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First-order impulsive ordinary differential equations with advanced arguments

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Abstract

This paper deals with impulsive advanced ordinary differential equations with boundary conditions. We investigate the existence of solutions and quasisolutions for advanced impulsive differential equations. To obtain such results we apply Schauder's fixed point theorem. Corresponding results are also formulated for differential inequalities.

Keywords: Impulsive differential equations; Equations with advanced arguments; Nonlinear boundary conditions; Existence results

1. Introduction

For $J = [0, T]$, $T > 0$, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. Put $J' = J \setminus \{t_1, t_2, \dots, t_m\}$. In this paper, we investigate first-order impulsive advanced differential equations of type

$$\begin{cases} x'(t) = f\bigl(t, x(t), x\bigl(\alpha(t)\bigr)\bigr) \equiv Fx(t), & t \in J', \\ \Delta x(t_k) = I_k\bigl(x(t_k)\bigr), & k = 1, 2, \dots, m, \\ 0 = g\bigl(x(0), x(T)\bigr), \end{cases} \tag{1}$$

where as usual $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of x at t_k, respectively, and

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(H_1) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $t \leqslant \alpha(t) \leqslant T$, $t \in J$, $I_k \in C(\mathbb{R}, \mathbb{R})$ for $k = 1, 2, \ldots, m$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and if there exists a point $\bar{t} \in J$ such that $\alpha(\bar{t}) \in \{t_1, t_2, \ldots, t_m\}$, then $\bar{t} \in \{t_1, t_2, \ldots, t_m\}$.

Put $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. Let us introduce the spaces:

$$PC(J) = PC(J, \mathbb{R}) = \begin{Bmatrix} x : J \to \mathbb{R},\ x|_{J_k} \in C(J_k, \mathbb{R}),\ k = 0, 1, \ldots, m, \\ \text{and there exist } x(t_k^+) \text{ for } k = 1, 2, \ldots, m \end{Bmatrix}$$

and

$$PC^1(J) = PC^1(J, \mathbb{R}) = \begin{Bmatrix} x \in PC(J),\ x|_{J_k} \in C^1(J_k, \mathbb{R}),\ k = 0, 1, \ldots, m, \\ \text{and there exist } x'(t_k^+) \text{ for } k = 1, 2, \ldots, m \end{Bmatrix}.$$

Indeed, $PC(J)$ and $PC^1(J)$ are Banach spaces with the respective norms:

$$\|x\|_{PC} = \sup_{t \in J} \|x(t)\|, \qquad \|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}.$$

By a solution of problem (1) we mean a function $x \in PC^1(J)$ which satisfies

- the differential equation in (1) for every $t \in J'$,
- the boundary condition in problem (1) and
- at every t_k, $k = 1, 2, \ldots, m$, the function x satisfies the second condition in problem (1).

Throughout this paper we assume that $\alpha(t) \not\equiv t$, $t \in J$.

An interesting and fruitful technique for proving existence results for nonlinear differential problems is the monotone iterative method, for details, see, for example, [11]. There exists a vast literature devoted to the applications of this method to differential equations with initial and boundary conditions. This technique can also be applied to impulsive differential equations, for details, see, for example, [12]. However, only a few papers have appeared where the monotone iterative technique is applied to delay impulsive differential problems, see, for example, [2,3, 6,14]. Usually, it is assumed that the function f satisfies a one-sided Lipschitz condition with corresponding Lipschitz constants. For problems with deviating arguments, it is better to assume that the above constants are replaced by corresponding Lipschitz functions. I know only a few papers where such assumptions appeared for differential equations without impulsive, see [7–10]. I do not know any paper where it is done for impulsive problems with deviating arguments. Just in this paper the function f from problem (1) satisfies a one-sided Lipschitz condition (with respect to the last two variables) with functional coefficients and argument α being of advanced type. Note that impulsive differential equations are also discussed in papers [1,4,5,13].

The plan of this paper is as follows. In Section 2, we formulate sufficient conditions which guarantee that problem (1) has a solution. To prove Theorem 2 we apply Schauder's theorem. It is assumed that a lower solution of (1) is bigger than an upper solution. Indeed, first impulsive differential inequalities are investigated. In Section 3, we discuss existence of quasisolutions of problem (1). Given are two examples to show that the assumptions of this paper are satisfied.

2. Lower and upper solutions of problem (1)

Let us introduce the following definition.

We say that $u \in PC^1(J)$ is a lower solution of (1) if

$$\begin{cases} u'(t) \leqslant Fu(t), & t \in J', \\ \Delta u(t_k) \leqslant I_k\big(u(t_k)\big), & k = 1, 2, \ldots, m, \\ g\big(u(0), u(T)\big) \leqslant 0, \end{cases} \tag{2}$$

and it is an upper solution of (1) if the above inequalities are reversed.

We assume that $z_0(t) \leqslant y_0(t)$, $t \in J$, and define the sector

$$[z_0, y_0]_* = \big\{v \in PC^1(J, \mathbb{R}){:}\ z_0(t) \leqslant v(t) \leqslant y_0(t),\ t \in J\big\}.$$

Lemma 1. *Assume that $K \in C(J, \mathbb{R})$, $\alpha \in C(J, J)$, $t \leqslant \alpha(t) \leqslant T$, $t \in J$, and $L_k \geqslant 0$, $k = 1, 2, \ldots, m$. Let $p \in PC^1(J)$ and*

$$\begin{cases} p'(t) \geqslant K(t)p(t) + M(t)p\big(\alpha(t)\big), & t \in J', \\ \Delta p(t_k) \geqslant L_k p(t_k), & k = 1, 2, \ldots, m, \\ p(T) \leqslant 0, \end{cases} \tag{3}$$

where M is nonnegative and $M \in PC(J)$.

In addition assume that

$$\int_0^T M^*(t)\,dt \left(\prod_{i=1}^m (1 + L_i)\right) \leqslant 1 \quad \text{with } M^*(t) = M(t)e^{\int_t^{\alpha(t)} K(s)\,ds}. \tag{4}$$

Then $p(t) \leqslant 0$, $t \in J$.

Proof. Put

$$q(t) = e^{\int_t^T K(s)\,ds} p(t), \quad t \in J.$$

Then $q(T) = p(T) \leqslant 0$, $\Delta q(t_k) \geqslant L_k q(t_k)$, $k = 1, 2, \ldots, m$, and

$$q'(t) = e^{\int_t^T K(s)\,ds}\big\{-K(t)p(t) + p'(t)\big\} \geqslant e^{\int_t^T K(s)\,ds} M(t)p\big(\alpha(t)\big), \quad t \in J'.$$

Then system (3) takes the form

$$\begin{cases} q'(t) \geqslant M^*(t)q\big(\alpha(t)\big), & t \in J', \\ q\big(t_k^+\big) \geqslant (1 + L_k)q(t_k), & k = 1, 2, \ldots, m, \\ q(T) \leqslant 0. \end{cases} \tag{5}$$

Note that if $q(t) \leqslant 0$, $t \in J$, then also $p(t) \leqslant 0$ on J.

We need to prove that $q(t) \leqslant 0$, $t \in J$. Suppose that the inequality $q(t) \leqslant 0$, $t \in J$, is not true. It means that we can find $t_1^* \in [0, T)$ such that $q(t_1^*) > 0$. Then $\inf_{[t_1^*, T]} q(t) = -\rho$. Indeed, $\rho \geqslant 0$ and there exists $t_0^* \in J_p$ for some fixed p such that $q(t_0^*) = -\rho$ or $q(t_p^+) = -\rho$. Below we discuss only the situation when $q(t_0^*) = -\rho$ because in the case when $q(t_p^+) = -\rho$, the proof is similar.

Let $t_1^* \in J_r$ for some r. Indeed, $t_1^* < t_0^*$, so $r \leqslant p$.

Now, for $\sigma \in PC(J)$, we consider the following inequalities

$$\begin{cases} q'(t) \geqslant \sigma(t), & t \in \big[t_1^*, T\big] \setminus \{t_{r+1}, \ldots, t_m\}, \\ q\big(t_k^+\big) \geqslant (1 + L_k)q(t_k), & k = r + 1, \ldots, m. \end{cases}$$

Then

$$q(t) \geqslant q\big(t_1^*\big) \prod_{i=r+1}^{k}{}' (1+L_i) + \sum_{i=r+1}^{k}{}' \int_{\bar{t}_{i-1}}^{\bar{t}_i} \sigma(s)\,ds \left(\prod_{j=i}^{k}{}' (1+L_j)\right) + \int_{\bar{t}_k}^{t} \sigma(s)\,ds \tag{6}$$

for $t \in \bar{J}_k$, $k = r, r+1, \ldots, m$. Here $\bar{J}_r = [\bar{t}_r, \bar{t}_{r+1}]$, $\bar{t}_r = t_1^*$, $\bar{t}_k = t_k$, $\bar{J}_k = (\bar{t}_k, \bar{t}_{k+1}]$ for $k = r+1, \ldots, m$, and $\sum_{i=a}^{b}{}' \cdots = 0$, $\prod_{i=a}^{b}{}' \cdots = 1$ if $a > b$.

Let $\sigma(t) = M^*(t)q(\alpha(t))$. It yields $\sigma(t) \geqslant -\rho M^*(t)$, $t \in [t_1^*, T]$. Put $t = t_0^*$. Then

$$\begin{aligned} q\big(t_0^*\big) &\geqslant q\big(t_1^*\big) \prod_{i=r+1}^{p}{}' (1+L_i) + \sum_{i=r+1}^{p}{}' \int_{\bar{t}_{i-1}}^{\bar{t}_i} \sigma(s)\,ds \left(\prod_{j=i}^{p}{}' (1+L_j)\right) + \int_{\bar{t}_p}^{t_0^*} \sigma(s)\,ds \\ &> \sum_{i=r+1}^{p}{}' \int_{\bar{t}_{i-1}}^{\bar{t}_i} \sigma(s)\,ds \left(\prod_{j=i}^{p}{}' (1+L_j)\right) + \int_{\bar{t}_p}^{t_0^*} \sigma(s)\,ds \\ &\geqslant -\rho \left\{ \sum_{i=r+1}^{p}{}' \int_{\bar{t}_{i-1}}^{\bar{t}_i} M^*(s)\,ds \left(\prod_{j=i}^{p}{}' (1+L_j)\right) + \int_{\bar{t}_p}^{t_0^*} M^*(s)\,ds \right\}. \end{aligned}$$

Hence, if $\rho > 0$, we have

$$\begin{aligned} 1 &< \sum_{i=r+1}^{p}{}' \int_{\bar{t}_{i-1}}^{\bar{t}_i} M^*(s)\,ds \left(\prod_{j=i}^{p}{}' (1+L_j)\right) + \int_{\bar{t}_p}^{t_0^*} M^*(s)\,ds \\ &\leqslant \int_0^T M^*(s)\,ds \left(\prod_{i=1}^{m} (1+L_i)\right). \end{aligned}$$

It contradicts (4).

If $\rho = 0$, then

$$0 \geqslant q\big(t_1^*\big) \prod_{i=r+1}^{p}{}' (1+L_i) > 0.$$

It is a contradiction too. The proof is complete. □

Remark 1. If $M(t) = 0$, $t \in J$, then Lemma 1 reduces to Lemma 4 of [4]. Note that condition (4) is satisfied if $K(t) \geqslant 0$, $t \in J$, and

$$\int_0^T M(t) e^{\int_t^T K(s)\,ds}\,dt \left(\prod_{i=1}^{m} (1+L_i)\right) \leqslant 1. \tag{7}$$

We see that condition (7) does not depend on the advanced argument α. If we extra assume that $K(t) = K > 0$, $M(t) = M > 0$, then condition (4) holds if

$$M\big(e^{KT} - 1\big) \prod_{i=1}^{m} (1+L_i) \leqslant K.$$

For example, if we take $m=2$, $L_1=\frac{1}{3}$, $L_2=\frac{1}{2}$, $T=\frac{2}{3}$, $K=\frac{3}{2}$, then from the last condition we have

$$M \leqslant \frac{3}{4(e-1)} \approx 0.43648.$$

Theorem 1. *Assume that $K \in C(J,\mathbb{R})$, $\eta \in PC(J)$, M is nonnegative, $M(t) \not\equiv 0$, and $M \in PC(J)$. Moreover, let $\alpha \in C(J,J)$, $t \leqslant \alpha(t) \leqslant T$, $\gamma_k, L_k \in \mathbb{R}$ and $L_k \geqslant 0$ for $k=1,2,\dots,m$. In addition, assume that*

$$\delta \equiv \int_0^T M^*(s)\,ds + \sum_{i=1}^m L_i < 1, \tag{8}$$

where M^* is defined as in Lemma 1. *Then the impulsive problem*

$$\begin{cases} v'(t) = K(t)v(t) + M(t)v\bigl(\alpha(t)\bigr) + \eta(t), & t \in J', \\ v\bigl(t_k^+\bigr) = (1+L_k)v(t_k) + \gamma_k, & k=1,2,\dots,m, \\ v(T) = k_0 \end{cases} \tag{9}$$

has a unique solution $v \in PC^1(J)$.

Proof. Put

$$z(t) = e^{\int_t^T K(s)\,ds} v(t), \quad t \in J.$$

Then problem (9) takes the form

$$\begin{cases} z'(t) = M(t)e^{\int_t^{\alpha(t)} K(s)\,ds} z(\alpha(t) + \eta(t)e^{\int_t^T K(s)\,ds} \equiv \mathcal{F}z(t), & t \in J', \\ z\bigl(t_k^+\bigr) = (1+L_k)z(t_k) + \gamma_k e^{\int_{t_k}^T K(s)\,ds} \equiv z(t_k) + \mathcal{P}_k z(t_k), & k=1,2,\dots,m, \\ z(T) = k_0. \end{cases} \tag{10}$$

Note that z is the solution of the following impulsive integral equation

$$z(t) = k_0 - \int_t^T \mathcal{F}z(s)\,ds - \sum_{i=k+1}^m{}' \mathcal{P}_i z(t_i) \equiv \mathcal{A}z(t), \quad t \in J_k, \tag{11}$$

for $k=0,1,\dots,m$.

To find a solution of problem (11) is equivalent to get a fixed point of the operator $\mathcal{A}: PC(J) \to PC(J)$. Let $x, y \in PC(J)$. Then

$$\begin{aligned} \|\mathcal{A}x - \mathcal{A}y\| &= \sup_{t\in J}\bigl|\mathcal{A}x(t) - \mathcal{A}y(t)\bigr| \\ &\leqslant \sup_{t\in J}\left[\int_t^T \bigl|\mathcal{F}x(s) - \mathcal{F}y(s)\bigr|\,ds\right] + \sum_{i=1}^m \bigl|\mathcal{P}_i x(t_i) - \mathcal{P}_i y(t_i)\bigr| \\ &\leqslant \sup_{t\in J}\int_t^T M^*(s)\bigl|x\bigl(\alpha(s)\bigr) - y\bigl(\alpha(s)\bigr)\bigr|\,ds + \sum_{i=1}^m L_i\bigl|x(t_i) - y(t_i)\bigr| \\ &= \delta\|x-y\|. \end{aligned}$$

Problem (11) has a unique solution, by the Banach fixed point theorem. It means that also problem (9) has a unique solution. This ends the proof. □

Remark 2. If $M(t) = 0$ on J, then condition (8) is superfluous. Note that in this case system (10) takes the form

$$\begin{cases} z'(t) = \bar{\eta}(t), & t \in J', \\ z(t_k^+) = (1 + L_k)z(t_k) + \bar{\gamma}_k, & k = 1, 2, \dots, m, \\ x(T) = k_0 \end{cases} \tag{12}$$

with

$$\bar{\eta}(t) = \eta(t)e^{\int_t^T K(s)\,ds}, \qquad \bar{\gamma}_k = \gamma_k e^{\int_{t_k}^T K(s)\,ds}.$$

It is easy to verify that the solution z of problem (12) has now the form

$$z(t) = z(0)\prod_{i=1}^{k}{}'(1 + L_i) + \sum_{j=1}^{k}{}'\left\{\int_{t_{j-1}}^{t_j} \bar{\eta}(s)\,ds \prod_{i=j}^{k}{}'(1 + L_i) + \bar{\gamma}_j \prod_{i=j+1}^{k}{}'(1 + L_i)\right\}$$
$$+ \int_{t_k}^{t} \bar{\eta}(s)\,ds, \quad t \in J_k, \text{ for } k = 0, 1, \dots, m,$$

where

$$z(0) = \frac{1}{V}\left\{k_0 - \sum_{j=1}^{m}\left[\int_{t_{j-1}}^{t_j} \bar{\eta}(s)\,ds \prod_{i=j}^{m}(1 + L_i) + \bar{\gamma}_j \prod_{i=j+1}^{m}{}'(1 + L_i)\right] - \int_{t_m}^{T} \bar{\eta}(s)\,ds\right\}$$

with

$$V = \prod_{i=1}^{m}(1 + L_i).$$

Now we give sufficient conditions when problem (1) has a solution.

Theorem 2. *Let assumption* (H_1) *hold. Moreover, assume that*

(H_2) *$y_0, z_0 \in PC^1(J)$ are lower and upper solutions of problem* (1), *respectively, and $z_0(t) \leqslant y_0(t)$ on J,*

(H_3) *there exist functions $K, M \in C(J, \mathbb{R})$, M is nonnegative and such that*

$$f(t, u, v) - f(t, \bar{u}, \bar{v}) \geqslant -K(t)(\bar{u} - u) - M(t)(\bar{v} - v)$$

for $z_0(t) \leqslant u \leqslant \bar{u} \leqslant y_0(t)$, $z_0(\alpha(t)) \leqslant v \leqslant \bar{v} \leqslant y_0(\alpha(t))$, $t \in J$,

(H_4) *there exist constants $L_k \in [0, 1)$, $k = 1, 2, \dots, m$, such that*

$$I_k(w(t_k)) - I_k(\bar{w}(t_k)) \geqslant -L_k[\bar{w}(t_k) - w(t_k)], \quad k = 1, 2, \dots, m,$$

for any $w, \bar{w}$ with $z_0(t_k) \leqslant w(t_k) \leqslant \bar{w}(t_k) \leqslant y_0(t_k)$, $k = 1, 2, \dots, m$,

(H_5) *conditions* (4) *and* (8) *hold,* (H$_6$) *there exists* $\gamma > 0$ *such that for any* $u, \bar{u} \in [z_0(0), y_0(0)]$ *with* $u \leqslant \bar{u}$ *and* $v, \bar{v} \in [z_0(T), y_0(T)]$ *with* $v \leqslant \bar{v}$ *we have*

$$g(u, v) \leqslant g(\bar{u}, v), \tag{13}$$

$$g(u, v) - g(u, \bar{v}) \leqslant \gamma(\bar{v} - v). \tag{14}$$

Then there exist solutions $v, w \in [z_0, y_0]_*$ *of problem* (1).

Proof. Some ideas are taken from paper [5]. Let $\eta, \xi \in [z_0, y_0]$, where

$$[z_0, y_0] = \{u \in PC(J, \mathbb{R}):\ z_0(t) \leqslant u \leqslant y_0(t),\ t \in J\}.$$

Put $\varphi(t) = \sup[\eta(t), \xi(t)]$, $\Phi(t) = \inf[\eta(t), \xi(t)]$. Consider the initial value problems

$$\begin{cases} v'(t) = F\varphi(t) + K(t)[v(t) - \varphi(t)] + M(t)[v(\alpha(t)) - \varphi(\alpha(t))], & t \in J', \\ \Delta v(t_k) = I_k(\varphi(t_k)) + L_k[v(t_k) - \varphi(t_k)], & k = 1, 2, \ldots, m, \\ v(T) = \varphi(T) + \dfrac{1}{\gamma} g(\varphi(0), \varphi(T)), \end{cases} \tag{15}$$

$$\begin{cases} w'(t) = F\Phi(t) + K(t)[w(t) - \Phi(t)] + M(t)[w(\alpha(t)) - \Phi(\alpha(t))], & t \in J', \\ \Delta w(t_k) = I_k(\Phi(t_k)) + L_k[w(t_k) - \Phi(t_k)], & k = 1, 2, \ldots, m, \\ w(T) = \Phi(T) + \dfrac{1}{\gamma} g(\Phi(0), \Phi(T)). \end{cases} \tag{16}$$

By Theorem 1, problems (15), (16) have a unique solution. Therefore, we can define the operator

$$B : \bar{\Omega} \to PC(J) \times PC(J), \qquad [z_0, y_0] \subset PC(J), \qquad B(\eta, \xi) = (v, w), \tag{17}$$

where v, w are solutions of (15), (16), $\bar{\Omega} = [z_0, y_0] \times [z_0, y_0]$.

Now, we want to show that

$$z_0(t) \leqslant w(t) \leqslant v(t) \leqslant y_0(t), \quad t \in J. \tag{18}$$

Put $p = z_0 - w$. Then

$$\begin{aligned} p'(t) &\geqslant Fz_0(t) - F\Phi(t) - K(t)[w(t) - \Phi(t)] - M(t)[w(\alpha(t)) - \Phi(\alpha(t))] \\ &\geqslant -K(t)[\Phi(t) - z_0(t)] - M(t)[\Phi(\alpha(t)) - z_0(\alpha(t))] - K(t)[w(t) - \Phi(t)] \\ &\quad - M(t)[w(\alpha(t)) - \Phi(\alpha(t))] \\ &= K(t)p(t) + M(t)p(\alpha(t)). \end{aligned}$$

Moreover,

$$\begin{aligned} p(T) &= z_0(T) - \Phi(T) - \frac{1}{\gamma} g(\Phi(0), \Phi(T)) \\ &\leqslant z_0(T) - \Phi(T) - \frac{1}{\gamma} g(z_0(0), \Phi(T)) \\ &= z_0(T) - \Phi(T) + \frac{1}{\gamma}[g(z_0(0), z_0(T)) - g(z_0(0), \Phi(T))] - \frac{1}{\gamma} g(z_0(0), z_0(T)) \\ &\leqslant z_0(T) - \Phi(T) + \frac{1}{\gamma}\gamma[\Phi(T) - z_0(T)] = 0, \end{aligned}$$

and

$$\Delta p(t_k) \geqslant I_k\big(z_0(t_k)\big) - I_k\big(\Phi(t_k)\big) - L_k\big[w(t_k) - \Phi(t_k)\big] \geqslant L_k p(t_k), \quad k = 1, 2, \ldots, m.$$

This and Lemma 1 show that $z_0(t) \leqslant w(t)$, $t \in J$. Similarly we can show that $v(t) \leqslant y_0(t)$, $t \in J$. To show that $w(t) \leqslant v(t)$, $t \in J$, we put $p = w - v$. Then

$$\begin{aligned}
p'(t) &= F\Phi(t) - F\varphi(t) + K(t)\big[w(t) - \Phi(t) - v(t) + \varphi(t)\big] \\
&\quad + M(t)\big[w\big(\alpha(t)\big) - \Phi\big(\alpha(t)\big) - v\big(\alpha(t)\big) + \varphi\big(\alpha(t)\big)\big] \\
&\geqslant -K(t)\big[\varphi(t) - \Phi(t)\big] - M(t)\big[\varphi\big(\alpha(t)\big) - \Phi\big(\alpha(t)\big)\big] \\
&\quad + K(t)\big[w(t) - \Phi(t) - v(t) + \varphi(t)\big] \\
&\quad + M(t)\big[w\big(\alpha(t)\big) - \Phi\big(\alpha(t)\big) - v\big(\alpha(t)\big) + \varphi\big(\alpha(t)\big)\big] \\
&= K(t)p(t) + M(t)p\big(\alpha(t)\big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
p(T) &= \Phi(T) - \varphi(T) + \frac{1}{\gamma}\big[g\big(\Phi(0), \Phi(T)\big) - g\big(\varphi(0), \varphi(T)\big)\big] \\
&\leqslant \Phi(T) - \varphi(T) + \frac{1}{\gamma}\gamma\big[\varphi(T) - \Phi(T)\big] = 0,
\end{aligned}$$

and, for $k = 1, 2, \ldots, m$, we have

$$\Delta p(t_k) = I_k\big(\Phi(t_k)\big) - I_k\big(\varphi(t_k)\big) + L_k\big[w(t_k) - \Phi(t_k) - v(t_k) + \varphi(t_k)\big] \geqslant L_k p(t_k).$$

This and Lemma 1 show that $w(t) \leqslant v(t)$, $t \in J$, so (18) holds.

Hence $B : \bar{\Omega} \to \bar{\Omega}$. In order to apply Schauder's fixed point theorem we need to show that the operator B is continuous and compact. Let $(v_n, w_n) \in \bar{\Omega}$, and $v_n \to v$, $w_n \to w$ in $PC(J)$. Put

$$\begin{aligned}
&\mathcal{D}_0 v(t) = F\varphi(t) + K(t)\big[v(t) - \varphi(t)\big] + M(t)\big[v\big(\alpha(t)\big) - \varphi\big(\alpha(t)\big)\big], \\
&\mathcal{D}_k v(t_k) = I_k\big(\varphi(t_k)\big) + L_k\big[v(t_k) - \varphi(t_k)\big], \quad k = 1, 2, \ldots, m, \\
&\bar{k}_0 = \varphi(T) + \frac{1}{\gamma} g\big(\varphi(0), \varphi(T)\big).
\end{aligned}$$

Then problem (15) takes the form

$$\begin{cases}
v'(t) = \mathcal{D}_0 v(t), & t \in J', \\
\Delta v(t_k) = \mathcal{D}_k v(t_k), & k = 1, 2, \ldots, m, \\
v(T) = \bar{k}_0.
\end{cases}$$

Similarly as in the proof of Theorem 1, v is the solution of the following impulsive integral equation

$$v(t) = \bar{k}_0 - \int_t^T \mathcal{D}_0 v(s)\, ds - \sum_{i=k+1}^{m}{}' \mathcal{D}_i v(t_i) \equiv \mathcal{D}v(t), \quad t \in J_k,\ k = 0, 1, \ldots, m.$$

Then, for $t \in J$, we have

$$
\begin{aligned}
\left|\mathcal{D}v_n(t)-\mathcal{D}v(t)\right| &\leqslant \int_t^T \left|\mathcal{D}_0 v_n(s)-\mathcal{D}_0 v(s)\right| ds + \sum_{i=1}^m \left|\mathcal{D}_i v_n(t_i)-\mathcal{D}_i v(t_i)\right| \\
&\leqslant \int_0^T \left[\left|K(s)\right| v_n(s)-v(s)\right] + \left[M(s)\left|v_n\big(\alpha(s)\big)-v\big(\alpha(s)\big)\right|\right] ds \\
&\quad + \sum_{i=1}^m L_i \left|v_n(t_i)-v(t_i)\right|.
\end{aligned}
$$

Thus the Lebesgue dominated convergence theorem implies

$$\sup_{t\in J}\left|\mathcal{D}v_n(t)-\mathcal{D}v(t)\right| \to 0 \quad \text{if } n\to\infty,$$

so operator $\mathcal{D}$ is continuous. Similar property holds for $w_n \to w$ too. As a result $B:\bar{\Omega}\to\bar{\Omega}$ is continuous. In view of (18), the operator $B:\bar{\Omega}\to\bar{\Omega}$ is bounded too.

Now we need to show that the operator $B:\bar{\Omega}\to\bar{\Omega}$ is compact. Note that

$$\left|\mathcal{D}v(t_1)-\mathcal{D}v(t_2)\right| \leqslant \left|\int_{t_1}^{t_2} \mathcal{D}_0 v(s)\, ds\right|.$$

Similar property also holds for the solution w. It proves that the operator $B:\bar{\Omega}\to\bar{\Omega}$ is equicontinuous on J. The Arzela–Ascoli theorem guarantees that B is compact. Hence, by Schauder's fixed point theorem, operator B has a fixed point, i.e. there exist $(v,w)\in\bar{\Omega}$ such that $B(v,w)=(v,w)$ and $v\leqslant w$.

Now, by (17), we see that v, w satisfy the following relations

$$
\begin{cases}
v'(t) = Fv(t) + K(t)\big[v(t)-v(t)\big] + M(t)\big[v\big(\alpha(t)\big)-v\big(\alpha(t)\big)\big], & t\in J',\\
\Delta v(t_k) = I_k\big(v(t_k)\big) + L_k\big[v(t_k)-v(t_k)\big], & k=1,2,\ldots,m,\\
v(T) = v(T) + \dfrac{1}{\gamma} g\big(v(0), v(T)\big),
\end{cases}
$$

$$
\begin{cases}
w'(t) = Fw(t) + K(t)\big[w(t)-w(t)\big] + M(t)\big[w\big(\alpha(t)\big)-w\big(\alpha(t)\big)\big], & t\in J',\\
\Delta w(t_k) = I_k\big(w(t_k)\big) + L_k\big[w(t_k)-w(t_k)\big], & k=1,2,\ldots,m,\\
w(T) = w(T) + \dfrac{1}{\gamma} g\big(w(0), w(T)\big).
\end{cases}
$$

It shows that $v, w\in PC^1(J)$ are solutions of problem (1). This ends the proof. □

Example 1. For $J=[0,T]$, we consider the problem

$$
\begin{cases}
x'(t) = \lambda_1(t)e^{x(t)} + \lambda_2(t)\sin\big(x\big(\alpha(t)\big)\big) - \lambda_1(t), & t\in J\setminus\{t_1\},\\
\Delta x(t_1) = Lx(t_1),\\
0 = 2x^2(0) + x(T) - k,
\end{cases}
\tag{19}
$$

where $\lambda_1,\lambda_2\in C(J,\mathbb{R}_+)$, $\mathbb{R}_+=[0,\infty)$, $\alpha\in C(J,J)$, $t\leqslant\alpha(t)\leqslant T$, $t\in J$, $0<t_1<T$, $L\geqslant 0$, $0\leqslant k\leqslant 1$.

Take $y_0(t)=0$, $z_0(t)=-1$, $t\in J$. Indeed, $z_0(t)<y_0(t)$ on J, and

$$\begin{aligned}
&Fy_0(t) = \lambda_1(t) - \lambda_1(t) = 0 = y_0'(t),\\
&Fz_0(t) = \lambda_1(t)\big[e^{-1} - 1\big] - \lambda_2(t)\sin 1 \leqslant 0 = z_0'(t),\\
&\Delta y_0(t_1) = L \cdot 0 = I_1\big(y_0(t_1)\big),\\
&\Delta z_0(t_1) = 0 \geqslant L(-1) = I_1\big(z_0(t_1)\big),\\
&g\big(y_0(0), y_0(T)\big) = g(0,0) = -k \leqslant 0,\\
&g\big(z_0(0), z_0(T)\big) = g(-1,-1) = 1 - k \geqslant 0.
\end{aligned}$$

It proves that y_0, z_0 are lower and upper solutions of problem (19), respectively. Moreover, $K(t) = \lambda_1(t)$, $M(t) = \lambda_2(t)$, $L_1 = L$, so assumptions (H_3), (H_4), (H_6) are satisfied. If we extra assume that

$$\int_0^T \lambda_2(t) e^{\int_t^{\alpha(t)} \lambda_1(s)\,ds}\,dt + L < 1, \tag{20}$$

then problem (19) has solutions in the segment $[-1, 0]_*$, by Theorem 2. Note that condition (20) guaranties that condition (4) is satisfied too.

For example, if we take $L = \frac{1}{2}$, $T = \pi$, $\lambda_1(t) = \lambda > 0$, $\lambda_2(t) = \beta e^{\lambda(t-T)} \sin t$ for $t \in J$, then condition (20) holds if $0 < \beta < \frac{1}{4}$.

3. Coupled lower and upper solutions of problem (1)

Let us introduce the following definition.

We say that $u, w \in PC^1(J, \mathbb{R})$ are coupled lower and upper solutions of (1) if

$$\begin{cases} u'(t) \leqslant Fu(t), & t \in J',\\ \Delta u(t_k) \leqslant I_k\big(u(t_k)\big), & k = 1, 2, \ldots, m,\\ g\big(u(0), w(T)\big) \leqslant 0, \end{cases}$$

$$\begin{cases} w'(t) \geqslant Fw(t), & t \in J',\\ \Delta w(t_k) \geqslant I_k\big(w(t_k)\big), & k = 1, 2, \ldots, m,\\ g\big(w(0), u(T)\big) \geqslant 0. \end{cases}$$

The next result deals with the case when problem (1) has quasisolutions.

Theorem 3. *Assume that assumptions* (H_1), (H_3)–(H_5) *hold, where y_0, z_0 are coupled lower and upper solutions of problem* (1) *and $z_0(t) \leqslant y_0(t)$ on J. In addition, we assume that*

(H_6') *there exists $\gamma > 0$ such that for any $u, \bar{u} \in [z_0(0), y_0(0)]$ with $u \leqslant \bar{u}$ and $v, \bar{v} \in [z_0(T), y_0(T)]$ with $v \leqslant \bar{v}$ we have*

$$\begin{aligned}
&g(u, v) \leqslant g(\bar{u}, v),\\
&g(u, v) - g(u, \bar{v}) \geqslant -\gamma(\bar{v} - v).
\end{aligned}$$

Then there exist $u, v \in [z_0, y_0]_*$ coupled quasisolutions of problem (1), *i.e. the pair (v, w) is a solution of the system*:

$$\begin{cases} \begin{cases} v'(t) = Fv(t), & t \in J', \\ \Delta v(t_k) = I_k\big(v(t_k)\big), & k = 1, 2, \ldots, m, \\ g\big(v(0), w(T)\big) = 0, \end{cases} \\ \begin{cases} w'(t) = Fw(t), & t \in J', \\ \Delta w(t_k) = I_k\big(w(t_k)\big), & k = 1, 2, \ldots, m, \\ g\big(w(0), v(T)\big) = 0. \end{cases} \end{cases}$$

Proof. Consider the initial value problems

$$\begin{cases} v'(t) = F\varphi(t) + K(t)\big[v(t) - \varphi(t)\big] + M(t)\big[v\big(\alpha(t)\big) - \varphi\big(\alpha(t)\big)\big], & t \in J', \\ \Delta v(t_k) = I_k\big(\varphi(t_k)\big) + L_k\big[v(t_k) - \varphi(t_k)\big], & k = 1, 2, \ldots, m, \\ v(T) = \varphi(T) - \dfrac{1}{\gamma} g\big(\Phi(0), \varphi(T)\big), \end{cases}$$

$$\begin{cases} w'(t) = F\Phi(t) + K(t)\big[w(t) - \Phi(t)\big] + M(t)\big[w\big(\alpha(t)\big) - \Phi\big(\alpha(t)\big)\big], & t \in J', \\ \Delta w(t_k) = I_k\big(\Phi(t_k)\big) + L_k\big[w(t_k) - \Phi(t_k)\big], & k = 1, 2, \ldots, m, \\ w(T) = \Phi(T) - \dfrac{1}{\gamma} g\big(\varphi(0), \Phi(T)\big), \end{cases}$$

where Φ and φ are defined as in the proof of Theorem 2. The proof is similar to the proof of Theorem 2 and therefore it is omitted. □

Example 2. Now we consider the problem

$$\begin{cases} x'(t) = bx(t) - bx^2\big(\alpha(t)\big) + (a-1)(b-5) \equiv Fx(t), \quad t \in J = [0, T], \\ \Delta x(t_i) = L_i x(t_i), \quad i = 1, 2, \ldots, m, \text{ with } 0 < t_1 < t_2 < \cdots < t_m < T, \\ 0 = \lambda\big[x(0) + x^2(0)\big] - x(T) - a, \end{cases} \tag{21}$$

where $a > 1$, $b \geqslant 5$, $L_i \geqslant 0$, $i = 1, 2, \ldots, m$, $\lambda > 0$, $\alpha \in C(J, J)$, $t \leqslant \alpha(t) \leqslant T$ and

$$a\big[\lambda(a-1) - 1\big] \geqslant 0. \tag{22}$$

In addition, we assume that

$$\sum_{i=1}^{m} L_i < 1. \tag{23}$$

Put $y_0(t) = 0$, $z_0(t) = -a$, $t \in J$. Then

$$\begin{aligned} &Fy_0(t) = (a-1)(b-5) \geqslant 0 = y_0'(t), \\ &Fz_0(t) = -ab - ba^2 + (a-1)(b-5) < 0 = z_0'(t), \\ &\Delta y_0(t_i) = 0 = 0L_i = I_i\big(y_0(t_i)\big), \quad i = 1, 2, \ldots, m, \\ &\Delta z_0(t_i) = 0 \geqslant -L_i a = I_i\big(z_0(t_i)\big), \quad i = 1, 2, \ldots, m, \\ &g\big(y_0(0), z_0(T)\big) = g(0, -a) = a - a = 0, \\ &g\big(z_0(0), y_0(T)\big) = g(-a, 0) = a\big[\lambda(a-1) - 1\big] \geqslant 0, \end{aligned}$$

by (22). It shows that y_0, z_0 are weakly coupled lower and upper solutions of (21). Note that $K(t) = b$, $M(t) = 0$, $t \in J$, so assumption (H_3) holds. Assumptions (H_4), (H_5), (H_6') are also satisfied. By Theorem 3, problem (21) has, in the sector $[z_0, y_0]_*$, coupled quasisolutions.

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