  $s \in \{1, 2, \dots, k-1\}$ .

Moreover, the intersection of  $\Lambda$  and  $\Xi$  in a sufficiently small neighbourhood of  $(0, \lambda_0)$  can be parametrized as follows

$$I_{\Lambda, \Xi} = \{(0, \lambda', f(\hat{\xi}(\lambda'), \lambda')) : |\lambda' - \lambda'_0| < \varrho\}$$

where  $0 < \varrho \leq r$  and  $\hat{\xi}$  is a  $C^p$ -smooth function such that  $\hat{\xi}(\lambda'_0) = 0$  and  $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$  for every  $s \in \{1, 2, \dots, k-1\}$ , which gives that  $(0, \lambda_0)$  is a bifurcation point of (1).

**Proof 3.8.** Let  $f: \Omega_0 \rightarrow \Omega$  be a function of Proposition 3.3. Fix  $r > 0$  such that  $(-r, r) \times B_r(\lambda'_0) \subset \Omega_0$ . Let  $\hat{x}: (-r, r) \times B_r(\lambda'_0) \rightarrow X$  be given by  $\hat{x}(\xi, \lambda') = x(\xi, \lambda', f(\xi, \lambda'))$ . Then  $f(0, \lambda'_0) = \lambda_{0k}$  and  $\hat{x}(0, \lambda'_0) = x(0, \lambda_0) = 0$ . Differentiating  $\hat{x}$  we get  $\hat{x}'_\xi(0, \lambda'_0) = e$  and  $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$  for every  $s \in \{1, 2, \dots, k-1\}$ . Moreover, differentiating the equality  $\psi(\xi, \lambda', f(\xi, \lambda')) = 0$  we obtain  $f'_\xi(0, \lambda'_0) = -\frac{\psi'_\xi(0, \lambda_0)}{\psi'_{\lambda_k}(0, \lambda_0)} = -\frac{1}{2} \frac{\varphi''_{\xi\xi}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$  and  $f'_{\lambda_s}(0, \lambda'_0) = -\frac{\psi'_{\lambda_s}(0, \lambda_0)}{\psi'_{\lambda_k}(0, \lambda_0)} = -\frac{\varphi''_{\xi\lambda_s}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$  for every  $s \in \{1, 2, \dots, k-1\}$ . From Theorem 3.5 it follows that there exist open sets  $\tilde{V}_0 \subset X$  and  $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$  such that  $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$  and  $\{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : F(x, \lambda) = 0\} = \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : x = 0\} \cup \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : \exists \xi \in (-r, r) x = x(\xi, \lambda', f(\xi, \lambda')) \wedge \lambda_k = f(\xi, \lambda')\} = (\Lambda \cup \Xi) \cap \tilde{V}_0 \times \tilde{V}$ . A point  $(x, \lambda) \in \Lambda \cap \Xi$  only if it satisfies the following system

$$\begin{cases} x = \hat{x}(\xi, \lambda'), \\ \lambda_k = f(\xi, \lambda'), \quad \xi \in (-r, r), \quad \lambda' \in B_r(\lambda'_0), \\ x = 0. \end{cases}$$

Since  $\hat{x}(0, \lambda'_0) = 0$  and  $\hat{x}'_\xi(0, \lambda'_0) = e \neq 0$ , there exist:  $0 < \varrho \leq r$ , an open set  $B \subset (-r, r)$  such that  $0 \in B$  and a  $C^p$ -smooth function  $\hat{\xi}: B_\varrho(\lambda'_0) \rightarrow B$  such that  $\hat{\xi}(\lambda'_0) = 0$  and for all  $(\xi, \lambda') \in B \times B_\varrho(\lambda'_0)$  we have  $\hat{x}(\xi, \lambda') = 0$  only if  $\xi = \hat{\xi}(\lambda')$ . Differentiating the equality  $\hat{x}(\hat{\xi}(\lambda'), \lambda') = 0$  we receive  $\hat{x}'_\xi(\hat{\xi}(\lambda'), \lambda') \hat{\xi}'_{\lambda_s}(\lambda') + \hat{x}'_{\lambda_s}(\hat{\xi}(\lambda'), \lambda') = 0$  for every  $s \in \{1, 2, \dots, k-1\}$ , and hence  $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$ . Summarizing  $I_{\Lambda, \Xi} \subset \Lambda \cap \Xi$  and in a sufficiently small neighbourhood of  $(0, \lambda_0)$  the intersection  $\Lambda \cap \Xi$  is equal to  $I_{\Lambda, \Xi}$ .

**Conclusion 3.9.** Assume that  $(I_1)$ – $(I_5)$  hold and  $k = 2$ . Then the solution set of (1) in a small neighbourhood of  $(0, \lambda_0) \in \Lambda$  is the union of two surfaces:  $\Lambda$  and  $\Xi$ . The surface  $\Xi$  can be parametrized as follows

$$\Xi = \{(\hat{x}(\xi, \lambda_1), \lambda_1, f(\xi, \lambda_1)) : (\xi, \lambda_1) \in (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r)\},$$

where  $\hat{x}: (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r) \rightarrow X$  and  $f: (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r) \rightarrow R^1$  are  $C^p$ -smooth functions such that  $\hat{x}(0, \lambda_{01}) = 0$ ,  $f(0, \lambda_{01}) = \lambda_{02}$ ,  $\hat{x}'_{\xi}(0, \lambda_{01}) = e$ ,  $\hat{x}'_{\lambda_1}(0, \lambda_{01}) = 0$ ,  $f'_{\xi}(0, \lambda_{01}) = -\frac{1}{2} \frac{\varphi''_{\xi\xi}(0, \lambda_{01})}{\varphi''_{\xi\lambda_2}(0, \lambda_{01})}$  and  $f'_{\lambda_1}(0, \lambda_{01}) = -\frac{\varphi''_{\xi\lambda_1}(0, \lambda_{01})}{\varphi''_{\xi\lambda_2}(0, \lambda_{01})}$ . In a sufficiently small neighbourhood of  $(0, \lambda_0)$  the surfaces  $\Lambda$  and  $\Xi$  intersect only along the curve

$$I_{\Lambda, \Xi} = \{(0, \lambda_1, f(\hat{\xi}(\lambda_1), \lambda_1)) : \lambda_1 \in (\lambda_{01} - \varrho, \lambda_{01} + \varrho)\},$$

where  $0 < \varrho \leq r$  and  $\hat{\xi}: (\lambda_{01} - \varrho, \lambda_{01} + \varrho) \rightarrow (-r, r)$  is a  $C^p$ -smooth function such that  $\hat{\xi}(\lambda_{01}) = \hat{\xi}'(\lambda_{01}) = 0$ , and hence  $(0, \lambda_0)$  is a bifurcation point of (1).

Let us consider the following condition:

(I'\_3)  $F: X \times R^k \rightarrow Y$  is a variational gradient of a certain functional  $E: X \times R^k \rightarrow R^1$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$ , i.e. for all  $x, y \in X$  and  $\lambda \in R^k$

$$E'_x(x, \lambda)y = \langle F(x, \lambda), y \rangle.$$

It is evident that (I'\_3) implies (I\_3). Furthermore, by formula (7) we obtain

$$\varphi''_{\xi\lambda_s}(0, \lambda_0) = -E^{(3)}_{xx\lambda_s}(0, \lambda_0)(e, e, 1) \tag{10}$$

for  $s \in \{1, 2, \dots, k\}$ . From this it follows that if  $F$  satisfies (I'\_3) then (I\_5) can be replaced by the equivalent condition:

(I'\_5)  $E^{(3)}_{xx\lambda_k}(0, \lambda_0)(e, e, 1) \neq 0$ .

By (7) we also obtain

$$\varphi''_{\xi\xi}(0, \lambda_0) = -E^{(3)}_{xxx}(0, \lambda_0)(e, e, e). \tag{11}$$

Summarizing, in a variational case we have the following result.

**Conclusion 3.10.** Under assumptions: (I\_1), (I\_2), (I'\_3), (I\_4) and (I'\_5), the solution set of equation (1) in a certain neighbourhood of  $(0, \lambda_0) \in \Lambda$  is the union of two sets:  $\Lambda$  and  $\Xi$ . The set  $\Xi$  is given by

$$\Xi = \{(\hat{x}(\xi, \lambda'), \lambda', f(\xi, \lambda')) : |\xi| < r, |\lambda' - \lambda'_0| < r\},$$

where  $\hat{x}$  and  $f$  are  $C^p$ -smooth functions such that  $\hat{x}(0, \lambda'_0) = 0$ ,  $f(0, \lambda'_0) = \lambda_{0k}$ ,  $\hat{x}'_{\xi}(0, \lambda'_0) = e$ ,  $f'_{\xi}(0, \lambda'_0) = -\frac{1}{2} \frac{E^{(3)}_{xxx}(0, \lambda_0)(e, e, e)}{E^{(3)}_{xx\lambda_k}(0, \lambda_0)(e, e, 1)}$ ,  $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$  and  $f'_{\lambda_s}(0, \lambda'_0) = -\frac{E^{(3)}_{xx\lambda_s}(0, \lambda_0)(e, e, 1)}{E^{(3)}_{xx\lambda_k}(0, \lambda_0)(e, e, 1)}$  for every  $s \in \{1, 2, \dots, k - 1\}$ .

Moreover, the intersection of  $\Lambda$  and  $\Xi$  in a sufficiently small neighbourhood of  $(0, \lambda_0)$  can be parametrized as follows

$$I_{\Lambda, \Xi} = \{(0, \lambda', f(\hat{\xi}(\lambda'), \lambda')) : |\lambda' - \lambda'_0| < \varrho\}$$

where  $0 < \varrho \leq r$  and  $\hat{\xi}$  is a  $C^p$ -smooth function such that  $\hat{\xi}(\lambda'_0) = 0$  and  $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$  for every  $s \in \{1, 2, \dots, k - 1\}$ , which gives that  $(0, \lambda_0)$  is a bifurcation point of (1).



## 4 Applications

It is obvious that if we assume that  $F$  is a map from a small neighbourhood of the point  $(0, \lambda_0)$  in  $X \times R^k$  to  $Y$ , our results remain true. After this remark we are ready to give an example of application of Conclusion 3.10 to mathematical physics. All the results of Section 4 were proved either in [12] or [15]. However, to make this exposition self-sufficient we give the main ideas of the proofs.

For every  $m \in N$  and  $\mu \in (0, 1)$ , let  $C^{m,\mu}(\bar{D})$  denote the real Hölder space of functions defined on  $D = \{(u, v) \in R^2: u^2 + v^2 < 1\}$  with the standard norm

$$\|x; C^{m,\mu}(\bar{D})\| = \max_{|\alpha| \leq m} \sup \{|D^\alpha x(u, v)|: (u, v) \in D\} + \max_{|\alpha| \leq m} \sup \left\{ \frac{|D^\alpha x(u, v) - D^\alpha x(\bar{u}, \bar{v})|}{|(u - \bar{u}, v - \bar{v})|^\mu} : (u, v), (\bar{u}, \bar{v}) \in D, (u, v) \neq (\bar{u}, \bar{v}) \right\},$$

where  $D^\alpha x = \frac{\partial^{|\alpha|} x}{\partial^{\alpha_1} u \partial^{\alpha_2} v}$ ,  $\alpha = (\alpha_1, \alpha_2) \in N_0 \times N_0$ ,  $N_0 = N \cup \{0\}$  and  $|\alpha| = \alpha_1 + \alpha_2$ . It is well-known that  $C^{m,\mu}(\bar{D})$  is a Banach space (see [1]). Let

- $C_{0,0}^{4,\mu}(\bar{D}) = \{f \in C^{4,\mu}(\bar{D}): \Delta f|_{\partial D} = f|_{\partial D} = 0\}$ ,
- $C_0^{2,\mu}(\bar{D}) = \{f \in C^{2,\mu}(\bar{D}): f|_{\partial D} = 0\}$ ,
- $X = C_{0,0}^{4,\mu}(\bar{D}) \times C_{0,0}^{4,\mu}(\bar{D})$ ,
- $Y = C^{0,\mu}(\bar{D}) \times C^{0,\mu}(\bar{D})$ .

The norms in  $X$  and  $Y$  are defined by coordinates. That is as the maximum (or the sum) of norms of both coordinates of a given element. The function given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \frac{1}{\pi} \iint_D (x_1 y_1 + x_2 y_2) du dv$$

is a scalar product in  $Y$ , which is continuous with respect to the norms in  $X$  and  $Y$ . We define  $F: X \times R_+^2 \rightarrow Y$  as follows

$$F(x, \lambda) = (\Delta^2 x_1 - [x_1, x_2] + 2\lambda_1 \Delta x_1 + \lambda_2 x_1 - \gamma x_1^3, -\Delta^2 x_2 - \frac{1}{2}[x_1, x_1]), \quad (12)$$

where  $R_+ = (0, +\infty)$ ,  $x = (x_1, x_2)$ ,  $\lambda = (\lambda_1, \lambda_2)$ ,  $\gamma$  is a positive constant and  $[\cdot, \cdot]: X \rightarrow Y$  is given by

$$[x_1, x_2] = \frac{\partial^2 x_1}{\partial u^2} \frac{\partial^2 x_2}{\partial v^2} - 2 \frac{\partial^2 x_1}{\partial u \partial v} \frac{\partial^2 x_2}{\partial u \partial v} + \frac{\partial^2 x_1}{\partial v^2} \frac{\partial^2 x_2}{\partial u^2}.$$

The equation

$$F(x, \lambda) = 0 \quad (13)$$

with  $F$  given by (12) is called the von Kármán equation for a thin circular elastic plate which lies on an elastic base and is uniformly radially compressed along its boundary. In mechanics  $x_1$  is a deflection function,  $x_2$  is a stress function,  $\lambda_1$  is a value of a compressing force,  $\lambda_2$  and  $\gamma$  are parameters of an elastic foundation. The solutions of (13) lying in a sufficiently small neighbourhood of the set of trivial solutions of (13) are called the forms



of equilibrium of a plate. The map  $F$  is  $C^\infty$ -smooth and an easy computation shows that for all  $y = (y_1, y_2) \in X$

$$F'_x(x, \lambda)y = (\Delta^2 y_1 - [y_1, x_2] - [x_1, y_2] + 2\lambda_1 \Delta y_1 + \lambda_2 y_1 - 3\gamma x_1^2 y_1, -\Delta^2 y_2 - [x_1, y_1]). \tag{14}$$

Let  $E: X \times R_+^2 \rightarrow R^1$  be given by

$$E(x, \lambda) = \frac{1}{2\pi} \iint_D ((\Delta x_1)^2 - (\Delta x_2)^2 - [x_1, x_1]x_2) dudv + \frac{1}{2\pi} \iint_D \left( -2\lambda_1 \left( \left( \frac{\partial x_1}{\partial u} \right)^2 + \left( \frac{\partial x_1}{\partial v} \right)^2 \right) + \lambda_2 x_1^2 - \frac{1}{2} \gamma x_1^4 \right) dudv. \tag{15}$$

$E$  is easily seen to be  $C^\infty$ -smooth.

**Theorem 4.1 (see Th. 2.4 of [12]).** The map  $F$  is a variational gradient of the functional  $E$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$ .

**Sketch of the proof 4.2.** For all  $x, y \in X$  and  $\lambda \in R_+^2$ , we have

$$\begin{aligned} E'_x(x, \lambda)y &= \frac{d}{dt} E(x + ty, \lambda)|_{t=0} = \frac{1}{\pi} \iint_D \Delta x_1 \Delta y_1 dudv - \frac{1}{\pi} \iint_D \Delta x_2 \Delta y_2 dudv \\ &\quad - \frac{1}{\pi} \iint_D [x_1, y_1]x_2 dudv - \frac{1}{2\pi} \iint_D [x_1, x_1]y_2 dudv \\ &\quad - \frac{1}{\pi} \iint_D 2\lambda_1 \left( \frac{\partial x_1}{\partial u} \frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v} \frac{\partial y_1}{\partial v} \right) dudv \\ &\quad + \frac{1}{\pi} \iint_D (\lambda_2 x_1 y_1 - \gamma x_1^3 y_1) dudv. \end{aligned}$$

Integrating by part we receive

$$\begin{aligned} \iint_D \Delta x_1 \Delta y_1 dudv &= \iint_D (\Delta^2 x_1) y_1 dudv, \\ \iint_D \Delta x_2 \Delta y_2 dudv &= \iint_D (\Delta^2 x_2) y_2 dudv, \\ \iint_D [x_1, y_1]x_2 dudv &= \iint_D [x_1, x_2]y_1 dudv \end{aligned}$$

and

$$\iint_D \left( \frac{\partial x_1}{\partial u} \frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v} \frac{\partial y_1}{\partial v} \right) dudv = - \iint_D (\Delta x_1) y_1 dudv.$$

Hence  $E'_x(x, \lambda)y = \langle F(x, \lambda), y \rangle$ , which completes the proof.

**Theorem 4.3 (see Th. 2.2 of [12]).** For every  $\lambda \in R_+^2$ ,  $F'_x(0, \lambda): X \rightarrow Y$  is a Fredholm map of index 0.

**Sketch of the proof 4.4.** Fix  $\lambda \in R_+^2$ . By (14) we get

$$F'_x(0, \lambda)y = (\Delta^2 y_1 + 2\lambda_1 \Delta y_1 + \lambda_2 y_1, -\Delta^2 y_2). \tag{16}$$

We can write (16) as

$$F'_x(0, \lambda)y = A(y) + B(y),$$

where  $A, B: X \rightarrow Y$  are given as follows:

$$A(y) = (\Delta^2 y_1, -\Delta^2 y_2), \quad B(y) = (2\lambda_1 \Delta y_1 + \lambda_2 y_1, 0).$$

It is known that  $\Delta: C^{2,\mu}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$  is an isomorphism. Moreover, it is a simple matter to check that  $B$  is compact, which finishes the proof.

Let  $J_k: R \rightarrow R, k \in N_0$ , denote the  $k$ -th Bessel function. It is well-known (see [8], [18]) that  $\alpha \in R$  is an eigenvalue of  $\Delta: C^{2,\mu}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$  if and only if  $\alpha < 0$  and there is  $k \in N_0$  such that  $J_k(\sqrt{-\alpha}) = 0$ . Furthermore, if  $J_0(\sqrt{-\alpha}) = 0$  then the eigenspace corresponding to  $\alpha$  is one-dimensional. If  $J_k(\sqrt{-\alpha}) = 0$  for a certain  $k \in N$  then the corresponding eigenspace is two-dimensional.

For  $\lambda = (\lambda_1, \lambda_2) \in R^2_+$ , let  $\delta = (\lambda_1)^2 - \lambda_2, a = -\lambda_1 - \sqrt{\delta}$  and  $b = -\lambda_1 + \sqrt{\delta}$ . Of course,  $a$  and  $b$  are determined on condition  $\delta \geq 0$ . Let  $\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I: C^{4,\mu}_{0,0}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$  and  $\Delta - aI, \Delta - bI: C^{2,\mu}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$ , where  $I(h) = h$  are natural embeddings of the appropriate Hölder spaces.

**Lemma 4.5 (see Lemmas 4.1-4.3 of [12]).** Under the above assumptions:

- (i) If  $\delta < 0$  then  $\ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) = \{0\}$ .
- (ii) If  $\delta = 0$  then  $\ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) = \ker(\Delta + \lambda_1 I)$ .
- (iii) If  $\delta > 0$  then  $\ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) = \ker(\Delta - aI) \oplus \ker(\Delta - bI)$ .

By (16),  $N(\lambda) = \ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) \times \{0\}$ . From this and Lemma 4.5 we obtain what follows.

**Theorem 4.6.**  $\dim N(\lambda) = 1$  if and only if one of the below conditions is satisfied:

- (I)  $\delta = 0$  and  $J_0(\sqrt{\lambda_1}) = 0$ ,
- (II)  $\delta > 0, J_0(\sqrt{-a}) = 0$  and  $J_k(\sqrt{-b}) \neq 0$  for every  $k \in N_0$ ,
- (III)  $\delta > 0, J_0(\sqrt{-b}) = 0$  and  $J_k(\sqrt{-a}) \neq 0$  for every  $k \in N_0$ .

Suppose that  $\lambda_0 = (\lambda_{01}, \lambda_{02})$  and  $\dim N(\lambda_0) = 1$ . Fix  $e = (e_1, 0) \in N(\lambda_0)$  such that  $\langle e, e \rangle = 1$ . Set

$$c_0 = \begin{cases} a_0 & \text{if (I) or (II),} \\ b_0 & \text{if (III),} \end{cases}$$

where  $a_0 = -\lambda_{01} - \sqrt{\delta_0}, b_0 = -\lambda_{01} + \sqrt{\delta_0}$  and  $\delta_0 = (\lambda_{01})^2 - \lambda_{02}$ . A trivial verification combining Theorem 4.1 with (14) shows that

$$E'''_{xx\lambda_1}(x, \lambda)(y, z, 1) = \frac{2}{\pi} \iint_D (\Delta y_1) z_1 dudv,$$

$$E'''_{xx\lambda_2}(x, \lambda)(y, z, 1) = \frac{1}{\pi} \iint_D y_1 z_1 dudv,$$

and

$$E'''_{xxx}(x, \lambda)(y, z, w) = -\frac{1}{\pi} \iint_D ([y_1, z_2] + [y_2, z_1] + 6\gamma x_1 y_1 z_1) w_1 dudv - \frac{1}{\pi} \iint_D [y_1, z_1] w_2 dudv,$$

where  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$ ,  $z = (z_1, z_2)$ ,  $w = (w_1, w_2)$ . From this and Lemma 4.5 we receive

$$E'''_{xx\lambda_1}(0, \lambda_0)(e, e, 1) = \frac{2}{\pi} \iint_D (\Delta e_1) e_1 dudv = 2c_0 \langle e, e \rangle = 2c_0,$$

$$E'''_{xx\lambda_2}(0, \lambda_0)(e, e, 1) = \frac{1}{\pi} \iint_D e_1^2 dudv = \langle e, e \rangle = 1,$$

$$E'''_{xxx}(0, \lambda_0)(e, e, e) = 0.$$

Applying Conclusion 3.10 we get the following theorem.

**Theorem 4.7.** Let  $\lambda_0 \in R^2_+$  satisfy the above assumptions. Then the solution set of equation (13) in a certain neighbourhood of  $(0, \lambda_0) \in X \times R^2_+$  is the union of two sets:  $\Lambda$  and  $\Xi$ . The set  $\Xi$  is given by

$$\Xi = \{(\hat{x}(\xi, \lambda_1), \lambda_1, f(\xi, \lambda_1)) : |\xi| < r, |\lambda_1 - \lambda_{01}| < r\},$$

where  $\hat{x}$  and  $f$  are  $C^\infty$ -smooth functions such that  $\hat{x}(0, \lambda_{01}) = 0$ ,  $f(0, \lambda_{01}) = \lambda_{02}$ ,  $\hat{x}'_\xi(0, \lambda_{01}) = e$ ,  $f'_\xi(0, \lambda_{01}) = 0$ ,  $\hat{x}'_{\lambda_1}(0, \lambda_{01}) = 0$  and  $f'_{\lambda_1}(0, \lambda_{01}) = -2c_0$ .

Moreover, the intersection of  $\Lambda$  and  $\Xi$  in a sufficiently small neighbourhood of  $(0, \lambda_0)$  can be parametrized as follows

$$I_{\Lambda, \Xi} = \{(0, \lambda_1, f(\hat{\xi}(\lambda_1), \lambda_1)) : |\lambda_1 - \lambda_{01}| < \varrho\},$$

where  $0 < \varrho \leq r$  and  $\hat{\xi}$  is a  $C^\infty$ -smooth function such that  $\hat{\xi}(\lambda_{01}) = 0$  and  $\hat{\xi}'(\lambda_{01}) = 0$ , which gives that  $(0, \lambda_0)$  is a bifurcation point of (13).

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