

## On the doubly connected domination number of a graph

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Received 3 January 2005; accepted 19 September 2005

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**Abstract:** For a given connected graph  $G = (V, E)$ , a set  $D \subseteq V(G)$  is a *doubly connected dominating set* if it is dominating and both  $\langle D \rangle$  and  $\langle V(G) - D \rangle$  are connected. The cardinality of the minimum doubly connected dominating set in  $G$  is the *doubly connected domination number*. We investigate several properties of doubly connected dominating sets and give some bounds on the doubly connected domination number.

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*Keywords:* Doubly connected domination number, connected domination number

*MSC (2000):* 05C69

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### 1 Introduction

Let  $G = (V, E)$  be a simple connected graph with  $|V(G)| = n(G)$  and  $|E(G)| = m(G)$ . The *neighbourhood*  $N_G(v)$  of a vertex  $v$  is the set of all vertices adjacent to  $v$  in  $G$  and the *closed neighbourhood*  $N_G[v] = N_G(v) \cup \{v\}$ . The *degree*  $d_G(v) = |N_G(v)|$  of a vertex  $v$  is the number of edges incident to  $v$  in  $G$ . The *minimum* and *maximum degrees* of vertices of  $V(G)$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. A vertex  $x$  such that  $d_G(x) = \Delta(G) = n(G) - 1$  we call a *universal vertex*. Let  $\Omega(G)$  be the set of all end-vertices of  $G$ , that is the set of vertices degree 1, and let  $n_1(G)$  be the cardinality of  $\Omega(G)$ .

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A vertex that is a neighbour of an end-vertex is called a *support*. Let  $S(G)$  be the set of supports in  $G$ .

The *corona*  $G = H \circ K_1$  is the graph constructed from a copy of  $H$ , where for each vertex  $v \in V(H)$ , a new vertex  $v'$  and a pendant edge  $vv'$  are added. For disjoint graphs  $G_1$  and  $G_2$ , the *join*  $G = G_1 + G_2$  is the graph  $G$  with  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \wedge v \in V(G_2)\}$ . Let us denote by  $G - v$  the graph obtained from  $G$  by removing the vertex  $v \in V(G)$  and all edges incident to  $v$ .

For any connected graph  $G$ , a vertex  $x \in V(G)$  is called a *cut-vertex* of  $G$  if  $G - x$  is no longer connected. The *vertex-connectivity* or simply *connectivity*  $\kappa(G)$  is the minimum number of vertices whose removal from  $G$  results in disconnected graph or a graph with only one vertex.

A set  $D \subseteq V(G)$  is a *dominating set* of  $G$  if for every vertex  $v \in V(G) - D$ , there exists a vertex  $u \in D$  such that  $v$  is adjacent to  $u$ . The minimum cardinality of a dominating set in  $G$  is the *domination number*  $\gamma(G)$ .

Sampathkumar and Walikar [7] defined a *connected dominating set*  $D$  to be a dominating set whose induced subgraph  $\langle D \rangle$  is connected. The minimum cardinality of a connected dominating set in  $G$  is the *connected domination number* of  $G$  and is denoted by  $\gamma_c(G)$ .

In this paper we introduce a new type of domination: a set  $D \subseteq V(G)$  is a *doubly connected dominating set* of  $G$  if it is dominating and both  $\langle D \rangle$  and  $\langle V(G) - D \rangle$  are connected. The cardinality of a minimum doubly connected dominating set of  $G$  is the *doubly connected domination number* of  $G$  and is denoted by  $\gamma_{cc}(G)$ . We define that for each connected graph  $G$  the set of all vertices of  $G$  is a doubly connected dominating set of  $G$ .

For unexplained terms and symbols see [1, 5].

## 2 Preliminary results

We begin with some basic properties of doubly connected dominating sets.

**Proposition 2.1.** *Let  $D$  be a minimum doubly connected dominating set of a connected graph  $G$  on  $n \geq 3$  vertices. Then*

- (i) *every cut-vertex is in  $D$ ;*
- (ii) *every support is in  $D$ ;*
- (iii) *at least  $n_1(G) - 1$  end-vertices are in  $D$ ;*
- (iv)  *$\gamma_{cc}(G) \geq n_1(G)$ , with equality if and only if  $G$  is a star  $K_{1,n-1}$ ;*
- (v)  *$\gamma_{cc}(G) \geq n_1(G) + |S(G)| - 1$ , with equality if and only if each vertex  $v \in V(G)$  is either an end-vertex or a support.*

**Proof.** (i) Assume  $v$  is a cut-vertex of  $G$  that does not belong to a minimum doubly connected dominating set  $D$ . As  $G - v$  is disconnected, it is not possible to choose a connected dominating set  $D \subseteq V(G) - \{v\}$ , a contradiction.

- (ii) As every support is a cut-vertex, by (i) our claim follows.
- (iii) If not, assume there are two end-vertices not belonging to  $D$ . As every support is in  $D$  it follows, that  $\langle V(G) - D \rangle$  is not connected, a contradiction.
- (iv) By (iii), at least  $n_1(G) - 1$  end-vertices are in  $D$ . If  $\Omega(G) \subseteq D$  our claim follows. Similarly, if there exists a vertex  $x \in \Omega(G)$  such that  $x \notin D$ , then  $\gamma_{cc}(G) = n(G) - 1$  and since  $n \geq 3$  we have  $n(G) - 1 \geq n_1(G)$ , which completes the proof of the bound.
- It is easy to see that  $\gamma_{cc}(K_{1,n-1}) = n_1(G)$ . Conversely, assume that  $\gamma_{cc}(G) = n_1(G)$ . In this case, by (ii) and (iii), each support and at least all end-vertices except one are in a minimum doubly connected dominating set  $D$ . Thus  $|S(G)| = 1$ ,  $|V(G) - D| = 1$  and  $|D| = n - 1 = n_1(G)$ . We conclude  $G$  is a star  $K_{1,n-1}$ .
- (v) By (ii) and (iii), the inequality is straightforward. If  $\Omega(G) \cup S(G) = V(G)$  then obviously  $n_1(G) + |S(G)| - 1 = \gamma_{cc}(G)$ . Conversely, assume that  $\gamma_{cc}(G) = n_1(G) + |S(G)| - 1$ . In this case, the minimum doubly connected dominating set  $D$  consists of all vertices of the set  $S(G)$  and all except one end-vertices. Thus  $\gamma_{cc}(G) = n - 1$  and  $V(G) = S(G) \cup \Omega(G)$ .

□

As an immediate consequence of Proposition 2.1 we have

**Corollary 2.2.** *If  $G = H \circ K_1$ , then  $\gamma_{cc}(G) = n(G) - 1$ .*

**Corollary 2.3.** *For a tree  $T$  on  $n \geq 3$  vertices,  $\gamma_{cc}(T) = n - 1$ .*

**Proof.** In a tree  $T$  each vertex is either a cut-vertex or an end-vertex. By Proposition 2.1, we conclude that  $\gamma_{cc}(T) \geq n - 1$ . On the other hand, if  $x$  is an end-vertex of a tree  $T$ , then  $D = V(T) - \{x\}$  is a doubly connected dominating set. Thus,  $\gamma_{cc}(T) = n - 1$ . □

Since every doubly connected dominating set is a connected dominating and every connected dominating set is dominating, we have the following inequality chain for every connected graph  $G$ :

$$\gamma(G) \leq \gamma_c(G) \leq \gamma_{cc}(G).$$

We characterize now some graphs for which the numbers  $\gamma_{cc}(G)$  and  $\gamma_c(G)$  are the same.

**Proposition 2.4.** *Let  $G$  be a connected graph on  $n \geq 3$  vertices.*

- (i) *If  $G$  is a cycle, then  $\gamma_{cc}(G) = \gamma_c(G) = n - 2$ .*
- (ii) *If  $\gamma_{cc}(G) = \gamma_c(G)$ , then  $\gamma_{cc}(G) \leq n - 2$ .*
- (iii) *If  $\gamma_{cc}(G) = \gamma_c(G)$ , then  $\delta(G) \geq 2$ .*
- (iv) *For any unicyclic graph  $G$  we have  $\gamma_c(G) = \gamma_{cc}(G)$  if and only if  $G$  is a cycle.*

**Proof.** (i) It is obvious.

(ii) It is known [7] that for every connected graph  $G$  with  $n \geq 3$  we have  $\gamma_c(G) \leq n - 2$ .

Thus, for the equality  $\gamma_{cc}(G) = \gamma_c(G)$  we conclude that  $\gamma_{cc}(G) \leq n - 2$ .

(iii) Suppose  $\gamma_{cc}(G) = \gamma_c(G)$  and  $x$  is an end-vertex in  $G$ . By (ii),  $\gamma_{cc}(G) \leq n - 2$ . Let  $D$

be a minimum doubly connected dominating set of  $G$  of cardinality  $|D| \leq n - 2$ . If  $x \in D$ , then  $D - \{x\}$  is also a connected dominating set of  $G$ , a contradiction with equality  $\gamma_{cc}(G) = \gamma_c(G)$ . If  $x \notin D$ , then  $x$  is the unique vertex in  $V(G) - D$ , because  $\langle V(G) - D \rangle$  is connected. Thus  $\gamma_{cc}(G) = n - 1$ , a contradiction.

(iv) If  $G$  is a cycle on  $n$  vertices, then by (i)  $\gamma_{cc}(G) = \gamma_c(G) = n - 2$ . Suppose now  $G$  is unicyclic with  $\gamma_c(G) = \gamma_{cc}(G)$  and  $G$  is not a cycle. By (ii),  $\gamma_{cc}(G) \leq n - 2$ . Moreover, there exists a vertex  $x \in V(G)$  such that  $d_G(x) = 1$ . Let  $D$  be a minimum doubly connected dominating set of  $G$ . If  $x \notin D$ , then  $\gamma_{cc}(G) = n - 1$ , a contradiction. On the other hand,  $x \in D$  implies, that  $D - \{x\}$  is a connected dominating set of  $G$ , a contradiction, as  $\gamma_c(G) = \gamma_{cc}(G)$ . □

We have shown that there exist graphs  $G$  for which the equality  $\gamma_c(G) = \gamma_{cc}(G)$  holds. However the difference between  $\gamma_{cc}(G)$  and  $\gamma_c(G)$  can be arbitrarily large.

**Lemma 2.5.** *The difference  $\gamma_{cc} - \gamma_c$  can be arbitrarily large.*

**Proof.** Consider a star  $K_{1,n-1}$  with  $n - 1$  end-vertices. Of course,  $\gamma_c(K_{1,n-1}) = 1$ . By Proposition 2.1,  $\gamma_{cc}(K_{1,n-1}) = n - 1$ . Thus  $\gamma_{cc}(K_{1,n-1}) - \gamma_c(K_{1,n-1}) = n - 2$ . □

**Observation 2.6.** Let  $G = K_{m_1, m_2, \dots, m_k}$  be the complete  $k$  partite graph,  $k \geq 3$  with  $m_1 \leq m_2 \leq \dots \leq m_k$ .

- If  $m_1 = 1$ , then  $\gamma_{cc}(G) = 1$ ;
- If  $m_1 \geq 2$ , then  $\gamma_{cc}(G) = 2$ .

**Observation 2.7.** If  $G_1$  and  $G_2$  are disjoint connected graphs, then

$$\gamma_{cc}(G_1 + G_2) = \begin{cases} 1 & \text{if } \gamma_{cc}(G_1) = 1 \text{ or } \gamma_{cc}(G_2) = 1; \\ 2 & \text{otherwise.} \end{cases}$$

A connected subgraph  $B$  of  $G$  is called a *block* if  $B$  has no cut-vertex and every subgraph  $B' \subseteq G$  with  $B \subseteq B'$  and  $B \neq B'$  has at least one cut-vertex. A connected graph  $G$  is called a *block graph* if every block in  $G$  is complete. A vertex  $v$  of a graph  $G$  is called a *simplicial vertex* if every two vertices of  $N_G(v)$  are adjacent in  $G$ .

**Theorem 2.8.** *If  $G$  is a block graph, then  $\gamma_{cc}(G) = n(G) - t$ , where  $t$  is the maximal number of simplicial vertices in a block with a largest number of simplicial vertices.*

**Proof.** Let  $D$  be a minimum doubly connected dominating set of a block graph  $G$ . By Proposition 2.1, each cut-vertex belongs to  $D$ . Hence  $\gamma_{cc}(G) \geq n(G) - t$ , where  $t$  is maximal number of simplicial vertices in a block with a largest number of simplicial vertices.

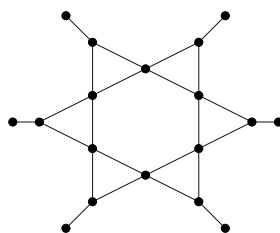
Conversely, let  $B$  be a block with a largest number of simplicial vertices. Denote by  $F$  the set of all simplicial vertices belonging to  $B$  and let  $|F| = t$ . Then  $V(G) - F$  is a doubly



connected dominating set of  $G$  and we have  $\gamma_{cc}(G) \leq n(G) - t$ . Thus  $\gamma_{cc}(G) = n(G) - t$ .  $\square$

### 3 Bounds

Now we find some bounds on the doubly connected domination number. For this purpose, denote by  $\mathcal{A}$  a family of graphs such that  $K_2 \in \mathcal{A}$  and  $G$  belongs to  $\mathcal{A}$  if and only if for each pair of adjacent non-cut-vertices  $u, v \in V(G)$ ,  $\langle V(G) - \{u, v\} \rangle$  is disconnected.



**Fig. 1** A graph  $G \in \mathcal{A}$ .

**Theorem 3.1.** For every connected graph  $G$  on  $n \geq 2$  vertices,

$$1 \leq \gamma_{cc}(G) \leq n - 1$$

with equality for the lower bound if and only if there exists a connected graph  $H$  such that  $G = H + K_1$  and equality for the upper bound if and only if  $G \in \mathcal{A}$ .

**Proof.** The inequality  $1 \leq \gamma_{cc}(G)$  is obvious. If  $G = H + K_1$  and  $H$  is connected, then obviously  $\gamma_{cc}(G) = 1$ . Assume now that  $\gamma_{cc}(G) = 1$  and let  $D = \{x\}$  be a minimum doubly connected dominating set of  $G$ . Since  $D$  is dominating,  $x$  must be a universal vertex. Moreover,  $\langle V(G) - D \rangle = \langle V(G) - \{x\} \rangle$  is connected, so  $x$  is a non-cut-vertex. We conclude that  $G = H + K_1$ , where  $H$  is connected.

Now we prove that  $\gamma_{cc}(G) \leq n - 1$ . The inequality and the equality are straightforward when  $G = K_2$ . Suppose  $n \geq 3$ . Then there exist in  $G$  at least two non-cut-vertices, for example two leaves of a spanning tree of  $G$ . Let  $x$  be a non-cut-vertex. Then  $D = V(G) - \{x\}$  is a doubly connected dominating set of  $G$ .

If  $G \in \mathcal{A}$ , then  $\gamma_{cc}(G) = n - 1$ , because every support of  $G$  is in  $D$  and for each pair of adjacent non-cut-vertices  $u, v \in V(G)$ , the induced subgraph  $\langle V(G) - \{u, v\} \rangle$  is disconnected. Now let  $G \notin \mathcal{A}$ . It suffices to show that  $\gamma_{cc}(G) \leq n - 2$ . If  $G \notin \mathcal{A}$ , then there exist adjacent non-cut-vertices  $u, v \in V(G)$  such that  $\langle V(G) - \{u, v\} \rangle$  is connected. In this case  $D = V(G) - \{u, v\}$  is a doubly connected dominating set of  $G$ , as  $n \geq 3$ ,  $G$  is connected and neither of  $u, v$  is a support.  $\square$

**Proposition 3.2.** Let  $G$  be a connected graph on  $n \geq 2$  vertices. Then  $\gamma_{cc}(G) \leq n - \kappa(G) + 1$ .

**Proof.** If  $\kappa(G) \leq 2$ , then by Theorem 3.1 our claim follows. Thus assume now  $\kappa(G) \geq 3$ . It is obvious that  $\kappa(G) \leq \delta(G)$ . Let  $A$  be a set of an arbitrary vertex  $x \in V(G)$  and  $\kappa(G) - 2$  of its neighbours. Obviously,  $\langle V(G) - A \rangle$  is connected. Observe that  $D = V(G) - A$  is dominating in  $G$ . Thus  $D$  is a doubly connected dominating set in  $G$  with  $|D| = n - \kappa(G) + 1$ .  $\square$

In [7] Sampathkumar and Walikar showed that for every connected graph  $G$  with  $n \geq 3$  vertices and  $m$  edges we have inequalities  $\frac{n}{\Delta(G)+1} \leq \gamma_c(G) \leq 2m - n$ . Now we present similar inequalities for the number  $\gamma_{cc}$ .

**Theorem 3.3.** For any connected graph  $G$  with  $n \geq 2$  vertices and  $m$  edges,

$$\frac{n}{\Delta(G) + 1} \leq \gamma_{cc}(G) \leq 2m - n + 1$$

with equality for the lower bound if and only if  $\gamma_{cc}(G) = 1$  and equality for the upper bound if and only if  $G$  is a tree.

**Proof.** Since  $\frac{n}{\Delta(G)+1} \leq \gamma_c(G) \leq \gamma_{cc}(G)$  the lower bound follows. If  $\gamma_{cc}(G) = 1$ , then by Theorem 3.1 there exists a vertex  $v \in V(G)$  such that  $d_G(v) = n - 1$ . Thus  $\frac{n}{\Delta(G)+1} = 1 = \gamma_{cc}(G)$ .

Conversely, let  $G$  be a graph such that  $\gamma_{cc}(G) = \frac{n}{\Delta(G)+1}$  and  $\gamma_{cc}(G) > 1$ . Let  $D$  be a minimum doubly connected dominating set of  $G$ . Since  $\langle D \rangle$  is connected, for each  $v \in D$  we have  $|N_G(v) \cap (V(G) - D)| \leq \Delta(G) - 1$ . Hence  $|V(G) - D| \leq (\Delta(G) - 1)|D|$  and  $n - \gamma_{cc}(G) \leq (\Delta(G) - 1)\gamma_{cc}(G)$ , which gives  $\gamma_{cc}(G) \geq \frac{n}{\Delta(G)}$ , a contradiction.

By Theorem 3.1,  $\gamma_{cc}(G) \leq n - 1 = 2(n - 1) - n + 1$  and since  $G$  is connected,  $m \geq n - 1$ . Thus  $\gamma_{cc}(G) \leq 2m - n + 1$ .

We now show that  $\gamma_{cc}(G) = 2m - n + 1$  if and only if  $G$  is a tree. If  $G$  is a tree, then  $m = n - 1$  and  $\gamma_{cc}(G) = n - 1 = 2m - n + 1$ . Conversely, let  $\gamma_{cc}(G) = 2m - n + 1$ . By Theorem 3.1 we have  $2m - n + 1 \leq n - 1$ , which implies  $m \leq n - 1$ , so  $G$  must be a tree with  $m = n - 1$ .  $\square$

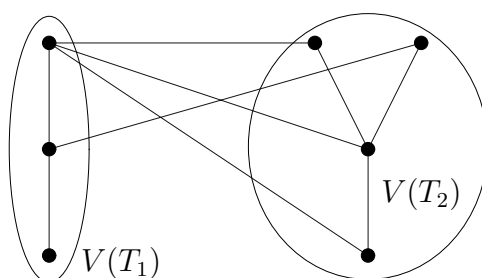
As an immediate consequence of the second paragraph of the proof of Theorem 3.3 we have what follows.

**Corollary 3.4.** For each connected graph  $G$  with  $\gamma_{cc}(G) > 1$  is  $\gamma_{cc}(G) \geq \frac{n}{\Delta(G)}$ .

Now we introduce the following notation: if  $T_1$  and  $T_2$  are vertex disjoint trees, then by  $\mathcal{P}(T_1, T_2)$  we denote the set of all graphs  $G$  that can be obtained from  $T_1$  and  $T_2$  by adding  $n(T_2)$  edges, one edge joining each vertex of  $T_2$  to one arbitrarily chosen vertex of  $T_1$ . We say that a graph  $G$  belongs to the family  $\mathcal{U}$  if there exist trees  $T_1$  and  $T_2$  such that  $G \in \mathcal{P}(T_1, T_2)$ .

**Theorem 3.5.** For any connected graph  $G$  on  $n \geq 2$  vertices and with  $m$  edges,

$$2n - m - 2 \leq \gamma_{cc}(G)$$



**Fig. 2** A graph  $G \in \mathcal{P}(T_1, T_2)$ .

with equality for the bound if and only if  $G$  belongs to the family  $\mathcal{U}$ .

**Proof.** Let  $D$  be a minimum doubly connected dominating set in  $G$ . Since  $\langle D \rangle$  and  $\langle V(G) - D \rangle$  are connected and  $D$  is dominating, we have the following inequalities:

$$\begin{aligned} m(\langle D \rangle) &\geq \gamma_{cc}(G) - 1, \\ m(\langle V(G) - D \rangle) &\geq n - \gamma_{cc}(G) - 1, \\ m_{\gamma_{cc}} &\geq n - \gamma_{cc}(G), \end{aligned}$$

where  $m_{\gamma_{cc}}$  is the number of the edges connecting vertices of  $V(G) - D$  to vertices of  $D$ . By summing the inequalities we obtain

$$m = m(\langle D \rangle) + m(\langle V - D \rangle) + m_{\gamma_{cc}} \geq 2n - \gamma_{cc}(G) - 2$$

and thus  $2n - m - 2 \leq \gamma_{cc}(G)$ .

We now show that  $\gamma_{cc}(G) = 2n - m - 2$  if and only if  $G$  belongs to the family  $\mathcal{U}$ . Let  $G \in \mathcal{U}$ . Then there exist trees  $T_1$  and  $T_2$  such that  $G \in \mathcal{P}(T_1, T_2)$ . In such a graph  $G$  the set  $V(T_1)$  is a doubly connected dominating set. Thus  $\gamma_{cc}(G) \leq n(T_1)$ . Of course  $n = n(T_1) + n(T_2)$  and

$$m = m(T_1) + m(T_2) + n(T_2) = n(T_1) - 1 + n(T_2) - 1 + n(T_2) = n(T_1) + 2n(T_2) - 2.$$

It follows that

$$2n - m - 2 = 2(n(T_1) + n(T_2)) - (n(T_1) + 2n(T_2) - 2) - 2 = n(T_1).$$

Consequently  $\gamma_{cc}(G) \geq n(T_1)$ , which together with  $\gamma_{cc}(G) \leq n(T_1)$  gives  $\gamma_{cc}(G) = n(T_1) = 2n - m - 2$ .

Conversely, suppose  $\gamma_{cc}(G) = 2n - m - 2$ . This implies that

$$\begin{aligned} m(\langle D \rangle) &= \gamma_{cc}(G) - 1 = n(\langle D \rangle) - 1, \\ m(\langle V(G) - D \rangle) &= n - \gamma_{cc}(G) - 1 = n(\langle V(G) - D \rangle) - 1, \\ m_{\gamma_{cc}} &= n - \gamma_{cc}(G). \end{aligned}$$

It follows that  $\langle D \rangle$  and  $\langle V(G) - D \rangle$  are trees and each vertex of  $V(G) - D$  has exactly one neighbour in  $D$ . Thus  $G$  is a graph obtained from two trees  $T_1$  and  $T_2$  by adding  $n(T_2)$  edges, one edge joining each vertex of  $T_2$  to one arbitrarily chosen vertex of  $T_1$ .  $\square$



Duchet and Meyniel [3] have shown that for any connected graph  $G$  is  $\gamma_c(G) \leq 2\beta_0(G) - 1$  and  $\gamma_c(G) \leq 2\Gamma(G) - 1$ , where  $\Gamma(G)$  is the maximum cardinality of a minimal dominating set of  $G$  and  $\beta_0$  is the maximum cardinality of an independent set of  $G$ . The next theorem shows that there is no similar result for the doubly connected domination number of a graph.

**Theorem 3.6.** *Each of the differences  $\gamma_{cc} - \beta_0$  and  $\gamma_{cc} - \Gamma$  can be arbitrarily large.*

**Proof.** We show a graph  $G$  for which  $\gamma_{cc}(G) - \beta_0(G) = \gamma_{cc}(G) - \Gamma(G) = k$  for any positive integer  $k$ . Let  $G$  be a corona  $K_{k+1} \circ K_1$ . It is easy to observe that the set of end-vertices  $\Omega(G)$  is the maximum independent set of  $G$  and thus  $\beta_0(G) = k + 1$ . The set  $\Omega(G)$  is also the maximum minimal dominating set of  $G$ , so  $\Gamma(G) = k + 1$ . Since  $G$  is a corona, from Corollary 2.2 we have  $\gamma_{cc}(G) = |V(G)| - 1 = 2(k + 1) - 1 = 2k + 1$ . It follows that  $\gamma_{cc}(G) - \beta_0(G) = \gamma_{cc}(G) - \Gamma(G) = k$ .  $\square$

## 4 Edge subdivision and vertex removing

Now we examine the effects on  $\gamma_{cc}(G)$  when  $G$  is modified by an edge subdivision.

An *edge subdivision* in a nonempty graph  $G$  is an operation of removal of an edge  $e = uv$  and the addition of a new vertex  $w$  and edges  $uw$  and  $vw$ . A graph obtained from  $G$  by subdividing the edge  $e = uv$  is denoted by  $G \oplus w_{uv}$ .

**Theorem 4.1.** *For every connected graph  $G$  we have  $\gamma_{cc}(G) \leq \gamma_{cc}(G \oplus w_{uv})$ .*

**Proof.** Let  $e = uv$  be the subdivided edge and let  $D_0$  be a minimum doubly connected dominating set of  $G \oplus w_{uv}$ . We consider two cases:

- $w \in D_0$ . Then, since  $\langle D_0 \rangle$  is connected,  $u$  or  $v$  belong to  $D_0$ . If both of these vertices belong to  $D_0$ , then  $D_0 - \{w\}$  is a doubly connected dominating set of  $G$  and thus  $\gamma_{cc}(G) < |D_0| = \gamma_{cc}(G \oplus w_{uv})$ . If  $u \in D_0$  and  $v \notin D_0$ , then  $D_0 - \{w\}$  is a doubly connected dominating set of  $G$  and we have the required inequality.
- $w \notin D_0$ . Then, since  $D_0$  is dominating,  $u$  or  $v$  belong to  $D_0$ . Then, similarly as in case a), we have  $\gamma_{cc}(G) \leq |D_0| = \gamma_{cc}(G \oplus w_{uv})$ .

$\square$

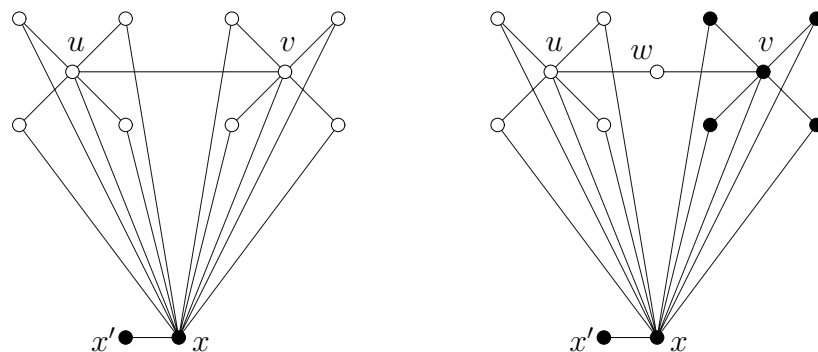
**Theorem 4.2.** *The difference  $\gamma_{cc}(G \oplus w_{uv}) - \gamma_{cc}(G)$  can be arbitrarily large.*

**Proof.** We construct graphs  $G$  and  $G \oplus w_{uv}$  for which  $\gamma_{cc}(G \oplus w_{uv}) - \gamma_{cc}(G) = k$  for a non-negative integer  $k \geq 2$ .

We begin with two stars  $K_{1,k-1}$ ,  $k \geq 2$  and denote their centers by  $u$  and  $v$ . Next we add a vertex  $x$  and edges joining  $x$  with all vertices of the stars. Finally, to obtain a graph  $G$ , we add an edge  $e = uv$  and a pendant edge  $xx'$  (see Fig. 3). It is easy to observe that the set  $D = \{x, x'\}$  is a minimum doubly connected dominating set of  $G$  and thus  $\gamma_{cc}(G) = 2$ .



For the graph  $G \oplus w_{uv}$  notice that the set  $D_u = N[v] \cup \{x'\} - \{w\}$  is a minimum doubly connected dominating set and the size of this set is  $k + 2$ . Thus  $\gamma_{cc}(G \oplus w_{uv}) - \gamma_{cc}(G) = k + 2 - 2 = k$ .



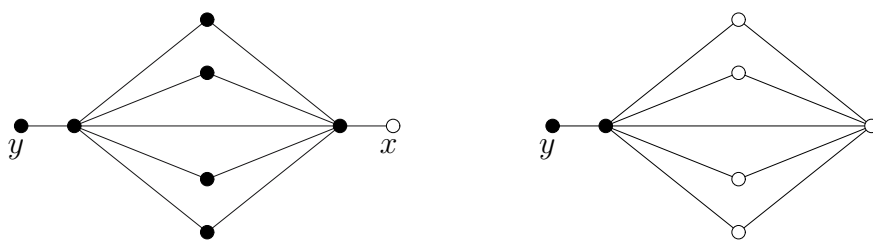
**Fig. 3** Graphs  $G$  and  $G \oplus w_{uv}$  for  $k = 5$ .

□

**Theorem 4.3.** *The difference  $\gamma_{cc}(G) - \gamma_{cc}(G - x)$  can be arbitrarily large.*

**Proof.** Let  $H$  be the join  $K_{1,k} + K_1$ ,  $k \geq 2$ , and let  $G$  be the graph that results if we add two pendant edges and two end-vertices  $x$  and  $y$  to the vertices of degree  $k + 1$  of the graph  $H$  (see Fig. 4). It is easy to observe that  $V(G) - \{x\}$  is a minimum doubly connected dominating set of  $G$ . Thus,  $\gamma_{cc}(G) = k + 3$ .

The set  $N_G[y]$  is a minimum doubly connected dominating set of  $G - x$ . Thus  $\gamma_{cc}(G - x) = 2$  and finally we have  $\gamma_{cc}(G) - \gamma_{cc}(G - x) = k + 1$ . □



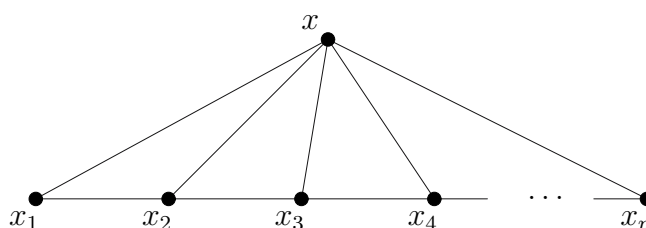
**Fig. 4** Graphs  $G$  and  $G - x$  for  $k = 4$ .

**Theorem 4.4.** *The difference  $\gamma_{cc}(G - x) - \gamma_{cc}(G)$  can be arbitrarily large.*

**Proof.** Let  $G$  be the join of a path  $P$  on  $n$  vertices and  $K_1$ . Let  $x$  be the vertex of  $K_1$ . Clearly we have  $\gamma_{cc}(G) = 1$ .

As  $G - x$  is a tree, by Corollary 2.3 we have  $\gamma_{cc}(G - x) = n - 1$ . Thus  $\gamma_{cc}(G - x) - \gamma_{cc}(G) = n - 2$ . □





**Fig. 5** Graph  $G$ .

## 5 Complexity issues for $\gamma_{cc}$

In this section we consider the decision problem of DOUBLY CONNECTED DOMINATING SET as follows

### DOUBLY CONNECTED DOMINATING SET (DCDS)

**INSTANCE:** A connected graph  $G = (V, E)$  and a positive integer  $k$ .

**QUESTION:** Does  $G$  have a doubly connected dominating set of size at most  $k$ ?

We show that the decision problem DCDS is NP-complete, even when restricted to connected bipartite graphs. We will use a well-known NP-completeness result, called DOMINATING SET, which is defined as follows.

### DOMINATING SET (DS)

**INSTANCE:** A graph  $G = (V, E)$  and a positive integer  $k$ .

**QUESTION:** Does  $G$  have a dominating set of size at most  $k$ ?

Garey and Johnson in [4] proved that DS is NP-complete.

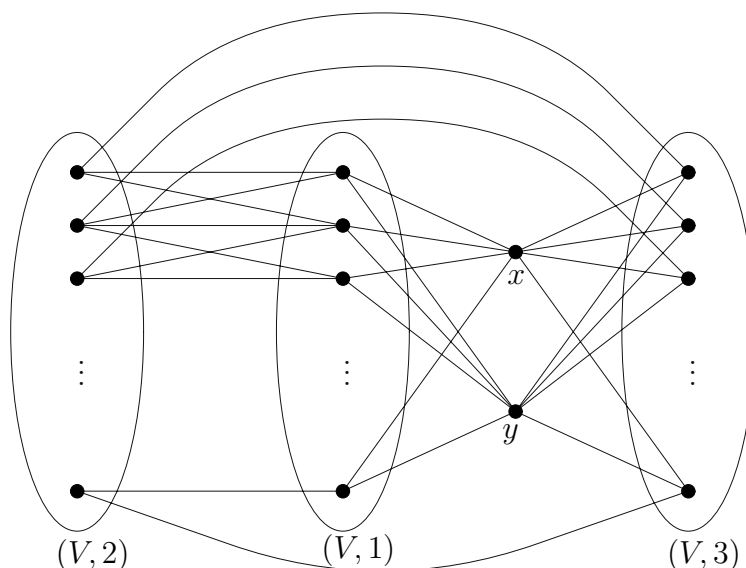
**Theorem 5.1.** *DCDS for bipartite graphs is NP-complete.*

**Proof.** We know that DCDS problem for bipartite graphs is in class NP of decision problems as it is easy to verify in polynomial time whether  $D$  is a doubly connected dominating set.

For any given instance for DS, which is a graph  $G = (V, E)$  and an integer  $k$ , we construct a graph  $H$  and an integer  $q$  as follows:

$$\begin{aligned}
V(H) &= V(G) \times \{1, 2, 3\} \cup \{x, y\}, \\
E(H) &= \{(v, 1)(v, 2) : v \in V(G)\} \\
&\cup \{(v, 2)(v, 3) : v \in V(G)\} \\
&\cup \{(v, 1)x : v \in V(G)\} \\
&\cup \{(v, 1)y : v \in V(G)\} \\
&\cup \{(v, 3)x : v \in V(G)\} \\
&\cup \{(v, 3)y : v \in V(G)\} \\
&\cup \{(v, 1)(w, 2) : vw \in E(G)\}, \\
q &= k + 1.
\end{aligned}$$

The graph  $H$  is connected and bipartite, as every cycle in  $H$  has even length. (See Figure 6).



**Fig. 6** Reduction from DS to DCDS for bipartite graphs.

Assume first that  $G$  has a dominating set  $D = \{v_1, v_2, \dots, v_{k'}\}$ ,  $k' \leq k$ , of size at most  $k$ . Let  $F = \{(v_1, 1), (v_2, 1), \dots, (v_{k'}, 1), x\}$ . Since  $x$  dominates all vertices in  $(V, 1) \cup (V, 3)$  and  $D$  is a dominating set in  $G$ , the set  $F$  is dominating in  $H$ . Moreover, from the construction of  $H$  we see that induced subgraphs  $\langle F \rangle$  and  $\langle V(H) - F \rangle$  are connected. Thus  $F$  is a doubly connected dominating set of  $H$  of size at most  $q = k + 1$ .

Conversely, assume that  $F$  is a doubly connected dominating set of cardinality at most  $q$  in  $H$ . We shall show that  $G$  contains a dominating set  $D$  of size at most  $k = q - 1$ . It is easy to see that if  $q > n(G)$ , answers for problems DCDS and DS are "yes". So assume  $q \leq n(G)$ . We claim that either vertex  $x$  or  $y$  is in every doubly connected dominating set of size  $q \leq n(G)$ , because a connected dominating set of size at most  $n(G)$  that dominates all vertices of  $(V, 3)$  and does not contain  $x$  nor  $y$  does not exist. (Observe

that in  $\langle V \times \{1, 2, 3\} \rangle$  the subset  $(V, 3)$  is a set of vertices of degree 1.) Thus assume  $x \in F$ . Moreover, every doubly connected dominating set  $F'$  of size  $q_1 \leq n(G)$  can be transformed into a doubly connected dominating set  $F \subseteq (V, 1) \cup \{x\}$  of size  $q \leq q_1$  as follows

- $x \in F$ ;
- if  $(v_i, 1) \in F'$ , then  $(v_i, 1) \in F$ ;
- if  $(v_i, 3) \in F'$ , then  $(v_i, 1) \in F$ ;
- if  $(v_i, 2) \in F'$ , then  $(v_i, 1) \in F$ .

Now, if  $F = \{(v_1, 1), (v_2, 1), \dots, (v_{q-1}, 1), x\}$  is a doubly connected dominating set of size  $q$ , then  $D = \{v_1, v_2, \dots, v_{q-1}\}$  is a dominating set in  $G$  of size  $k = q - 1$ .

It is obvious that the transformation used is polynomial, as  $H$  has  $3n(G) + 2$  vertices and  $4n(G) + 2m(G)$  edges.  $\square$

## References

- [1] J.A. Bondy and U.S.R. Murty: *Graph Theory with Applications*, Macmillan, London, 1976.
- [2] C. Bo and B. Liu: “Some inequalities about connected domination number”, *Disc. Math.*, Vol. 159, (1996), pp. 241–245.
- [3] P. Duchet and H. Meyniel: “On Hadwiger’s number and the stability number”, *Ann. Disc. Math.*, Vol. 13, (1982), pp. 71–74.
- [4] M.R. Garey and D.S. Johnson: *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman, San Francisco, 1979.
- [5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater: *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [6] S.T. Hedetniemi and R. Laskar: *Connected domination in graphs*, Graph Theory and Combinatorics, Academic Press, London, 1984, pp. 209–217.
- [7] E. Sampathkumar and H.B. Walikar: “The connected domination number of a graph”, *J. Math. Phys. Sci.*, Vol. 13, (1979), pp. 607–613.

