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ON THE CONLEY INDEX IN HILBERT SPACES — A MULTIVALUED CASE

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Abstract. We introduce the cohomological Conley type index theory for multivalued flows generated by vector fields which are compact and convex-valued perturbations of some linear operators.

1. Introduction. In 1999, K. Gęba, M. Izydorek and A. Pruszko [6] constructed an invariant which is a version of Conley index for flows determined by compact perturbations of special linear operators in an infinite-dimensional Hilbert space. Their method of construction is very similar to the definition of the Leray–Schauder degree. This invariant has been then applied to obtain existence and multiplicity results in variational problems with strongly indefinite functionals. On the other hand, M. Mrozek [13] considered a cohomological index for multivalued flows on compact spaces. The aim of this paper is to give an infinite-dimensional version of Mrozek's index by use of the above method. Nontriviality of the obtained invariant gives existence results of invariant sets for inclusions in Hilbert spaces. We can expect also further applications to differential inclusions coming from e.g. non-smooth analysis. One has to mention also the other definitions of Conley index for multivalued flows by M. Kunze [11], [12] and G. Gabor [5] consisting in approximation of the generators of the flow by more smooth one (locally Lipschitz). Other applications and equivariant version will be a subject of further research. We have limited our attention to the simplest case of flows. A local version of all considerations here is natural.

2. Multivalued mappings. A standard reference for this section can be [7].

Let X, Y be two topological spaces. A multivalued mapping $\varphi : X \to Y$ is a mapping which assigns to every $x \in X$ a non-empty compact subset $\varphi(x)$ of Y. The graph of a

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multivalued mapping $\varphi : X \to Y$ is the set

$$\Gamma(\varphi) = \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

The image of a subset $A \subseteq X$ under φ is $\varphi(A) = \bigcup\{\varphi(x) : x \in A\}$. The (small) inverse-image of a subset $B \subseteq Y$ under φ is the set $\varphi^{-1}(B) = \{x \in X : \varphi(x) \subseteq B\}$.

The multivalued mapping φ is called *upper semicontinuous* (*usc*) iff for every open $V \subseteq Y$ the inverse-image $\varphi^{-1}(V)$ is open in X.

The following properties of usc mappings are easy exercises.

PROPOSITION 2.1. *If $\varphi : X \to Y$ is usc then the graph $\Gamma(\varphi)$ is a closed subset of $X \times Y$.*

PROPOSITION 2.2. *If $\varphi : X \to Y$ is usc and $K \subseteq X$ is compact then $\varphi(K)$ is compact.*

The functor of the Alexander–Spanier cohomology will be considered as a functor $H^* : Top_2 \to GMod$, where $GMod$ denotes the category of graded $\mathbb{Z}$-modules and linear maps of degree zero. A non-empty compact space X is called *acyclic* iff $H^0(X) = \mathbb{Z}$ and $H^n(X) = 0$ for $n \neq 0$.

Recall that a mapping $f : X \to Y$ is called *proper* iff for every compact $K \subseteq Y$, $f^{-1}(K)$ is compact. It is called *a Vietoris map* iff it is proper and for every $y \in Y$, $f^{-1}(y)$ is acyclic. Note that every Vietoris map is a closed mapping. The following observation is a consequence of the Vietoris–Begle Theorem (see [14], Chap. 6.9, Thm. 15).

PROPOSITION 2.3. *Composition of Vietoris maps is a Vietoris map.*

The map of pairs $f : (P_1, P_2) \to (Q_1, Q_2)$ is called *Vietoris* iff $f : P_1 \to Q_1$ and $f_{|P_2} : P_2 \to Q_2$ are Vietoris maps.

Using the Five Lemma one can easily deduce a version of the Vietoris–Begle Theorem for pairs.

THEOREM 2.4. *If $f : (P_1, P_2) \to (Q_1, Q_2)$ is a Vietoris map, then $f^* : H^*(Q_1, Q_2) \to H^*(P_1, P_2)$ is an isomorphism.*

DEFINITION 2.5. An usc mapping $\varphi : X \to Y$ is called *admissible* provided there exist: a space Γ, and continuous mappings $p : \Gamma \to X$, $q : \Gamma \to Y$ such that p is Vietoris and for every $x \in X$, $\varphi(x) = q(p^{-1}(x))$.

We call such a pair of mappings (p, q) *a Vietoris pair*. We say then that the Vietoris pair of maps (p, q) *determines* the multivalued mapping φ. Obviously such pairs are non-unique for a given mapping φ. One can define a relation between such pairs (p, q) (see e.g. [13] for details).

A multivalued mapping $\varphi : X \to Y$ is called *acyclic*, if it is usc and $\varphi(x)$ is acyclic for every $x \in X$. A good example of such maps are compact convex-valued mappings in Banach spaces. One can easily check that the projections from the graph of an acyclic mapping φ form a Vietoris pair and they determine it. Therefore acyclic mappings are admissible. Moreover we have the following.

PROPOSITION 2.6. *A composition of acyclic mappings is an admissible mapping.*

3. Multivalued flows. Let X be a metric space.

DEFINITION 3.1. An usc mapping $\varphi : X \times \mathbb{R} \to X$ is a *multivalued flow* on X provided for every $s, t \in \mathbb{R}$, $x, y \in X$

(i) $\varphi(x, 0) = \{x\}$,

(ii) $st \geq 0 \;\Rightarrow\; \varphi(x, t + s) = \varphi(\varphi(x, t) \times s)$,

(iii) $y \in \varphi(x, t) \;\Leftrightarrow\; x \in \varphi(y, -t)$.

The flow is called *admissible*, if there exists $T > 0$ such that the restriction of φ to $X \times [0, T]$ is an admissible mapping.

If we admit the empty set as a value we can say about a partial multivalued flow.

Let $\Delta \subseteq \mathbb{R}$.

DEFINITION 3.2. A continuous mapping $\sigma : \Delta \to X$ is a *Δ-solution* if for every $t, s \in \Delta$, $\sigma(t) \in \varphi(\sigma(s), t - s)$.

The set of all Δ-solutions in $N \subseteq X$ originating in x (i. e. such that $0 \in \Delta$, $\sigma(0) = x$) is denoted by $Sltn_N(\Delta, x)$.

A *connection* from x to y in N is a $[0, t]$-solution σ in N such that $\sigma(0) = x$ and $\sigma(t) = y$. The set of all such connections is denoted by $Conn_N(t, x, y)$.

Let N be a compact subset of X. We can define a mapping $\varphi_N : N \times \mathbb{R} \to N$ by the formula $\varphi_N(x, t) = \{y : Conn_N(t, x, y) \neq \emptyset\}$. It is easy to verify that φ_N is a partial multivalued flow on N.

DEFINITION 3.3. Let $A \subseteq X$ be an arbitrary subset. Define *invariant, right-invariant, left-invariant* part of A to be:

$$\begin{aligned} \operatorname{Inv} A &:= \{x \in A : Sltn_A(\mathbb{R}, x) \neq \emptyset\}, \\ \operatorname{Inv}^+ A &:= \{x \in A : Sltn_A(\mathbb{R}^+, x) \neq \emptyset\}, \\ \operatorname{Inv}^- A &:= \{x \in A : Sltn_A(\mathbb{R}^-, x) \neq \emptyset\}, \end{aligned}$$

respectively.

DEFINITION 3.4. A subset $A \subseteq X$ is *invariant* (resp. *positively* (*negatively*) *invariant*) iff $\operatorname{Inv} A = A$ (resp. $\operatorname{Inv}^+ A = A$ ($\operatorname{Inv}^- A = A$)).

Notice that the set $\operatorname{Inv} A$ is a maximal invariant subset of A. There is also a stronger version of invariance which is equivalent to the above in the single-valued case.

DEFINITION 3.5. The set $A \subseteq X$ is *strongly* (*positively*, *negatively*) *invariant* if for every $x \in A$ we have $\varphi(x, \mathbb{R}) \subseteq A$ (resp. $\varphi(x, \mathbb{R}^+) \subseteq A$, $\varphi(x, \mathbb{R}^-) \subseteq A$).

Let now X be a locally compact metric space with an admissible flow φ.

A subset $K \subseteq X$ is *an isolated invariant set* iff there exists a compact neighbourhood N of K in X such that $K = \operatorname{Inv} N$. The set N is called then an *isolating neighbourhood* of K.

DEFINITION 3.6. A pair (P_1, P_2) of subsets of $N \subseteq X$ is an *index pair* in N provided

(i) P_1, P_2 are compact and strongly positively invariant with respect to φ_N,

(ii) $\operatorname{Inv}^- N \subseteq \operatorname{Int}_N P_1$, $\operatorname{Inv}^+ N \subseteq N \setminus P_2$,

(iii) $\operatorname{cl}(P_1 \setminus P_2) \subseteq \operatorname{Int} N$.

Theorem 3.7 ([13]). *Let φ be a multivalued flow on a locally compact space X and let K be an isolated invariant set with an isolating neighbourhood N. Then for every neighbourhood W of K there exists an index pair in N such that* $\mathrm{cl}(P_1 \setminus P_2) \subseteq W$.

Theorem 3.8 ([13]). If the flow is admissible and K is an isolated invariant set, then the Alexander–Spanier cohomology groups $H^*(P_1, P_2)$ do not depend on the choice of the isolating neighbourhood N and of the index pair in N.

Theorem 3.8 assures that the following notion is well-defined:

Definition 3.9 ([13]). We define the *cohomological Conley index* of an isolated invariant set $K \subseteq X$ to be

$$\mathrm{CH}(K) = H^*(P_1, P_2)$$

where (P_1, P_2) is an index pair for K.

This index has at least two fundamental properties:

(i) If $\mathrm{CH}(K)$ is non-trivial, then $K \neq \emptyset$,

(ii) The index is invariant under continuation of admissible flows.

Example 3.10. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an usc mapping with compact convex values, and let F be bounded (or of at most a linear growth). Then the solutions of the differential inclusion $x'(t) \in -F(x(t))$ a.e. form an admissible flow in $\mathbb{R}^n$. This follows from the fact that the set of all solutions of the Cauchy problem for this inclusion is an acyclic set (comp. e.g. [1] for a detailed formulation) and therefore the mapping associating with t the set of values at t of solutions starting at t_0 is admissible as a composition of an acyclic-valued one and a continuous evaluation at t. See [13] for more detailed description of this flow. We use the sign "$-$" in the inclusion in order to be consistent with the notation used in [8] (see also Remark 4.7).

4. LS-flows in a Hilbert space. Let $E = (E, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $L : E \to E$ a linear bounded operator with spectrum $\sigma(L)$. We assume the following

- $E = \bigoplus_{k=0}^{\infty} E_k$ with all subspaces E_k being mutually orthogonal and of finite dimension,
- $L(E_0) \subseteq E_0$. E_0 is the invariant subspace of L corresponding to the part of spectrum $\sigma_0(L) = i\mathbb{R} \cap \sigma(L)$ lying on the imaginary axis,
- $L(E_k) = E_k$ for all $k > 0$,
- $\sigma_0(L)$ is isolated in $\sigma(L)$, i.e. $\sigma_0(L) \cap \mathrm{cl}(\sigma(L) \setminus \sigma_0(L)) = \emptyset$.

Definition 4.1. A multivalued flow $\varphi : E \times \mathbb{R} \to E$ is called an *LS-flow* if it can be written in the form

$$\varphi(x, t) = e^{tL} + U(t, x),$$

where $U : E \times \mathbb{R} \to E$ is an admissible mapping which is completely continuous, i.e. maps bounded sets to relatively compact sets.

Definition 4.2. A bounded and closed subset $X \subset E$ is an *isolating neighbourhood* for a flow φ, if $\mathrm{Inv}(X) \subset \mathrm{Int}\, X$.

Let Λ be a compact metric space.

DEFINITION 4.3. An *LS-family of flows* is a multivalued flow $\eta : E \times \mathbb{R} \times \Lambda \to E$ of the form

$$\eta(x, t, \lambda) = e^{tL} + U(x, t, \lambda),$$

where $U : E \times \mathbb{R} \times \Lambda \to E$ is an admissible completely continuous mapping.

PROPOSITION 4.4. *Let $\eta : E \times \mathbb{R} \times \Lambda \to E$ be an LS-family of flows. If $X \subset E$ is bounded and closed, then the set*

$$S = \operatorname{Inv}(X \times \Lambda, \eta) := \{(x, \lambda) : x \in \operatorname{Inv}(X)\}$$

is a compact subset of $X \times \Lambda$.

Proof. Denote by E_- (E_+) the L-invariant subspace corresponding to all the eigenvalues of L with negative (resp. positive) real part. Then E clearly splits into the direct sum $E = E_- \oplus E_0 \oplus E_+$.

Denote by $P_- : E \to E_-$, $P_+ : E \to E_+$, $P_0 : E \to E_0$ the orthogonal projections. We assumed that $\sigma_0(L)$ is isolated in $\sigma(L)$. Therefore for every $\varrho > 0$ there exists $t_0 > 0$ such that

$$\heartsuit \qquad \|e^{tL}x\| \geq \varrho\|x\| \quad \text{for all } t \geq t_0,\ x \in E_+,$$

and

$$\spadesuit \qquad \|e^{tL}x\| \geq \varrho\|x\| \quad \text{for all } t \leq -t_0,\ x \in E_-.$$

It is clear from the definition that S is a closed subset of $E \times \Lambda$. Suppose that it is not compact.

Consider the set $S_0 := \{x \in X : (x, \lambda) \in S \text{ for some } \lambda \in \Lambda\}$. Observe that $S_0 \subset \operatorname{cl} P_-(S_0) \times \operatorname{cl} P_0(S_0) \times \operatorname{cl} P_+(S_0)$. The set $\operatorname{cl} P_0(S_0)$ is compact being a closed and bounded subset of a finite-dimensional space E_0. Therefore either $\operatorname{cl} P_-(S_0)$ or $\operatorname{cl} P_+(S_0)$ is non-compact. Without loss of generality we can assume that $P_+(S_0)$ is not relatively compact. Since E_+ is a complete metric space, there exists an $\varepsilon > 0$ such that $P_+(S_0)$ does not admit a finite ε-net. Hence, we can choose a sequence $(y_n, \lambda_n) \in S$ such that the projections $x_n = P_+(y_n)$ satisfy $\|x_i - x_j\| \geq \varepsilon$ whenever $i \neq j$. Now we can choose $\delta > 0$ such that $X \subset B(0, \delta)$ and t_0 large enough that the inequality $\heartsuit$ holds with $\varrho = 3\delta/\varepsilon$. Let us take $u_n := e^{t_0 L} y_n$ and an arbitrary $v_n \in U(y_n, t_0, \lambda_n)$. Then

$$u_n + v_n \in e^{t_0 L} y_n + U(y_n, t_0, \lambda_n) = \eta(y_n, t_0, \lambda_n) \subset X \subset B(0, \delta).$$

Thus

$$3\delta \leq \|u_i - u_j\| \leq \|u_i + v_i\| + \|v_i - v_j\| + \|u_j + v_j\| < 2\delta + \|v_i - v_j\|$$

and consequently

$$\|v_i - v_j\| > \delta \quad \text{whenever} \quad i \neq j.$$

But $v_n \in U(\{t_0\} \times S)$ and the last set is relatively compact because U is completely continuous. Thus v_n should have a convergent subsequence and we obtain a contradiction. In the other case one uses $\spadesuit$ in the similar argument. ∎

PROPOSITION 4.5. *Let Λ be a compact metric space and let $\eta : E \times \mathbb{R} \times \Lambda \to E$ be a family of LS-flows. If X is an isolating neighbourhood for some η_{λ_0}, then it is an isolating neighbourhood for all λ in some open neighbourhood V of λ_0 in Λ.*

Proof. From Proposition 4.4 it follows that the set $\mathrm{Inv}(X \times \Lambda, \eta)$ is a compact subset of $X \times \Lambda$. Since $\mathrm{Inv}(X \times \Lambda, \eta) \cap (X \times \{\lambda_0\}) \subset \mathrm{Int}\, X$ there exists an open neighbourhood V of λ_0 such that $\mathrm{Inv}(X \times \Lambda) \cap (X \times V) \subset \mathrm{Int}\, X \times V$, and thus $\mathrm{Inv}(X, \eta_\lambda) \subset \mathrm{Int}\, X$ for every $\lambda \in V$. ∎

Definition 4.6. An usc mapping $f : E \to E$ is an *LS-vector field* provided it is of the form $f(x) = L(x) + K(x)$, where $K : E \to E$ is completely continuous with compact convex values, and if f generates an LS-flow φ on the whole E.

Remark 4.7. We can assume for simplicity that $K(E) \subset E$ is bounded. Then the solutions of the inclusion $x'(t) \in -f(x(t))$ are defined on the whole $\mathbb{R}$ and form an LS-flow on E (comp. [4]).

5. LS-Conley index. As in the previous section we work with a convex-valued LS-vector field $f : E \to E$, $f(x) = Lx + K(x)$ which induces an LS-flow φ on E.

Denote by $P_n : E \to E$ the orthogonal projection onto $E^n = \bigoplus_{k=0}^n E_k$. Define a sequence of vector fields $f_n : E^n \to E^n$ by the formula $f_n(x) = Lx + P_n(K(x))$, and $F_n : E^{n+1} \times [0,1] \to E^{n+1}$ by

$$F_n(x,s) = Lx + (1-s)P_n(K(x)) + sP_{n+1}(K(x)).$$

One easily checks that for each n, f_n and F_n are compact, convex-valued usc mappings which are of linear growth (since $K(E)$ is bounded and P_n are linear). Therefore, for every n, f_n generates an admissible flow φ_n, and F_n generates a family ξ_n of admissible flows.

Lemma 5.1. *Let $X \subset E$ be an isolating neighbourhood for φ. There exists $n_0 \in N$ such that $X_n = X \cap E^n$ is an isolating neighbourhood for φ_n and ξ_{n-1}, whenever $n \geq n_0$.*

Proof. Define a family of LS-vector fields $F : E \times [0,1] \to E$ by

$$F(x,0) = f(x),$$
$$F(x,s) = Lx + (1+n)(1-ns)P_{n+1}(K(x)) + n[(n+1)s-1]P_n(K(x)) \quad \text{if } s \in \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right].$$

This family generates a family of multivalued LS-flows ξ. By Proposition 4.4 the set $\mathrm{Inv}(X \times [0,1], \xi)$ is compact in $X \times [0,1]$ and $\mathrm{Inv}(X \times [0,1], \xi) \cap X \times \{0\} \subset \mathrm{Int}\, X$. Therefore for some s_0 we have $\mathrm{Inv}(X \times [0,1], \xi) \cap X \times [0, s_0] \subset \mathrm{Int}(X) \times [0, s_0]$. One takes $n_0 > 1/s_0$. ∎

Choose $n \geq n_0$ from Lemma 5.1. The invariant set $S_n := \mathrm{Inv}(X_n, \varphi_n)$ admits an index pair (Y_n, Z_n) and its cohomological Conley index $\mathrm{CH}(S_n) = H^*(Y_n, Z_n)$ is well-defined by Definition 3.9.

Remark 5.2. For the Alexander–Spanier cohomology we have an isomorphism

$$H^*(Y_n, Z_n) = H^*((Y_n/Z_n), *).$$

Let $D_n^+ := \{x \in E_n^+ : \|x\| \leq 1\}$, $D_n^- := \{x \in E_n^- : \|x\| \leq 1\}$, $\partial D_n^- := \{x \in E_n^- : \|x\| = 1\}$. Consider a family of flows $\theta_n : E^{n+1} \times \mathbb{R} \times [0,1] \to E^{n+1}$ generated by $h_n(x,s) = Lx + P_n(K(P_n x + s(x - P_n x)))$.

One can easily check that the pair

$$(Y_n \times D_{n+1}^+ \times D_{n+1}^-, Z_n \times D_{n+1}^+ \times D_{n+1}^- \cup Y_n \times D_{n+1}^+ \times \partial D_{n+1}^-)$$

is an index pair for the isolated invariant set $\operatorname{Inv}(X_{n+1}, \theta_n(\cdot,\cdot,0)) = S_n$.

Let us think of the circle as $S^1 = [0,1]/\{0,1\}$. Recall that the *suspension functor* is defined by the smash product $SX := S^1 \wedge X$ and for every $m \in \mathbb{N}$ we define $S^m X = S(S^{m-1}X)$ (comp. [15]).

Denote by $\nu(n)$ the dimension of $E_n^- = E_n \cap E^-$. One observes that the quotient space

$$(Y_n \times D_{n+1}^+ \times D_{n+1}^- / Z_n \times D_{n+1}^+ \times D_{n+1}^- \cup Y_n \times D_{n+1}^+ \times \partial D_{n+1}^-)$$

has the homotopy type of the suspension $S^{\nu(n)}(Y_n/Z_n)$. Thus their cohomology groups are naturally isomorphic. On the other hand, one easily sees that X_{n+1} is an isolating neighbourhood for both families of admissible flows $\theta_n(\cdot,\cdot,s)$ and $\xi_n(\cdot,\cdot,s)$, $s \in [0,1]$. Moreover $\theta_n(\cdot,\cdot,1) = \xi_n(\cdot,\cdot,0)$. Therefore, by the continuation property of the cohomological Conley index (see [13]) and Remark 5.2 we obtain an isomorphism

$$c^* : H^*(Y_{n+1}/Z_{n+1}, *) \to H^*(S^{\nu(n)}(Y_n/Z_n), *)$$

Remark 5.3. Observe that in the single-valued case we obtain here a homotopy equivalence, because we can use a homotopy index then (see [8]).

Since we have also an isomorphism $H^*(S^{\nu(n)}(Y_n/Z_n), *) \cong H^{*+\nu(n)}(Y_n/Z_n, *)$, we conclude that for each $n \geq n_0$ there is an isomorphism

$$\gamma_n : H^{k+\nu(n)}(Y_{n+1}, Z_{n+1}) \to H^k(Y_n, Z_n).$$

Define a map $\rho : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by

$$\rho(0) = 0 \quad \text{and} \quad \rho(n) = \sum_{i=0}^{n-1} \nu(i) \quad \text{for} \quad n \geq 1.$$

Consider for a fixed $q \in \mathbb{Z}$ a sequence of cohomology groups $H^{q+\rho(n)}(Y_n, Z_n)$ and define a sequence of homomorphisms

$$h_n : H^{q+\rho(n+1)}(Y_{n+1}, Z_{n+1}) \to H^{q+\rho(n+1)}(S^{\nu(n)}(Y_n, Z_n)) \to H^{q+\rho(n)}(Y_n, Z_n)$$

as described above. Since h_n are isomorphisms for large n, we can define the cohomological LS-index as follows.

Definition 5.4.

$$\mathrm{CH}^q(X) := \varprojlim \{H^{q+\rho(n)}(Y_n, Z_n), h_n\}.$$

As was pointed out in [8], these groups may be non-trivial both for negative and positive q.

We formulate now two basic properties. They are obvious consequences of respective properties in finite dimension ([13]).

Proposition 5.5 (non-triviality). *Let X be an isolating neighbourhood for an LS-flow φ. If $\mathrm{CH}^q(X) \neq \{0\}$ for some q, then $\operatorname{Inv}(X, \varphi) \neq \emptyset$. Therefore we have a bounded solution of the inclusion generating φ.*

Proposition 5.6. *Let Λ be a compact, connected and locally contractible metric space. Assume that $\eta : E \times \mathbb{R} \to E$ is a family of flows generated by a family of LS-vector fields $f : E \times \Lambda \to E$. Let X be an isolating neighbourhood for the flow η_λ for some $\lambda \in \Lambda$. Then* *there exists a compact neighbourhood $C \subset \Lambda$ of λ such that* $\mathrm{CH}^*(X, \eta_\mu) = \mathrm{CH}^*(X, \eta_\nu)$ *for all $\mu, \nu \in C$.*

EXAMPLE 5.7. Let $G : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz mapping which is 2π-periodic in the second variable and satisfies some growth condition in respect to the first one. We consider the Hamiltonian system of differential inclusions

$$\dot{z} \in J\partial G(z, t)$$

where J is the standard symplectic matrix and ∂ denotes the Clarke generalized gradient with respect to $z \in \mathbb{R}^{2n}$ (comp. [2]). If one is concerned with the existence of 2π-periodic solutions (in an appropriate sense), it is natural to work in the Sobolev space $E = H^{1/2}(S^1, \mathbb{R}^{2n})$. One can prove that the generalized gradient of an action functional is an LS-vector field and we can use our invariant to obtain some multiplicity results (comp. [8]) in non-smooth situation. Details (not all obvious) will be given in a forthcoming paper.

One can also define the notions of attractor, repeller and Morse decomposition in the context of admissible LS-flows. The properties are then analogous to those described in [8].

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