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# Homoclinic solutions for a class of the second order Hamiltonian systems

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## Abstract

We study the existence of homoclinic orbits for the second order Hamiltonian system  $\ddot{q} + V_q(t, q) = f(t)$ , where  $q \in \mathbb{R}^n$  and  $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ ,  $V(t, q) = -K(t, q) + W(t, q)$  is  $T$ -periodic in  $t$ . A map  $K$  satisfies the “pinching” condition  $b_1|q|^2 \leq K(t, q) \leq b_2|q|^2$ ,  $W$  is superlinear at the infinity and  $f$  is sufficiently small in  $L^2(\mathbb{R}, \mathbb{R}^n)$ . A homoclinic orbit is obtained as a limit of  $2kT$ -periodic solutions of a certain sequence of the second order differential equations.

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## 1. Introduction

In this paper, we shall be concerned with the existence of homoclinic orbits for the second order Hamiltonian system:

$$\ddot{q} + V_q(t, q) = f(t), \tag{HS}$$

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where  $t \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$  and functions  $V: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}^n$  satisfy:

(H<sub>1</sub>)  $V(t, q) = -K(t, q) + W(t, q)$ , where  $K, W: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $C^1$ -maps,  $T$ -periodic with respect to  $t$ ,  $T > 0$ ,

(H<sub>2</sub>) there are constants  $b_1, b_2 > 0$  such that for all  $(t, q) \in \mathbb{R} \times \mathbb{R}^n$

$$b_1|q|^2 \leq K(t, q) \leq b_2|q|^2,$$

(H<sub>3</sub>) for all  $(t, q) \in \mathbb{R} \times \mathbb{R}^n$ ,  $K(t, q) \leq (q, K_q(t, q)) \leq 2K(t, q)$ ,

(H<sub>4</sub>)  $W_q(t, q) = o(|q|)$ , as  $|q| \rightarrow 0$  uniformly with respect to  $t$ ,

(H<sub>5</sub>) there is a constant  $\mu > 2$  such that for every  $t \in \mathbb{R}$  and  $q \in \mathbb{R}^n \setminus \{0\}$

$$0 < \mu W(t, q) \leq (q, W_q(t, q)),$$

(H<sub>6</sub>)  $f: \mathbb{R} \rightarrow \mathbb{R}^n$  is a continuous and bounded function.

Here and subsequently,  $(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  denotes the standard inner product in  $\mathbb{R}^n$  and  $|\cdot|$  is the induced norm.

We will say that a solution  $q$  of (HS) is *homoclinic* (to 0) if  $q(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ . In addition, if  $q \not\equiv 0$  then  $q$  is called a *nontrivial homoclinic solution*.

For each  $k \in \mathbb{N}$ , let  $E_k := W_{2kT}^{1,2}(\mathbb{R}, \mathbb{R}^n)$ , the Hilbert space of  $2kT$ -periodic functions on  $\mathbb{R}$  with values in  $\mathbb{R}^n$  under the norm

$$\|q\|_{E_k} := \left( \int_{-kT}^{kT} (|\dot{q}(t)|^2 + |q(t)|^2) dt \right)^{1/2}.$$

Furthermore, let  $L_{2kT}^\infty(\mathbb{R}, \mathbb{R}^n)$  denote a space of  $2kT$ -periodic essentially bounded (measurable) functions from  $\mathbb{R}$  into  $\mathbb{R}^n$  equipped with the norm

$$\|q\|_{L_{2kT}^\infty} := \text{ess sup}\{|q(t)| : t \in [-kT, kT]\}.$$

We begin with a result which is a direct consequence of estimations made by Rabinowitz in [12].

**Proposition 1.1.** *There is a positive constant  $C$  such that for each  $k \in \mathbb{N}$  and  $q \in E_k$  the following inequality holds:*

$$\|q\|_{L_{2kT}^\infty} \leq C \|q\|_{E_k}. \quad (1)$$

One can easily show that the inequality (1) holds true with constant  $C = \sqrt{2}$  if  $T \geq \frac{1}{2}$  (see Fact 2.8).



Set  $M := \sup\{W(t, q) : t \in [0, T], |q| = 1\}$ ,  $\bar{b}_1 := \min\{1, 2b_1\}$ ,  $\bar{b}_2 := \max\{1, 2b_2\}$  and suppose that:

(H7)  $2M < \bar{b}_1$  and  $(\int_{\mathbb{R}} |f(t)|^2 dt)^{1/2} \leq \frac{\beta}{2C}$ , where  $0 < \beta < \bar{b}_1 - 2M$  and  $C$  is a constant from Proposition 1.1.

We will prove the following theorem:

**Theorem 1.2.** *If the conditions (H1)–(H7) are satisfied then the system (HS) possesses a nontrivial homoclinic solution  $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$  such that  $\dot{q}(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ .*

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. In particular, the second order systems were considered in [1,3,5–7,11–13,16], and those of the first order in [4,8–10,14,15]. Our study is motivated by a paper of Rabinowitz [12] in which the existence of a nontrivial homoclinic solution for the second order Hamiltonian system

$$\ddot{q} + V_q(t, q) = 0$$

was proved. The function  $V$  considered by the author is of the form

$$V(t, q) = -\frac{1}{2}(L(t)q, q) + \bar{W}(t, q), \tag{2}$$

where  $L$  is a continuous  $T$ -periodic matrix valued function such that  $L(t)$  is positive definite and symmetric for all  $t \in [0, T]$ ,  $\bar{W}$  satisfies (H4) and (H5). Let us note that conditions (H2) and (H3) are satisfied if  $K(t, q) = \frac{1}{2}(L(t)q, q)$ . On the other hand, one can easily check that if

$$K(t, x) = \begin{cases} \left(1 + \frac{1}{1+x^2}\right)x^2 & \text{for } x \geq 0, \\ \left(1 + \frac{2}{1+x^2}\right)x^2 & \text{for } x < 0 \end{cases}$$

and  $W(t, x) = x^4$ , where  $t, x \in \mathbb{R}$ , then  $V(t, x) = -K(t, x) + W(t, x)$  cannot be represented in the form (2) with  $\bar{W}$  satisfying (H4), (H5) while  $V$  satisfies conditions (H1)–(H5). Hence, our theorem extends the result from [12] even if  $f(t) = 0$ . It follows from our assumptions that  $q(t) = 0$  is a solution of (HS) only if  $f(t) = 0$ . Therefore, if  $f$  is a nonzero function the existence of a homoclinic solution of (HS) implies its nontriviality.

Similarly to [12] a homoclinic solution of (HS) is obtained as a limit, as  $k \rightarrow +\infty$ , of a certain sequence of functions  $q_k \in E_k$ . However, in our approach, we consider a sequence of systems of differential equations:

$$\ddot{q} + V_q(t, q) = f_k(t), \tag{HS}_k$$

where for each  $k \in \mathbb{N}$ ,  $f_k: \mathbb{R} \rightarrow \mathbb{R}^n$  is a  $2kT$ -periodic extension of the restriction of  $f$  to the interval  $[-kT, kT]$  and  $q_k$  is a  $2kT$ -periodic solution of  $(HS_k)$  obtained via the Mountain Pass Theorem.

Part of the difficulty in treating  $(HS)$  is caused by the fact that in order to get appropriate convergence of the sequence of approximative functions  $\{q_k\}_{k \in \mathbb{N}}$  we need the constants  $\rho$  and  $\alpha$  appearing in the condition (iii) of the Mountain Pass Theorem (see Theorem 2.5) to be independent of  $k$ .

## 2. Proof of Theorem 1.2

At first let us recall some properties of the function  $W(t, q)$  from [12]. They all are necessary to the proof of Theorem 1.2.

**Fact 2.1.** For every  $t \in [0, T]$  the following inequalities hold:

$$W(t, q) \leq W\left(t, \frac{q}{|q|}\right) |q|^\mu \quad \text{if } 0 < |q| \leq 1, \quad (3)$$

$$W(t, q) \geq W\left(t, \frac{q}{|q|}\right) |q|^\mu \quad \text{if } |q| \geq 1. \quad (4)$$

To prove this fact it suffices to show that for every  $q \neq 0$  and  $t \in [0, T]$  the function  $(0, +\infty) \ni \zeta \rightarrow W(t, \zeta^{-1}q)\zeta^\mu$  is nonincreasing. It is an immediate consequence of  $(H_5)$ .

**Fact 2.2.** Set  $m := \inf\{W(t, q) : t \in [0, T], |q| = 1\}$ . Then for every  $\zeta \in \mathbb{R} \setminus \{0\}$  and  $q \in E_k \setminus \{0\}$  we have

$$\int_{-kT}^{kT} W(t, \zeta q(t)) dt \geq m |\zeta|^\mu \int_{-kT}^{kT} |q(t)|^\mu dt - 2kTm. \quad (5)$$

**Proof.** Fix  $\zeta \in \mathbb{R} \setminus \{0\}$  and  $q \in E_k \setminus \{0\}$ . Set  $A_k = \{t \in [-kT, kT] : |\zeta q(t)| \leq 1\}$ , and  $B_k = \{t \in [-kT, kT] : |\zeta q(t)| \geq 1\}$ . From (4) we obtain

$$\begin{aligned} \int_{-kT}^{kT} W(t, \zeta q(t)) dt &\geq \int_{B_k} W(t, \zeta q(t)) dt \geq \int_{B_k} W\left(t, \frac{\zeta q(t)}{|\zeta q(t)|}\right) |\zeta q(t)|^\mu dt \\ &\geq m \int_{B_k} |\zeta q(t)|^\mu dt \\ &\geq m \int_{-kT}^{kT} |\zeta q(t)|^\mu dt - m \int_{A_k} |\zeta q(t)|^\mu dt \\ &\geq m |\zeta|^\mu \int_{-kT}^{kT} |q(t)|^\mu dt - 2kTm. \quad \square \end{aligned}$$



**Fact 2.3.** Let  $Y: [0, +\infty) \rightarrow [0, +\infty)$  be given as follows:  $Y(0) = 0$  and

$$Y(s) = \max_{\substack{t \in [0, T] \\ 0 < |q| \leq s}} \frac{(q, W_q(t, q))}{|q|^2} \tag{6}$$

for  $s > 0$ . Then  $Y$  is continuous, nondecreasing,  $Y(s) > 0$  for  $s > 0$  and  $Y(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ .

It is easy to verify this fact applying  $(H_4)$ ,  $(H_5)$  and (4).

Assumptions  $(H_4)$  and  $(H_5)$  imply that  $W(t, q) = o(|q|^2)$  as  $q \rightarrow 0$  uniformly for  $t \in [0, T]$  and  $W(t, 0) = 0, W_q(t, 0) = 0$ . Moreover, from  $(H_2)$  we conclude that  $K(t, 0) = 0, K_q(t, 0) = 0$ .

Before we will prove Theorem 1.2, we have to introduce more notation and some necessary definitions. For each  $k \in \mathbb{N}$ , let  $L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$  denote the Hilbert space of  $2kT$ -periodic functions on  $\mathbb{R}$  with values in  $\mathbb{R}^n$  under the norm  $\|q\|_{L^2_{2kT}} = \left( \int_{-kT}^{kT} |q(t)|^2 dt \right)^{1/2}$ . Let  $f_k: \mathbb{R} \rightarrow \mathbb{R}^n$  be a  $2kT$ -periodic extension of  $f_{[-kT, kT]}$  onto  $\mathbb{R}$ . From  $(H_7)$  it follows that  $\|f_k\|_{L^2_{2kT}} \leq \beta/2C$ . Consider the second order Hamiltonian system:

$$\ddot{q} + V_q(t, q) = f_k(t). \tag{HS}_k$$

Let  $\eta_k: E_k \rightarrow [0, +\infty)$  be given by

$$\eta_k(q) = \left( \int_{-kT}^{kT} [|\dot{q}(t)|^2 + 2K(t, q(t))] dt \right)^{1/2}. \tag{7}$$

By  $(H_2)$ ,

$$\bar{b}_1 \|q\|_{E_k}^2 \leq \eta_k^2(q) \leq \bar{b}_2 \|q\|_{E_k}^2. \tag{8}$$

It is worth pointing out that if the function  $K(t, q)$  is of the form  $\frac{1}{2}(L(t)q, q)$  with a matrix valued function  $L$  satisfying the same conditions as in [12] then  $\eta_k$  determined by (7) is a norm in  $E_k$  equivalent to the norm  $\|\cdot\|_{E_k}$ . Let  $I_k: E_k \rightarrow \mathbb{R}$  be defined by

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT} \left[ \frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) \right] dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt \\ &= \frac{1}{2} \eta_k^2(q) - \int_{-kT}^{kT} W(t, q(t)) dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt. \end{aligned} \tag{9}$$

Then  $I_k \in C^1(E_k, \mathbb{R})$  and it is easy to check that

$$I'_k(q)v = \int_{-kT}^{kT} [(\dot{q}(t), \dot{v}(t)) - (V_q(t, q(t)), v(t))] dt + \int_{-kT}^{kT} (f_k(t), v(t)) dt, \tag{10}$$

and

$$I'_k(q)q \leq \eta_k^2(q) - \int_{-kT}^{kT} (W_q(t, q(t)), q(t)) dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt. \quad (11)$$

Moreover, it is clear that critical points of  $I_k$  are classical  $2kT$ -periodic solutions of  $(HS_k)$ .

We have divided the proof of Theorem 1.2 into a sequence of lemmas.

**Lemma 2.4.** *If  $V$  and  $f$  satisfy  $(H_1)$ – $(H_7)$  then for every  $k \in \mathbb{N}$  the system  $(HS_k)$  possesses a  $2kT$ -periodic solution.*

We will obtain a critical point of  $I_k$  by the use of a standard version of the Mountain Pass Theorem (see [2]). It provides the minimax characterisation for the critical value which is important for what follows. Therefore, we state this theorem precisely.

**Theorem 2.5** (see Ambrosetti and Rabinowitz [2]). *Let  $E$  be a real Banach space and  $I: E \rightarrow \mathbb{R}$  be a  $C^1$ -smooth functional. If  $I$  satisfies the following conditions:*

- (i)  $I(0) = 0$ ,
- (ii) every sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $E$  such that  $\{I(u_j)\}_{j \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  and  $I'(u_j) \rightarrow 0$  in  $E^*$ , as  $j \rightarrow +\infty$ , contains a convergent subsequence (the Palais-Smale condition),
- (iii) there exist constants  $\varrho, \alpha > 0$  such that  $I|_{\partial B_\varrho(0)} \geq \alpha$ ,
- (iv) there exists  $e \in E \setminus \overline{B_\varrho(0)}$  such that  $I(e) \leq 0$ ,

where  $B_\varrho(0)$  is an open ball in  $E$  of radius  $\varrho$  centred at 0, then  $I$  possesses a critical value  $c \geq \alpha$  given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

**Proof of Lemma 2.4.** In our case it is clear that  $I_k(0) = 0$ . We show that  $I_k$  satisfies the Palais-Smale condition. Assume that  $\{u_j\}_{j \in \mathbb{N}}$  in  $E_k$  is a sequence such that  $\{I_k(u_j)\}_{j \in \mathbb{N}}$  is bounded and  $I'_k(u_j) \rightarrow 0$  as  $j \rightarrow +\infty$ . Then there exists a constant  $C_k > 0$  such that

$$|I_k(u_j)| \leq C_k, \quad \|I'_k(u_j)\|_{E_k^*} \leq C_k \quad (12)$$



for every  $j \in \mathbb{N}$ . We first prove that  $\{u_j\}_{j \in \mathbb{N}}$  is bounded. By (9) and (H<sub>5</sub>),

$$\begin{aligned} \eta_k^2(u_j) &\leq 2I_k(u_j) + \frac{2}{\mu} \int_{-kT}^{kT} (W_q(t, u_j(t)), u_j(t)) dt \\ &\quad - 2 \int_{-kT}^{kT} (f_k(t), u_j(t)) dt. \end{aligned} \tag{13}$$

From (13) and (11) we obtain

$$\begin{aligned} \left(1 - \frac{2}{\mu}\right) \eta_k^2(u_j) &\leq 2I_k(u_j) - \frac{2}{\mu} I'_k(u_j) u_j \\ &\quad - \left(2 - \frac{2}{\mu}\right) \int_{-kT}^{kT} (f_k(t), u_j(t)) dt. \end{aligned} \tag{14}$$

From (14) and (8) it follows that

$$\begin{aligned} \left(1 - \frac{2}{\mu}\right) \bar{b}_1 \|u_j\|_{E_k}^2 &\leq 2I_k(u_j) \\ &\quad + \left(\frac{2}{\mu} \|I'_k(u_j)\|_{E_k^*} + \left(2 - \frac{2}{\mu}\right) \|f_k\|_{L^2_{2kT}}\right) \|u_j\|_{E_k}. \end{aligned} \tag{15}$$

Combining (15) with (H<sub>7</sub>) and (12) we get

$$\left(1 - \frac{2}{\mu}\right) \bar{b}_1 \|u_j\|_{E_k}^2 - \left(\frac{2C_k}{\mu} + \left(2 - \frac{2}{\mu}\right) \frac{\beta}{2C}\right) \|u_j\|_{E_k} - 2C_k \leq 0. \tag{16}$$

Since  $\mu > 2$ , (16) shows that  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $E_k$ . Going if necessary to a subsequence, we can assume that there exists  $u \in E_k$  such that  $u_j \rightharpoonup u$ , as  $j \rightarrow +\infty$ , in  $E_k$ , which implies  $u_j \rightarrow u$  uniformly on  $[-kT, kT]$ . Hence  $(I'_k(u_j) - I'_k(u))(u_j - u) \rightarrow 0$ ,  $\|u_j - u\|_{L^2_{2kT}} \rightarrow 0$  and

$$\int_{-kT}^{kT} (V_q(t, u_j(t)) - V_q(t, u(t)), u_j(t) - u(t)) dt \rightarrow 0,$$

as  $j \rightarrow +\infty$ . Moreover, an easy computation shows that

$$\begin{aligned} (I'_k(u_j) - I'_k(u))(u_j - u) &= \|\dot{u}_j - \dot{u}\|_{L^2_{2kT}}^2 \\ &\quad - \int_{-kT}^{kT} (V_q(t, u_j(t)) - V_q(t, u(t)), u_j(t) - u(t)) dt, \end{aligned}$$

and so  $\|\dot{u}_j - \dot{u}\|_{L^2_{2kT}}^2 \rightarrow 0$ . Consequently,  $\|u_j - u\|_{E_k} \rightarrow 0$ .



We now show that there exist constants  $\varrho, \alpha > 0$  independent of  $k$  such that every  $I_k$  satisfies the assumption (iii) of Theorem 2.5 with these constants. Assume that  $0 < \|q\|_{L^\infty_{2kT}} \leq 1$ . By (3) we have

$$\begin{aligned} \int_{-kT}^{kT} W(t, q(t)) dt &\leq \int_{-kT}^{kT} W\left(t, \frac{q(t)}{|q(t)|}\right) |q(t)|^\mu dt \\ &\leq M \int_{-kT}^{kT} |q(t)|^2 dt \leq M \|q\|_{E_k}^2, \end{aligned}$$

and, in consequence, combining this with (8) and (H7) we obtain

$$\begin{aligned} I_k(q) &\geq \frac{1}{2} \bar{b}_1 \|q\|_{E_k}^2 - M \|q\|_{E_k}^2 - \|f_k\|_{L^2_{2kT}} \|q\|_{L^2_{2kT}} \\ &\geq \frac{1}{2} \bar{b}_1 \|q\|_{E_k}^2 - M \|q\|_{E_k}^2 - \frac{\beta}{2C} \|q\|_{E_k} \\ &= \frac{1}{2} (\bar{b}_1 - \beta - 2M) \|q\|_{E_k}^2 + \frac{\beta}{2} \|q\|_{E_k}^2 - \frac{\beta}{2C} \|q\|_{E_k}. \end{aligned} \quad (17)$$

Note that (H7) implies  $\bar{b}_1 - \beta - 2M > 0$ . Set

$$\varrho = \frac{1}{C}, \quad \alpha = \frac{\bar{b}_1 - \beta - 2M}{2C^2}.$$

By (1), if  $\|q\|_{E_k} = \varrho$  then  $0 < \|q\|_{L^\infty_{2kT}} \leq 1$  and (17) gives  $I_k(q) \geq \alpha$ .

It remains to prove that for every  $k \in \mathbb{N}$  there exists  $e_k \in E_k$  such that  $\|e_k\|_{E_k} > \varrho$  and  $I_k(e_k) \leq 0$ . By the use of (5), (9) and (8) we have that for every  $\zeta \in \mathbb{R} \setminus \{0\}$  and  $q \in E_k \setminus \{0\}$  the following inequality holds:

$$\begin{aligned} I_k(\zeta q) &\leq \frac{\bar{b}_2 \zeta^2}{2} \|q\|_{E_k}^2 - m |\zeta|^\mu \int_{-kT}^{kT} |q(t)|^\mu dt \\ &\quad + |\zeta| \cdot \|f_k\|_{L^2_{2kT}} \|q\|_{L^2_{2kT}} + 2kTm. \end{aligned} \quad (18)$$

Take  $Q \in E_1$  such that  $Q(\pm T) = 0$ . Since  $\mu > 2$  and  $m > 0$ , (18) implies that there exists  $\zeta \in \mathbb{R} \setminus \{0\}$  such that  $\|\zeta Q\|_{E_1} > \varrho$  and  $I_1(\zeta Q) < 0$ . Set  $e_1(t) = \zeta Q(t)$  and

$$e_k(t) = \begin{cases} e_1(t) & \text{for } |t| \leq T, \\ 0 & \text{for } T < |t| \leq kT \end{cases} \quad (19)$$

for  $k > 0$ . Then  $e_k \in E_k$ ,  $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \varrho$  and  $I_k(e_k) = I_1(e_1) < 0$  for every  $k \in \mathbb{N}$ . By Theorem 2.5,  $I_k$  possesses a critical value  $c_k \geq \alpha$  given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k(g(s)), \quad (20)$$





where

$$\Gamma_k = \{g \in C([0, 1], E_k) : g(0) = 0, g(1) = e_k\}.$$

Hence, for every  $k \in \mathbb{N}$ , there is  $q_k \in E_k$  such that

$$I_k(q_k) = c_k, \quad I'_k(q_k) = 0. \tag{21}$$

The function  $q_k$  is a desired classical  $2kT$ -periodic solution of  $(HS_k)$ . Since  $c_k > 0$ ,  $q_k$  is a nontrivial solution even if  $f_k(t) = 0$ .  $\square$

Let  $C^p_{loc}(\mathbb{R}, \mathbb{R}^n)$ , where  $p \in \mathbb{N} \cup \{0\}$ , denote the space of  $C^p$  functions on  $\mathbb{R}$  with values in  $\mathbb{R}^n$  under the topology of almost uniformly convergence of functions and all derivatives up to the order  $p$ . Using the Arzelà-Ascoli theorem we prove what follows.

**Lemma 2.6.** *Let  $\{q_k\}_{k \in \mathbb{N}}$  be the sequence given by (21). There exist an increasing function  $\varphi: \mathbb{N} \rightarrow \mathbb{N}$  and a  $C^1$  function  $q_0: \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $q_{\varphi(k)} \rightarrow q_0$ , as  $k \rightarrow +\infty$ , in  $C^1_{loc}(\mathbb{R}, \mathbb{R}^n)$ .*

**Proof.** The first step in the proof is to show that the sequences  $\{c_k\}_{k \in \mathbb{N}}$  and  $\{\|q_k\|_{E_k}\}_{k \in \mathbb{N}}$  are bounded. For every  $k \in \mathbb{N}$ , let  $g_k: [0, 1] \rightarrow E_k$  be a curve given by  $g_k(s) = se_k$ , where  $e_k$  is determined by (19). Then  $g_k \in \Gamma_k$  and  $I_k(g_k(s)) = I_1(g_1(s))$  for all  $k \in \mathbb{N}$  and  $s \in [0, 1]$ . Therefore, by (20),

$$c_k \leq \max_{s \in [0, 1]} I_1(g_1(s)) \equiv M_0 \tag{22}$$

independently of  $k \in \mathbb{N}$ . As  $I'_k(q_k) = 0$ , we receive from (9), (11) and  $(H_5)$  that

$$\begin{aligned} c_k &= I_k(q_k) - \frac{1}{2} I'_k(q_k) q_k \\ &\geq \left(\frac{\mu}{2} - 1\right) \int_{-kT}^{kT} W(t, q_k(t)) dt + \frac{1}{2} \int_{-kT}^{kT} (f_k(t), q_k(t)) dt, \end{aligned}$$

and hence

$$\int_{-kT}^{kT} W(t, q_k(t)) dt \leq \frac{1}{\mu - 2} \left( 2c_k - \int_{-kT}^{kT} (f_k(t), q_k(t)) dt \right).$$

Combining the above with (8), (9) and (22) we have

$$\bar{b}_1 \|q_k\|_{E_k}^2 \leq \frac{2\mu M_0}{\mu - 2} + \frac{2\mu - 2}{\mu - 2} \|f_k\|_{L^2_{2kT}} \|q_k\|_{L^2_{2kT}},$$

and, in consequence, by (H<sub>7</sub>)

$$\bar{b}_1 \|q_k\|_{E_k}^2 - \frac{\beta(\mu-1)}{C(\mu-2)} \|q_k\|_{E_k} - \frac{2\mu M_0}{\mu-2} \leq 0. \quad (23)$$

Since  $\bar{b}_1 > 0$  and all coefficients of (23) are independent of  $k$ , we see that there is  $M_1 > 0$  independent of  $k$  such that

$$\|q_k\|_{E_k} \leq M_1. \quad (24)$$

We now observe that the sequences  $\{q_k\}_{k \in \mathbb{N}}$ ,  $\{\dot{q}_k\}_{k \in \mathbb{N}}$  and  $\{\ddot{q}_k\}_{k \in \mathbb{N}}$  are uniformly bounded. By (1),

$$\|q_k\|_{L_{2kT}^\infty} \leq CM_1 \equiv M_2 \quad (25)$$

for every  $k \in \mathbb{N}$ . Since  $q_k$  satisfies (HS<sub>k</sub>), if  $t \in [-kT, kT]$  we have

$$|\ddot{q}_k(t)| \leq |f_k(t)| + |V_q(t, q_k(t))| = |f(t)| + |V_q(t, q_k(t))|.$$

Therefore (25), (H<sub>1</sub>) and (H<sub>6</sub>) imply that there is  $M_3 > 0$  independent of  $k$  such that

$$\|\ddot{q}_k\|_{L_{2kT}^\infty} \leq M_3. \quad (26)$$

From the Mean Value Theorem it follows that for every  $k \in \mathbb{N}$  and  $t \in \mathbb{R}$  there exists  $\tau_k \in [t-1, t]$  such that

$$\dot{q}_k(\tau_k) = \int_{t-1}^t \ddot{q}_k(s) ds = q_k(t) - q_k(t-1).$$

In consequence, combining the above with (25) and (26)

$$\begin{aligned} |\dot{q}_k(t)| &= \left| \int_{\tau_k}^t \ddot{q}_k(s) ds + \dot{q}_k(\tau_k) \right| \\ &\leq \int_{t-1}^t |\ddot{q}_k(s)| ds + |q_k(t) - q_k(t-1)| \leq M_3 + 2M_2 \equiv M_4, \end{aligned}$$

and hence for every  $k \in \mathbb{N}$

$$\|\dot{q}_k\|_{L_{2kT}^\infty} \leq M_4. \quad (27)$$



The task is now to show that  $\{q_k\}_{k \in \mathbb{N}}$  and  $\{\dot{q}_k\}_{k \in \mathbb{N}}$  are equicontinuous. Of course, it suffices to prove that both sequences satisfy the Lipschitz condition with some constants independent of  $k$ . Let  $k \in \mathbb{N}$  and  $t, t_0 \in \mathbb{R}$ . Then

$$|q_k(t) - q_k(t_0)| = \left| \int_{t_0}^t \dot{q}_k(s) ds \right| \leq \left| \int_{t_0}^t |\dot{q}_k(s)| ds \right| \leq M_4 |t - t_0|,$$

by (27), and analogously,

$$|\dot{q}_k(t) - \dot{q}_k(t_0)| \leq M_3 |t - t_0|,$$

by (26). Since  $\{q_k\}_{k \in \mathbb{N}}$  and  $\{\dot{q}_k\}_{k \in \mathbb{N}}$  are bounded in  $L^\infty_{2kT}(\mathbb{R}, \mathbb{R}^n)$  and equicontinuous, we obtain the existence of a subsequence  $\{q_{\varphi(k)}\}_{k \in \mathbb{N}}$  convergent to a certain  $q_0$  in  $C^1_{loc}(\mathbb{R}, \mathbb{R}^n)$  by using the Arzelà-Ascoli theorem.  $\square$

Our next goal is to show that  $q_0$  is the desired homoclinic solution of (HS). For this purpose, we need the following observations.

**Fact 2.7.** *Let  $q: \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous mapping. If a weak derivative  $\dot{q}: \mathbb{R} \rightarrow \mathbb{R}^n$  is continuous at  $t_0$ , then  $q$  is differentiable at  $t_0$  and*

$$\lim_{t \rightarrow t_0} \frac{q(t) - q(t_0)}{t - t_0} = \dot{q}(t_0).$$

**Proof.** Fix  $\varepsilon > 0$ . By the assumption, there exists  $\delta > 0$  such that for every  $t \in \mathbb{R}$ , if  $|t - t_0| < \delta$  then  $|\dot{q}(t) - \dot{q}(t_0)| < \varepsilon$ . Hence

$$\left| \frac{q(t) - q(t_0)}{t - t_0} - \dot{q}(t_0) \right| = \left| \frac{\int_{t_0}^t (\dot{q}(s) - \dot{q}(t_0)) ds}{t - t_0} \right| \leq \frac{\int_{t_0}^t |\dot{q}(s) - \dot{q}(t_0)| ds}{|t - t_0|} \leq \varepsilon$$

provided that  $0 < |t - t_0| < \delta$ .  $\square$

Let  $L^2_{loc}(\mathbb{R}, \mathbb{R}^n)$  denote the space of functions on  $\mathbb{R}$  with values in  $\mathbb{R}^n$  locally square integrable.

**Fact 2.8.** *Let  $q: \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous mapping such that  $\dot{q} \in L^2_{loc}(\mathbb{R}, \mathbb{R}^n)$ . For every  $t \in \mathbb{R}$  the following inequality holds:*

$$|q(t)| \leq \sqrt{2} \left( \int_{t-1/2}^{t+1/2} (|q(s)|^2 + |\dot{q}(s)|^2) ds \right)^{1/2}. \tag{28}$$

**Proof.** Fix  $t \in \mathbb{R}$ . For every  $\tau \in \mathbb{R}$ ,

$$|q(t)| \leq |q(\tau)| + \left| \int_{\tau}^t \dot{q}(s) ds \right|. \quad (29)$$

Integrating (29) over  $[t - \frac{1}{2}, t + \frac{1}{2}]$  and using the Hölder inequality we obtain

$$\begin{aligned} |q(t)| &\leq \int_{t-1/2}^{t+1/2} \left( |q(\tau)| + \left| \int_{\tau}^t \dot{q}(s) ds \right| \right) d\tau \\ &\leq \left( \int_{t-1/2}^{t+1/2} \left( |q(\tau)| + \left| \int_{\tau}^t \dot{q}(s) ds \right| \right)^2 d\tau \right)^{1/2} \\ &\leq \left( 2 \int_{t-1/2}^{t+1/2} \left( |q(\tau)|^2 + \left| \int_{\tau}^t \dot{q}(s) ds \right|^2 \right) d\tau \right)^{1/2} \\ &\leq \sqrt{2} \left( \int_{t-1/2}^{t+1/2} |q(\tau)|^2 d\tau + \int_{t-1/2}^{t+1/2} |\dot{q}(s)|^2 ds \right)^{1/2}. \quad \square \end{aligned}$$

**Lemma 2.9.** *The function  $q_0$  determined by Lemma 2.6 is the desired homoclinic solution of (HS).*

**Proof.** The proof will be divided into four steps.

*Step 1:* We show that  $q_0$  is a solution of (HS). For every  $k \in \mathbb{N}$  and  $t \in \mathbb{R}$  we have

$$\ddot{q}_{\varphi(k)}(t) = f_{\varphi(k)}(t) - V_q(t, q_{\varphi(k)}(t)). \quad (30)$$

Since  $q_{\varphi(k)} \rightarrow q_0$  and  $f_{\varphi(k)} \rightarrow f$  almost uniformly on  $\mathbb{R}$ , we obtain that  $\ddot{q}_{\varphi(k)} \rightarrow w$  almost uniformly on  $\mathbb{R}$ , where  $w(t) = f(t) - V_q(t, q_0(t))$ . Fix  $a, b \in \mathbb{R}$  such that  $a < b$ . There is  $k_0 \in \mathbb{N}$  such that for every  $k \geq k_0$  and  $t \in [a, b]$ , (30) becomes

$$\ddot{q}_{\varphi(k)}(t) = f(t) - V_q(t, q_{\varphi(k)}(t)).$$

Hence, if  $k \geq k_0$  then the restriction of  $\ddot{q}_{\varphi(k)}$  onto  $[a, b]$  is continuous. From Fact 2.7 it follows that  $\ddot{q}_{\varphi(k)}$  is a derivative of  $\dot{q}_{\varphi(k)}$  in  $(a, b)$  for every  $k \geq k_0$ . Since  $\ddot{q}_{\varphi(k)} \rightarrow w$  and  $\dot{q}_{\varphi(k)} \rightarrow \dot{q}_0$  almost uniformly on  $\mathbb{R}$ , we have  $w = \ddot{q}_0$  in  $(a, b)$ . By the above, we conclude that  $w = \ddot{q}_0$  in  $\mathbb{R}$  and  $q_0$  satisfies (HS). Moreover, note that we have actually proved that  $\{q_{\varphi(k)}\}_{k \in \mathbb{N}}$  converges to  $q_0$  in the topology of  $C_{loc}^2(\mathbb{R}, \mathbb{R}^n)$ .



Step 2: We prove that  $q_0(t) \rightarrow 0$ , as  $t \rightarrow \pm\infty$ . We have

$$\begin{aligned} \int_{-\infty}^{+\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt &= \lim_{i \rightarrow +\infty} \int_{-iT}^{iT} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \\ &= \lim_{i \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{-iT}^{iT} (|q_{\varphi(k)}(t)|^2 + |\dot{q}_{\varphi(k)}(t)|^2) dt. \end{aligned}$$

Clearly, for every  $i \in \mathbb{N}$  there exists  $k_i \in \mathbb{N}$  such that for all  $k \geq k_i$  we have

$$\int_{-iT}^{iT} (|q_{\varphi(k)}(t)|^2 + |\dot{q}_{\varphi(k)}(t)|^2) dt \leq \|q_{\varphi(k)}\|_{E_{\varphi(k)}}^2 \leq M_1^2,$$

by (24). Letting  $k \rightarrow +\infty$ , we get

$$\int_{-iT}^{iT} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \leq M_1^2,$$

and now, letting  $i \rightarrow +\infty$ , we have

$$\int_{-\infty}^{+\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \leq M_1^2,$$

and so

$$\int_{|t| \geq r} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \rightarrow 0, \tag{31}$$

as  $r \rightarrow +\infty$ . Combining (31) with (28) we receive our claim.

Step 3: We now show that  $\dot{q}_0(t) \rightarrow 0$ , as  $t \rightarrow \pm\infty$ . To do this, observe that

$$|\dot{q}_0(t)|^2 \leq 2 \int_{t-1/2}^{t+1/2} (|q_0(s)|^2 + |\dot{q}_0(s)|^2) ds + 2 \int_{t-1/2}^{t+1/2} |\ddot{q}_0(s)|^2 ds, \tag{32}$$

by (28). Since we have (31) and (32) it suffices to prove that

$$\int_r^{r+1} |\ddot{q}_0(s)|^2 ds \rightarrow 0, \tag{33}$$

as  $r \rightarrow \pm\infty$ . By (HS) we obtain

$$\begin{aligned} \int_r^{r+1} |\ddot{q}_0(s)|^2 ds &= \int_r^{r+1} (|V_q(s, q_0(s))|^2 + |f(s)|^2) ds \\ &\quad - 2 \int_r^{r+1} (V_q(s, q_0(s)), f(s)) ds. \end{aligned}$$



Since  $V_q(t, 0) = 0$  for all  $t \in \mathbb{R}$ ,  $q_0(t) \rightarrow 0$ , as  $t \rightarrow \pm\infty$  and  $\int_r^{r+1} |f(s)|^2 ds \rightarrow 0$ , as  $r \rightarrow \pm\infty$ , (33) follows.

*Step 4:* In the end, we have to show that if  $f \equiv 0$  then  $q_0 \not\equiv 0$ . For this purpose, as Rabinowitz we use the properties of  $Y$  given by (6). The definition of  $Y$  implies

$$\int_{-kT}^{kT} (W_q(t, q_k(t)), q_k(t)) dt \leq Y(\|q_k\|_{L_{2kT}^\infty}) \|q_k\|_{E_k}^2 \quad (34)$$

for every  $k \in \mathbb{N}$ . Since  $I'_k(q_k)q_k = 0$ , (10) gives

$$\int_{-kT}^{kT} (W_q(t, q_k(t)), q_k(t)) dt = \int_{-kT}^{kT} |\dot{q}_k(t)|^2 dt + \int_{-kT}^{kT} (K_q(t, q_k(t)), q_k(t)) dt. \quad (35)$$

Substituting (35) into (34), and next applying  $(H_3)$  and  $(H_2)$  we obtain

$$Y(\|q_k\|_{L_{2kT}^\infty}) \|q_k\|_{E_k}^2 \geq \min\{1, b_1\} \|q_k\|_{E_k}^2,$$

and hence

$$Y(\|q_k\|_{L_{2kT}^\infty}) \geq \min\{1, b_1\} > 0. \quad (36)$$

The remainder of the proof is the same as in [12]. If  $\|q_k\|_{L_{2kT}^\infty} \rightarrow 0$ , as  $k \rightarrow +\infty$ , we would have  $Y(0) \geq \min\{1, b_1\} > 0$ , a contradiction. Thus there is  $\gamma > 0$  such that

$$\|q_k\|_{L_{2kT}^\infty} \geq \gamma \quad (37)$$

for every  $k \in \mathbb{N}$ . Clearly,  $q_k(t + jT)$  is a  $2kT$ -periodic solution of  $(HS_k)$  for every  $j \in \mathbb{Z}$ . By replacing earlier, if necessary,  $q_k$  by  $q_k(t + jT)$  for some  $j \in [-k, k] \cap \mathbb{Z}$ , one can assume that the maximum of  $q_k$  occurs in  $[-T, T]$ . Suppose, contrary to our claim, that  $q_0 \equiv 0$ . Then, by Lemma 2.6,

$$\|q_{\varphi(k)}\|_{L_{2\varphi(k)T}^\infty} = \max_{t \in [-T, T]} |q_{\varphi(k)}(t)| \rightarrow 0,$$

which contradicts (37).  $\square$

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