

## DEGREE OF $T$ -EQUIVARIANT MAPS IN $\mathbb{R}^n$

JOANNA JANCZEWSKA

*Department of Technical Physics and Applied Mathematics  
Gdańsk University of Technology  
Narutowicza 11/12, 80-952 Gdańsk, Poland  
E-mail: janczewska@mif.pg.gda.pl*

MARCIN STYBORSKI

*Department of Technical Physics and Applied Mathematics  
Gdańsk University of Technology  
Narutowicza 11/12, 80-952 Gdańsk, Poland  
and Ph.D. student of the Institute of Mathematics, Polish Academy of Sciences  
E-mail: marcins@impan.gov.pl*

**Abstract.** A special case of  $G$ -equivariant degree is defined, where  $G = \mathbb{Z}_2$ , and the action is determined by an involution  $T : \mathbb{R}^p \oplus \mathbb{R}^q \rightarrow \mathbb{R}^p \oplus \mathbb{R}^q$  given by  $T(u, v) = (u, -v)$ . The presented construction is self-contained. It is also shown that two  $T$ -equivariant gradient maps  $f, g : (\mathbb{R}^n, S^{n-1}) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$  are  $T$ -homotopic iff they are gradient  $T$ -homotopic. This is an equivariant generalization of the result due to Parusiński.

**1. Introduction.** Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$ . Consider a continuous map  $f : \overline{\Omega} \rightarrow \mathbb{R}^n$  such that  $f$  is not equal to 0 at any point on the boundary of  $\Omega$ . Then an integer  $\deg(f, \Omega)$  called the Brouwer degree can be associated to  $f$ . The classical works on this subject are [3], [12], [13], and a modern one is [11]. It is well known that the Brouwer degree is an invariant of homotopy. This means that if  $h : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$  is a homotopy nowhere vanishing on  $\partial\Omega \times [0, 1]$  then  $\deg(h_t, \Omega) = \deg(h_0, \Omega)$  for all  $t \in [0, 1]$ , where  $h_t(x) = h(x, t)$ .

Let  $G$  be a compact Lie group. Assume that  $V$  is a real finite-dimensional representation of  $G$  and  $n = \dim V$ . Take  $\Omega \subset V$  and  $f : \overline{\Omega} \rightarrow V$  as above. In addition, suppose that  $\Omega$  is  $G$ -invariant ( $gx \in \Omega$  for all  $x \in \Omega$ ,  $g \in G$ ) and  $f$  is  $G$ -equivariant ( $f(gx) = gf(x)$ )

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for all  $x \in \overline{\Omega}$ ,  $g \in G$ ). In this case, the  $G$ -equivariant degree  $\text{Deg}_G(f, \Omega) \in B(G)$  is defined, where  $B(G)$  stands for the Burnside ring of  $G$ . This degree was introduced by Ize, Massabó and Vignoli in [10]. Up till now, it was considered by many authors. See for instance [6], [11] and [15]. Of course,  $G$ -equivariant degree is an invariant of  $G$ -equivariant homotopy ( $h(gx, t) = gh(x, t)$  for all  $x \in \overline{\Omega}$ ,  $t \in [0, 1]$ ,  $g \in G$ ).

Let  $G$  be equal to  $\mathbb{Z}_2$ . The action of  $\mathbb{Z}_2$  on  $\mathbb{R}^n$  is determined by a decomposition of  $\mathbb{R}^n$  onto the direct sum  $\mathbb{R}^p \oplus \mathbb{R}^q$  and the involution  $T(u, v) = (u, -v)$ , where  $n = p + q$ ,  $p, q \in \mathbb{N} \cup \{0\}$  and  $u \in \mathbb{R}^p$ ,  $v \in \mathbb{R}^q$ . In fact, to define the  $\mathbb{Z}_2$ -equivariant degree we do not need to use the representation theory. In this work we would like to describe a construction of this degree. We will call it  $T$ -equivariant degree.

Our approach is alternative to the one by Granas and Dugundji in [8]. There are two basic differences between our and their approach. Contrary to Granas and Dugundji, from the beginning we work with the family of  $T$ -equivariant maps (see Sec. 6, §20, Theorem 1.2, pp. 551–552 in [8], and Lemma 3.2, Conclusion 3.3 here). We also introduce notions of  $T$ -equivariant normal maps and homotopies, which are different from ones in [8]. Moreover, the proofs of all lemmas and propositions needed to define the degree are complete.

Our construction is divided into five main steps. Each step is a separate section.

In [14], Parusiński showed that if we have two gradient vector fields on the unit ball in  $\mathbb{R}^n$  and nowhere vanishing on the sphere, then they are homotopic if and only if they are gradient homotopic. In the last section we will prove this theorem in  $T$ -equivariant case. Namely, consider two  $T$ -equivariant gradient vector fields  $f$  and  $g$  on the unit ball in  $\mathbb{R}^n$  and nowhere vanishing on the sphere. It is shown that if there is a  $T$ -equivariant homotopy joining  $f$  to  $g$  then there is a  $T$ -equivariant gradient homotopy joining  $f$  to  $g$ . Our result suggests that there is no interesting generalization of  $T$ -equivariant degree on gradient vector fields. The proof is based on the latest results by Ferrario (see [4]) and Dancer, Gęba and Rybicki (see [1]).

**2.  $T$ -equivariant maps and homotopy.** Let  $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ , where  $n \in \mathbb{N}$ ,  $p, q \in \mathbb{N} \cup \{0\}$  and  $n = p + q$ . For every  $x \in \mathbb{R}^n$  we write  $x = (u, v)$ , where  $u \in \mathbb{R}^p$  and  $v \in \mathbb{R}^q$ . Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be given by

$$T(u, v) := (u, -v).$$

The map  $T$  is a linear isomorphism and an involution, i.e.  $T^2 = \text{Id}_{\mathbb{R}^n}$ .

DEFINITION 2.1. A set  $X \subset \mathbb{R}^n$  is  $T$ -invariant if  $T(X) \subset X$ .

If a set  $X$  is  $T$ -invariant then  $T(X) = X$  and  $T|_X : X \rightarrow X$  is an involution onto  $X$ .

DEFINITION 2.2. Let  $X \subset \mathbb{R}^n$  be  $T$ -invariant.

1. A map  $f : X \rightarrow \mathbb{R}^n$  is called  $T$ -equivariant if  $f(Tx) = Tf(x)$  for all  $x \in X$ .
2.  $h : X \times [0, 1] \rightarrow \mathbb{R}^n$  is called a  $T$ -equivariant homotopy if it is continuous and  $h(Tx, t) = Th(x, t)$  for all  $x \in X$  and  $t \in [0, 1]$ .
3. A function  $\tau : X \rightarrow \mathbb{R}$  is called  $T$ -equivariant if  $\tau(Tx) = \tau(x)$  for all  $x \in X$ .



Remark that if  $f = (f_1, f_2)$ , where  $f_1 : X \rightarrow \mathbb{R}^p$  and  $f_2 : X \rightarrow \mathbb{R}^q$ , then  $f$  is  $T$ -equivariant if and only if  $f_1(u, -v) = f_1(u, v)$  and  $f_2(u, -v) = -f_2(u, v)$  for all  $(u, v) \in X$ .

From now on, every open bounded  $T$ -invariant subset of  $\mathbb{R}^n$  is said to be  $T$ -admissible.

Assume that  $\Omega \subset \mathbb{R}^n$  is  $T$ -admissible. It is obvious that  $\bar{\Omega}$  is  $T$ -invariant. We say that  $f : \bar{\Omega} \rightarrow \mathbb{R}^n$  is  $T$ -admissible if  $f$  is continuous,  $T$ -equivariant and  $f(x) \neq 0$  for all  $x \in \partial\Omega$ . We will denote by  $\mathcal{A}_T(\Omega)$  the family of all  $T$ -admissible maps from  $\bar{\Omega}$  into  $\mathbb{R}^n$ . In the same spirit we generalize the notion of homotopy. We say that a homotopy  $h : \bar{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$  is  $T$ -admissible if  $h$  is  $T$ -equivariant and  $h(x, t) \neq 0$  for all  $x \in \partial\Omega$  and  $t \in [0, 1]$ . We will denote by  $\mathcal{HA}_T(\Omega)$  the family of all  $T$ -admissible homotopies from  $\bar{\Omega} \times [0, 1]$  into  $\mathbb{R}^n$ .

DEFINITION 2.3. We say that  $f$  is homotopic to  $g$  in  $\mathcal{A}_T(\Omega)$  and write  $f \sim g$  in  $\mathcal{A}_T(\Omega)$  if there exists  $h \in \mathcal{HA}_T(\Omega)$  joining  $f$  to  $g$ , i.e.  $h(x, 0) = f(x)$  and  $h(x, 1) = g(x)$  for all  $x \in \bar{\Omega}$ .

It is easy to check that  $\sim$  is an equivalence relation in  $\mathcal{A}_T(\Omega)$ . The homotopy class of  $f \in \mathcal{A}_T(\Omega)$  under  $\sim$  will be denoted by  $[f]$ . Finally, the set of all homotopy classes of the relation  $\sim$  will be denoted by  $\mathcal{A}_T[\Omega]$ .

**3.  $T$ -equivariant generic maps.** Let  $\Omega \subset \mathbb{R}^n$  be  $T$ -admissible. Here and subsequently,

$$\begin{aligned} \mathcal{A}_T^\infty(\Omega) &:= \{f \in \mathcal{A}_T(\Omega) : f|_\Omega \text{ is smooth}\}, \\ \mathcal{HA}_T^\infty(\Omega) &:= \{h \in \mathcal{HA}_T(\Omega) : h_t|_\Omega \text{ is smooth for } t \in [0, 1]\}, \end{aligned}$$

where  $h_t : \bar{\Omega} \rightarrow \mathbb{R}^n$  is defined by  $h_t(x) := h(x, t)$ .

DEFINITION 3.1. We say that  $f$  is homotopic to  $g$  in  $\mathcal{A}_T^\infty(\Omega)$  and write  $f \simeq g$  in  $\mathcal{A}_T^\infty(\Omega)$  if there exists  $h \in \mathcal{HA}_T^\infty(\Omega)$  joining  $f$  to  $g$ .

The relation  $\simeq$  is easily seen to be an equivalence relation in  $\mathcal{A}_T^\infty(\Omega)$ .

A map  $f \in \mathcal{A}_T^\infty(\Omega)$  is said to be generic if  $0 \in \mathbb{R}^n$  is a regular value of  $f|_\Omega$ , i.e. the derivative  $Df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an isomorphism for all  $x \in f^{-1}(\{0\})$ .

In this section we show that under some restrictions on  $\Omega$ , every homotopy class in  $\mathcal{A}_T[\Omega]$  possesses a  $T$ -equivariant generic map. For this purpose we prove now a few lemmas.

Let  $K$  be a compact subset of  $\mathbb{R}^n$ . A map  $f : K \rightarrow \mathbb{R}^n$  is called smooth if there exists an open set  $X \subset \mathbb{R}^n$  such that  $K \subset X$  and there exists a smooth map  $\tilde{f} : X \rightarrow \mathbb{R}^n$  such that  $\tilde{f}|_K = f$ . Let  $\{U_i\}_{i=1}^k$  be an open covering of  $K$ . Set  $U = \bigcup_{i=1}^k U_i$ . We call a family of smooth functions  $\lambda_i : U \rightarrow [0, 1]$ , where  $i = 1, 2, \dots, k$ , a smooth partition of unity subordinate to the covering  $\{U_i\}_{i=1}^k$ , if this family satisfies the following conditions:

- $\text{supp } \lambda_i = \overline{\{x \in \mathbb{R}^n : \lambda_i(x) \neq 0\}} \subset U_i$  for every  $i = 1, 2, \dots, k$ ,
- $\sum_{i=1}^k \lambda_i(y) = 1$  for every  $y \in K$ .

It is well known that such a partition exists (see [16]). Additionally, if  $K$  and every  $U_i$  are  $T$ -invariant sets and every  $\lambda_i$  is a  $T$ -equivariant function then we say that  $\{U_i\}_{i=1}^k$  is a  $T$ -invariant covering of  $K$  and  $\{\lambda_i\}_{i=1}^k$  is a  $T$ -equivariant partition of unity.

LEMMA 3.1. *Assume that  $K \subset \mathbb{R}^n$  is compact and  $T$ -invariant, and  $\{U_i\}_{i=1}^k$  is an open  $T$ -equivariant covering of  $K$ . Then there exists a smooth  $T$ -equivariant partition of unity subordinate to the covering  $\{U_i\}_{i=1}^k$ .*

*Proof.* Let  $\{\lambda_i\}_{i=1}^k$  be a smooth partition of unity subordinate to the covering  $\{U_i\}_{i=1}^k$  of  $K$ . For every  $i = 1, 2, \dots, k$ , let  $\widehat{\lambda}_i$  be given by  $\widehat{\lambda}_i = \frac{1}{2}(\lambda_i + \lambda_i T)$ . It is obvious that each function  $\widehat{\lambda}_i$  is smooth and  $T$ -equivariant. The family  $\{\widehat{\lambda}_i\}_{i=1}^k$  is a desired one. ■

Let  $K \subset \mathbb{R}^n$  be compact and  $T$ -invariant. We say that  $T$  acts freely on  $K$  if  $Tx \neq x$  for every  $x \in K$ , i.e.  $K \cap \mathbb{R}^p = \emptyset$ . Then

$$\text{dist}(K, \mathbb{R}^p) := \inf\{|x - y| : x \in K, y \in \mathbb{R}^p\}$$

is a positive number.

LEMMA 3.2. *Let  $K \subset \mathbb{R}^n$  be a compact  $T$ -invariant set such that  $T$  acts freely on  $K$ . If a map  $f : K \rightarrow \mathbb{R}^n$  is continuous and  $T$ -equivariant then for every  $\varepsilon > 0$  there is a smooth  $T$ -equivariant map  $g : K \rightarrow \mathbb{R}^n$  such that*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon.$$

From now on,  $B(a, r)$  stands for an open ball of radius  $r$ , centered at a point  $a \in \mathbb{R}^n$ .

*Proof.* Fix  $\varepsilon > 0$ . Since  $K$  is compact,  $f$  is uniformly continuous. Hence, there is  $\delta > 0$  such that if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \varepsilon$ . Set  $\delta' = \min\{\delta, \text{dist}(K, \mathbb{R}^p)\}$ , and  $U_x = B(x, \delta')$  for every  $x \in K$ . Then

- $TU_x = B(Tx, \delta')$ ,
- $U_x \cap TU_x = \emptyset$ ,
- $K \subset \bigcup_{x \in K} (U_x \cup TU_x)$ .

The compactness of  $K$  implies that there exist points  $x_1, x_2, \dots, x_k \in K$  such that  $K \subset \bigcup_{i=1}^k (U_i \cup TU_i)$ , where  $U_i = U_{x_i}$ . Consider a smooth  $T$ -equivariant partition of unity  $\{\lambda_i\}_{i=1}^k$  subordinate to the covering  $\{U_i \cup TU_i\}_{i=1}^k$  of  $K$ . Set  $U = \bigcup_{i=1}^k (U_i \cup TU_i)$ . For every  $i \in \{1, 2, \dots, k\}$ , let  $\pi_i : U \rightarrow \mathbb{R}^n$  be a map such that  $\pi_i(U_i) = \{x_i\}$  and  $\pi_i(TU_i) = \{Tx_i\}$ . The function  $g : U \rightarrow \mathbb{R}^n$  is defined by

$$g(x) = \sum_{i=1}^k \lambda_i(x) f(\pi_i(x)).$$

Take  $x \in U$ . If  $x \in U_i \cup TU_i$  then in a sufficiently small neighbourhood of  $x$  the map  $f \circ \pi_i$  is constant. If  $x \in \partial U_i \cup \partial TU_i$  then in a sufficiently small neighbourhood of  $x$  the function  $\lambda_i$  is equal to 0. Hence  $g$  is smooth.

Take  $x \in K$ . If  $x \in U_i$  then  $\pi_i(x) = x_i$  and  $|\pi_i(x) - x| < \delta$ . If  $x \in TU_i$  then  $\pi_i(x) = Tx_i$  and  $|\pi_i(x) - x| < \delta$ . Finally, if  $x \notin U_i \cup TU_i$  then  $\lambda_i(x) = 0$ . From this it follows that

$$|g(x) - f(x)| = \left| \sum_{i=1}^k \lambda_i(x) f(\pi_i(x)) - \sum_{i=1}^k \lambda_i(x) f(x) \right| \leq \sum_{i=1}^k \lambda_i(x) |f(\pi_i(x)) - f(x)| < \varepsilon.$$

Moreover,

$$g(Tx) = \sum_{i=1}^k \lambda_i(Tx) f(\pi_i(Tx)) = \sum_{i=1}^k \lambda_i(x) f(T\pi_i(x)) = \sum_{i=1}^k \lambda_i(x) T f(\pi_i(x)) = Tg(x),$$

which completes the proof. ■

CONCLUSION 3.3. *Let  $\Omega \subset \mathbb{R}^n$  be a  $T$ -admissible set such that  $T$  acts freely on  $\overline{\Omega}$ . Then for every  $f \in \mathcal{A}_T(\Omega)$  there exists  $g \in \mathcal{A}_T^\infty(\Omega)$  such that  $f \sim g$  in  $\mathcal{A}_T(\Omega)$ .*

*Proof.* By the assumption,  $\overline{\Omega}$  is a  $T$ -invariant compact set and  $T$  acts freely on  $\overline{\Omega}$ . Set  $d = \inf\{|f(x)| : x \in \partial\Omega\}$ . From Lemma 3.2 it follows that there exists  $g \in \mathcal{A}_T^\infty(\Omega)$  such that

$$\sup_{x \in \overline{\Omega}} |f(x) - g(x)| < d.$$

Consider the linear homotopy  $h : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$  joining  $f$  to  $g$ , i.e.

$$h(x, t) = tg(x) + (1 - t)f(x).$$

It is trivial that  $h$  is continuous and  $h(Tx, t) = Th(x, t)$  for all  $x \in \overline{\Omega}$  and  $t \in [0, 1]$ . Take  $x \in \partial\Omega$  and  $t \in [0, 1]$ . Then

$$|h(x, t)| = |f(x) - t(f(x) - g(x))| \geq |f(x)| - t|f(x) - g(x)| \geq |f(x)| - |f(x) - g(x)| > 0.$$

Hence  $h \in \mathcal{HA}_T(\Omega)$ . ■

Let  $U \subset \mathbb{R}^n$  be an open bounded set, and  $K \subset U$  be compact. It is well known that there exists a smooth function  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in K, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus U. \end{cases}$$

In the mathematical literature,  $\eta$  is called *the Urysohn function* (see [8]).

LEMMA 3.4. *Let  $U$  and  $U_0$  be  $T$ -admissible subsets of  $\mathbb{R}^n$ . Assume that  $\overline{U_0} \subset U$ . Then there exists a smooth  $T$ -equivariant function  $\tilde{\eta} : \mathbb{R}^n \rightarrow [0, 1]$  such that  $\tilde{\eta}(x) = 1$  for every  $x \in \overline{U_0}$  and  $\tilde{\eta}(x) = 0$  for every  $x \in \mathbb{R}^n \setminus U$ .*

The proof is similar to that of Lemma 3.1. We leave it to the reader.

LEMMA 3.5. *Let  $\Omega_0$  and  $\Omega$  be  $T$ -admissible subsets of  $\mathbb{R}^n$  such that  $\Omega_0 \subset \Omega$ . Suppose that  $f_0 \simeq g_0$  in  $\mathcal{A}_T^\infty(\Omega_0)$  and there is an  $f \in \mathcal{A}_T^\infty(\Omega)$  such that  $f|_{\Omega_0} = f_0$ . Then there exist a map  $g \in \mathcal{A}_T^\infty(\Omega)$  and a  $T$ -admissible set  $U_0 \subset \Omega_0$  satisfying the following conditions:*

1.  $f \simeq g$  in  $\mathcal{A}_T^\infty(\Omega)$ ,
2.  $g(x) = f(x)$  for every  $x \in \overline{\Omega} \setminus \Omega_0$ ,
3.  $g(x) = g_0(x)$  for every  $x \in U_0$ ,
4.  $g_0^{-1}(\{0\}) \cap \Omega_0 = g^{-1}(\{0\}) \cap \Omega_0 \subset U_0$ .

*Proof.* Let  $\bar{h} \in \mathcal{HA}_T^\infty(\Omega_0)$  be a homotopy joining  $f_0$  to  $g_0$ . Take an open  $T$ -invariant subset  $U_0$  of  $\mathbb{R}^n$  such that  $\overline{U_0} \subset \Omega_0$  and  $\bar{h}(x, t) \neq 0$  for every  $(x, t) \in (\overline{\Omega_0} \setminus U_0) \times [0, 1]$ . Consider an open  $T$ -invariant subset  $U$  of  $\mathbb{R}^n$  such that  $\overline{U_0} \subset U \subset \overline{U} \subset \Omega_0$ . Let  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  be



a smooth  $T$ -equivariant Urysohn function for the pair of sets  $U_0$  and  $U$ , i.e.  $\eta(x) = 1$  for every  $x \in \overline{U}_0$  and  $\eta(x) = 0$  for every  $x \in \mathbb{R}^n \setminus U$ . Let  $h : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$  be defined by

$$h(x, t) = \begin{cases} f(x) & \text{for } x \in \overline{\Omega} \setminus \overline{U}, \\ \bar{h}(x, t\eta(x)) & \text{for } x \in \Omega_0. \end{cases}$$

We check that  $h \in \mathcal{H}\mathcal{A}_T^\infty(\Omega)$  and  $g(x) := h(x, 1)$ ,  $x \in \overline{\Omega}$ , satisfies the claim of our lemma.

Remark that  $(\overline{\Omega} \setminus \overline{U}) \cap \Omega_0 = \Omega_0 \setminus \overline{U}$ . If  $x \in \Omega_0 \setminus \overline{U}$  then  $\eta(x) = 0$ , and hence  $\bar{h}(x, t\eta(x)) = \bar{h}(x, 0) = f(x)$  for all  $t \in [0, 1]$ . In consequence,  $h$  is smooth and  $h(x, 0) = f(x)$  for every  $x \in \overline{\Omega}$ . If  $x \in \partial\Omega$  then  $h(x, t) = f(x) \neq 0$  for all  $t \in [0, 1]$ . Moreover, for  $x \in \Omega_0$  and  $t \in [0, 1]$  we have  $h(Tx, t) = \bar{h}(Tx, t\eta(Tx)) = \bar{h}(Tx, t\eta(x)) = T\bar{h}(x, t\eta(x)) = Th(x, t)$ . Thus  $h$  is  $T$ -equivariant. Summarizing,  $h \in \mathcal{H}\mathcal{A}_T^\infty(\Omega)$  and it joins  $f$  to  $g$ .

Take  $x \in \overline{\Omega} \setminus \Omega_0$ . Since  $\overline{\Omega} \setminus \Omega_0 \subset \overline{\Omega} \setminus U$ , we get  $g(x) = h(x, 1) = f(x)$ .

If  $x \in U_0$  then  $\eta(x) = 1$  and  $g(x) = h(x, 1) = \bar{h}(x, 1) = g_0(x)$ .

Finally, fix  $x \in \Omega_0$ . If  $x \in \Omega_0 \setminus \overline{U}$  then  $g(x) = \bar{h}(x, 0) = f_0(x)$ . If  $x \in \overline{U}$  then  $g(x) = \bar{h}(x, \eta(x))$ . Since  $\{x \in \overline{\Omega}_0 : \bar{h}(x, t) = 0 \text{ for any } t \in [0, 1]\} \subset U_0$ , we have  $g_0^{-1}(\{0\}) \cap \Omega_0 = g^{-1}(\{0\}) \cap \Omega_0 \subset U_0$ , which completes the proof. ■

Let  $K \subset \mathbb{R}^n$  be nonempty, compact and  $T$ -admissible. Set  $k \in \mathbb{N}$ . We call a family of open sets  $\{U_i\}_{i=1}^k$  a  $(T, k)$ -simple covering of  $K$  if it satisfies the following conditions:

1.  $U_i \cap TU_i = \emptyset$  for every  $i \in \{1, 2, \dots, k\}$ ,
2.  $K \subset \bigcup_{i=1}^k (U_i \cup TU_i)$ .

We say that  $K$  is a  $(T, k)$ -simple set if it possesses a  $(T, k)$ -simple covering. If  $K = \emptyset$ , it is said to be  $(T, 0)$ -simple.

**PROPOSITION 3.6.** *Every nonempty compact  $T$ -invariant subset  $K$  of  $\mathbb{R}^n$  such that  $T$  acts freely on  $K$  is  $(T, k)$ -simple for a certain  $k \in \mathbb{N}$ .*

*Proof.* Since  $T$  acts freely on  $K$ ,  $K \cap \mathbb{R}^p = \emptyset$ . Set  $l = \text{dist}(K, \mathbb{R}^p)$ . We have

$$K \subset \bigcup_{x \in K} B(x, l)$$

By compactness of  $K$ , there are  $x_1, x_2, \dots, x_k \in K$  such that

$$K \subset \bigcup_{i=1}^k B(x_i, l).$$

Let  $U_i = B(x_i, l)$  for  $i = 1, 2, \dots, k$ . It is evident that  $U_i \cap TU_i = \emptyset$  and

$$K \subset \bigcup_{i=1}^k (U_i \cup TU_i). \quad \blacksquare$$

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set. For every  $f : \overline{\Omega} \rightarrow \mathbb{R}^n$  such that  $f|_\Omega$  is  $C^r$ -smooth, where  $r \geq 1$ , and  $f(x) \neq 0$  for all  $x \in \partial\Omega$ , set

$$R(f) = \{x \in f^{-1}(\{0\}) : Df(x) \in \text{GL}(\mathbb{R}^n)\}.$$

LEMMA 3.7. Assume that  $\Omega \subset \mathbb{R}^n$  is  $T$ -admissible,  $T$  acts freely on  $\overline{\Omega}$ , which is  $(T, k)$ -simple for a certain  $k \in \mathbb{N}$ . Let  $f \in \mathcal{A}_T^\infty(\Omega)$ . Then there exists  $g \in \mathcal{A}_T^\infty(\Omega)$  such that

- (i)  $f \simeq g$  in  $\mathcal{A}_T^\infty(\Omega)$ ,
- (ii)  $g^{-1}(\{0\}) \setminus R(g)$  is  $(T, k - 1)$ -simple.

*Proof.* Let  $\{U_i\}_{i=1}^k$  be a  $(T, k)$ -simple covering of  $\overline{\Omega}$ . Set

$$K = \overline{\Omega} \setminus \bigcup_{i=2}^k (U_i \cup TU_i), \quad K_1 = K \cap U_1.$$

Let us remark that  $K$  is  $T$ -invariant,  $K \subset (U_1 \cup TU_1)$ ,  $K$  and  $K_1$  are compact. Thus  $K$  is  $(T, 1)$ -simple and  $K = K_1 \cup TK_1$ .

From the Sard theorem it follows that there exists a regular value  $y_0$  of  $f|_{\Omega \cap U_1}$  such that  $|y_0| < \inf\{|f(x)| : x \in \partial\Omega\}$ . Since  $f \circ T = T \circ f$ , we have  $Df(Tx) = T \circ Df(x) \circ T$  for every  $x \in \Omega$ . Hence  $Ty_0$  is also a regular value of  $f|_{\Omega \cap TU_1}$ . Moreover,  $|Ty_0| = |y_0|$ .

Let  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  be a smooth function such that  $\eta(x) = 1$  for all  $x \in K_1$  and  $\eta(x) = 0$  for all  $x \in \mathbb{R}^n \setminus U_1$ . Let  $g : \overline{\Omega} \rightarrow \mathbb{R}^n$  be given by

$$g(x) = \begin{cases} f(x) - \eta(x)y_0 & \text{for } x \in U_1 \cap \overline{\Omega}, \\ f(x) - \eta(Tx)Ty_0 & \text{for } x \in TU_1 \cap \overline{\Omega}, \\ f(x) & \text{for } x \in \overline{\Omega} \setminus (U_1 \cup TU_1), \end{cases}$$

The map  $g|_\Omega$  is easily seen to be smooth. Let

$$h(x, t) = f(x) + t(g(x) - f(x))$$

for all  $(x, t) \in \overline{\Omega} \times [0, 1]$ . Take  $x \in \overline{\Omega}$ . If  $x \in U_1 \cap \overline{\Omega}$  then  $g(Tx) = f(Tx) - \eta(T^2x)Ty_0 = Tf(x) - \eta(x)Ty_0 = T(f(x) - \eta(x)y_0) = Tg(x)$ . If  $x \in TU_1 \cap \overline{\Omega}$  then  $g(Tx) = f(Tx) - \eta(Tx)y_0 = Tf(x) - \eta(Tx)y_0 = Tf(x) - T^2\eta(Tx)y_0 = T(f(x) - \eta(Tx)Ty_0) = Tg(x)$ . Finally, if  $x \in \overline{\Omega} \setminus (U_1 \cup TU_1)$  then  $g(Tx) = f(Tx) = Tf(x) = Tg(x)$ . Consequently,  $g$  is  $T$ -equivariant.

Since  $|g(x) - f(x)| < |y_0|$  for all  $x \in \overline{\Omega}$ , we conclude that  $h$  is a homotopy joining  $f$  to  $g$  in  $\mathcal{A}_T^\infty(\Omega)$ .

Remark that  $g^{-1}(\{0\}) \setminus R(g)$  is a compact set and  $g^{-1}(\{0\}) \subset \bigcup_{i=2}^k (U_i \cup TU_i) \cup K$ . Take  $x \in K$ . If  $x \in K_1$  then  $g(x) = f(x) - y_0$ . If  $x \in TK_1$  then  $g(x) = f(x) - Ty_0$ . From this  $K \cap g^{-1}(\{0\}) \subset R(g)$ , and so  $g^{-1}(\{0\}) \setminus R(g) \subset \bigcup_{i=2}^k (U_i \cup TU_i)$  is  $(T, k - 1)$ -simple. ■

LEMMA 3.8. Let  $\Omega \subset \mathbb{R}^n$  be a  $T$ -admissible set such that  $T$  acts freely on  $\overline{\Omega}$ . Assume that  $f \in \mathcal{A}_T^\infty(\Omega)$  and  $f^{-1}(\{0\}) \setminus R(f)$  is  $(T, k)$ -simple for a certain  $k \in \mathbb{N}$ . Then there exists a map  $g \in \mathcal{A}_T^\infty(\Omega)$  such that

- (i)  $f \simeq g$  in  $\mathcal{A}_T^\infty(\Omega)$ ,
- (ii)  $g^{-1}(\{0\}) \setminus R(g)$  is  $(T, k - 1)$ -simple.

*Proof.* Since  $f^{-1}(\{0\}) \setminus R(f)$  is  $(T, k)$ -simple, there is an open and  $T$ -invariant subset  $\Omega_0$  of  $\Omega$  such that

- (a)  $f^{-1}(\{0\}) \setminus R(f) \subset \Omega_0$ ,
- (b)  $R(f) \subset \Omega \setminus \overline{\Omega}_0$ ,

(c)  $\overline{\Omega}_0$  is  $(T, k)$ -simple.

Set  $f_0 = f|_{\Omega_0}$ . Combining (a) with (b), we see that  $f_0 \in \mathcal{A}_T^\infty(\Omega_0)$ . By Lemma 3.7 it follows that there is  $g_0 \in \mathcal{A}_T^\infty(\Omega_0)$  such that  $f_0 \simeq g_0$  in  $\mathcal{A}_T^\infty(\Omega_0)$  and  $g_0^{-1}(\{0\}) \setminus R(g_0)$  is  $(T, k - 1)$ -simple. From Lemma 3.5 we have that there is  $g \in \mathcal{A}_T^\infty(\Omega)$  such that  $f \simeq g$  in  $\mathcal{A}_T^\infty(\Omega)$  and  $g^{-1}(\{0\}) \setminus R(g) = g_0^{-1}(\{0\}) \setminus R(g_0)$ . Thus  $g^{-1}(\{0\}) \setminus R(g)$  is  $(T, k - 1)$ -simple. ■

Applying the mathematical induction, Lemma 3.8 and Conclusion 3.3, one can immediately prove the next theorem.

**THEOREM 3.9.** *Let  $\Omega \subset \mathbb{R}^n$  be a  $T$ -admissible set such that  $T$  acts freely on  $\overline{\Omega}$ . If  $f \in \mathcal{A}_T(\Omega)$  then there exists a generic map  $g \in \mathcal{A}_T^\infty(\Omega)$  such that  $f \sim g$  in  $\mathcal{A}_T(\Omega)$ .*

**CONCLUSION 3.10.** *Let  $\Omega \subset \mathbb{R}^n$  be a  $T$ -admissible set such that  $T$  acts freely on  $\overline{\Omega}$ . If  $f \in \mathcal{A}_T(\Omega)$  then  $\deg(f, \Omega) \in 2\mathbb{Z}$ .*

Here and subsequently,  $\deg(f, \Omega)$  stands for the Brouwer degree of  $f$  on  $\Omega$ .

*Proof.* Fix  $f \in \mathcal{A}_T(\Omega)$ . By Theorem 3.9 there is a generic map  $g \in \mathcal{A}_T^\infty(\Omega)$  such that  $f \sim g$  in  $\mathcal{A}_T(\Omega)$ . Hence  $\deg(f, \Omega) = \deg(g, \Omega)$ . Since  $T \circ g = g \circ T$ , we have  $Dg(x) = T \circ Dg(Tx) \circ T$  for every  $x \in \Omega$ . From this

$$g^{-1}(\{0\}) \cap \Omega = \{x_1, x_2, \dots, x_m\} \cup \{Tx_1, Tx_2, \dots, Tx_m\}$$

and  $\text{sign det } Dg(x_i) = \text{sign det } Dg(Tx_i)$  for  $i = 1, 2, \dots, m$ . In consequence,

$$\deg(g, \Omega) = \sum_{i=1}^m \text{sign det } Dg(x_i) + \sum_{i=1}^m \text{sign det } Dg(Tx_i) = 2 \sum_{i=1}^m \text{sign det } Dg(x_i),$$

which completes the proof. ■

**4.  $T$ -equivariant normal maps.** Let  $\Omega \subset \mathbb{R}^n$  be  $T$ -admissible and let  $\varepsilon > 0$ . Define

$$\Omega(\varepsilon) = \{(u, v) \in \Omega : |v| < \varepsilon\}.$$

**DEFINITION 4.1.** Let  $f = (f_1, f_2) \in \mathcal{A}_T(\Omega)$ , where  $f_1 : \overline{\Omega} \rightarrow \mathbb{R}^p$ , and  $f_2 : \overline{\Omega} \rightarrow \mathbb{R}^q$ .

1. A map  $f$  is said to be  $\varepsilon$ -normal if there exists  $\varepsilon > 0$  such that

$$f(u, v) = (f_1(u, 0), v)$$

for all  $(u, v) \in \Omega(\varepsilon)$ .

2. A map  $f$  is called normal if there exists  $\varepsilon > 0$  such that  $f$  is  $\varepsilon$ -normal.

We will denote by  $\mathcal{NA}_T(\Omega)$  the family of all normal maps from  $\overline{\Omega}$  into  $\mathbb{R}^n$ .

**DEFINITION 4.2.** Let  $\Omega \subset \mathbb{R}^n$  be a  $T$ -admissible set.

1. A homotopy  $h \in \mathcal{HA}_T(\Omega)$  is called normal if there exists  $\varepsilon > 0$  such that  $h_t : \overline{\Omega} \rightarrow \mathbb{R}^n$  is  $\varepsilon$ -normal for every  $t \in [0, 1]$ .
2. We say that  $f$  is homotopic to  $g$  in  $\mathcal{NA}_T(\Omega)$  and write  $f \approx g$  in  $\mathcal{NA}_T(\Omega)$  if there exists a normal homotopy joining  $f$  to  $g$ .

We will denote by  $\mathcal{HNA}_T(\Omega)$  the family of all normal homotopies from  $\overline{\Omega} \times [0, 1]$  into  $\mathbb{R}^n$ . The homotopy class of  $f \in \mathcal{NA}_T(\Omega)$  under  $\approx$  will be denoted by  $[[f]]$ . Finally, the set of all homotopy classes of the relation  $\approx$  will be denoted by  $\mathcal{NA}_T[\Omega]$ .





The construction of the degree for maps in  $\mathcal{A}_T(\Omega)$ , which will be described in the next section, is based on the following theorem.

**THEOREM 4.1.** *The map  $\tau : \mathcal{N}\mathcal{A}_T[\Omega] \rightarrow \mathcal{A}_T[\Omega]$ ,  $[[f]] \mapsto [f]$  is a bijection.*

*Proof.*

*Step 1.* We show that  $\tau$  is a surjection.

Fix  $f = (f_1, f_2) \in \mathcal{A}_T(\Omega)$ . We show that there is  $g \in \mathcal{N}\mathcal{A}_T(\Omega)$  such that  $f \sim g$  in  $\mathcal{A}_T(\Omega)$ . Set  $d = \inf\{|f(x)| : x \in \partial\Omega\}$ . Since  $f$  is  $T$ -equivariant,  $f_2(u, 0) = 0$  for every  $(u, 0) \in \overline{\Omega}$ . By the continuity of  $f$ , there is  $0 < \varepsilon \leq d/12$  such that if  $x, y \in \overline{\Omega}$  and  $|x - y| < 2\varepsilon$  then  $|f_i(x) - f_i(y)| < d/6$  for  $i = 1, 2$ .

Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\eta(t) = 1$  for every  $|t| \leq \varepsilon$  and  $\eta(t) = 0$  for every  $|t| \geq 2\varepsilon$ . Let  $h : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$  be defined by

$$h(u, v, t) = (1 - t\eta(|v|))f(u, v) + t\eta(|v|)(f_1(u, 0), v).$$

Then

$$\begin{aligned} h(T(u, v), t) &= h(u, -v, t) = (1 - t\eta(|v|))f(u, -v) + t\eta(|v|)(f_1(u, 0), -v) \\ &= (1 - t\eta(|v|))Tf(u, v) + t\eta(|v|)T(f_1(u, 0), v) = Th(u, v, t), \end{aligned}$$

for every  $(u, v) \in \overline{\Omega}$  and  $t \in [0, 1]$ .

Take  $(u, v) \in \partial\Omega$  and  $t \in [0, 1]$ . If  $|v| \geq 2\varepsilon$  then  $h(u, v, t) = f(u, v) \neq 0$ . If  $|v| < 2\varepsilon$  then

$$\begin{aligned} |h(u, v, t)| &= |f(u, v) - t\eta(|v|)(f_1(u, v) - f_1(u, 0), f_2(u, v) - v)| \\ &\geq |f(u, v)| - t\eta(|v|)|f_1(u, v) - f_1(u, 0)| - t\eta(|v|)|f_2(u, v) - v| \\ &\geq |f(u, v)| - |f_1(u, v) - f_1(u, 0)| - |f_2(u, v) - v| \\ &\geq |f(u, v)| - (|f_1(u, v) - f_1(u, 0)| + |f_2(u, v)| + |v|) \\ &> d - 3\frac{d}{6} = \frac{d}{2} > 0. \end{aligned}$$

In consequence,  $h \in \mathcal{H}\mathcal{A}_T(\Omega)$ . Set  $g := h_1$ . If  $|v| \leq \varepsilon$  then  $g(u, v) = h(u, v, 1) = (f_1(u, 0), v)$ . Thus  $g$  is normal.

*Step 2.* We show that  $\tau$  is an injection.

Take  $f = (f_1, f_2) \in \mathcal{N}\mathcal{A}_T(\Omega)$  and  $g = (g_1, g_2) \in \mathcal{N}\mathcal{A}_T(\Omega)$  such that  $f \sim g$  in  $\mathcal{A}_T(\Omega)$ . We prove that  $f \approx g$  in  $\mathcal{N}\mathcal{A}_T(\Omega)$ . Let  $h = (h_I, h_{II}) \in \mathcal{A}_T(\Omega)$  be a homotopy joining  $f$  to  $g$  in  $\mathcal{A}_T(\Omega)$ . Set  $d = \inf\{|h(x, t)| : x \in \partial\Omega \wedge t \in [0, 1]\}$ . Since  $h$  is  $T$ -equivariant, we get  $h_{II}(u, 0, t) = 0$  for every  $(u, 0) \in \overline{\Omega}$  and  $t \in [0, 1]$ . Take  $0 < \varepsilon \leq d/12$  such that  $f, g$  are  $2\varepsilon$ -normal, and if  $x, y \in \overline{\Omega}$  and  $|x - y| < 2\varepsilon$  then  $|h(x, t) - h(y, t)| < d/6$ . Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\eta(t) = 1$  for every  $|t| \leq \varepsilon$  and  $\eta(t) = 0$  for every  $|t| \geq 2\varepsilon$ . Let  $\hat{h} : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$  be given by

$$\hat{h}(u, v, t) = (1 - \eta(|v|))h(u, v, t) + \eta(|v|)(h_I(u, 0, t), v).$$

We check at once that  $\hat{h}$  is a normal homotopy joining  $f$  to  $g$ . ■

**5.  $T$ -equivariant degree.** In this section we introduce the degree of  $T$ -equivariant maps in  $\mathbb{R}^n$ , called the  $T$ -equivariant degree. First we define this degree for  $T$ -equivariant normal maps, and next for all  $T$ -admissible ones.

Let  $\Omega \subset \mathbb{R}^n$  be a  $T$ -admissible set and let  $f = (f_1, f_2) \in \mathcal{NA}_T(\Omega)$ , where  $f_1 : \overline{\Omega} \rightarrow \mathbb{R}^p$ ,  $f_2 : \overline{\Omega} \rightarrow \mathbb{R}^q$ . Set  $\Omega_0 = \Omega \cap \mathbb{R}^p$ . Assume that  $\Omega_0 \neq \emptyset$ . The map  $g_0 : \overline{\Omega}_0 \rightarrow \mathbb{R}^p$  is given by  $g_0(u) = f_1(u, 0)$ . Since  $f(u, v) \neq 0$  for all  $(u, v) \in \partial\Omega$  and  $f_2(u, 0) = 0$  for all  $(u, 0) \in \overline{\Omega}$ , we conclude that  $g_0(u, v) \neq 0$  for all  $(u, v) \in \partial\Omega_0$ . Define

$$d_0 = \begin{cases} \deg(g_0, \Omega_0) & \text{if } \Omega_0 \neq \emptyset, \\ 0 & \text{if } \Omega_0 = \emptyset. \end{cases}$$

Since  $f$  is normal, there is  $\varepsilon > 0$  such that  $f(x) \neq 0$  for all  $x \in \partial\Omega(\varepsilon)$ . Set  $\Omega_1 = \Omega \setminus \overline{\Omega(\varepsilon)}$ . Let us remark that  $T$  acts freely on  $\overline{\Omega}_1$ . Define

$$g_1(x) = f(x),$$

where  $x \in \overline{\Omega}_1$ . It is evident that  $g_1 \in \mathcal{A}_T(\Omega_1)$ . By Conclusion 3.10 there exists an integer  $d_1$  such that  $\deg(g_1, \Omega_1) = 2d_1$ . The  $T$ -equivariant degree of  $f$  on  $\Omega$  is given as follows:

$$\deg_T(f, \Omega) = (d_0, d_1) \in \mathbb{Z} \oplus \mathbb{Z}.$$

Let us denote by  $\mathcal{N}$  the set of all pairs  $(f, \Omega)$  such that  $f \in \mathcal{NA}_T(\Omega)$  and  $\Omega \subset \mathbb{R}^n$  is  $T$ -admissible.

**THEOREM 5.1.** *The map  $\deg_T : \mathcal{N} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ ,  $(f, \Omega) \mapsto \deg_T(f, \Omega)$ , possesses the following properties:*

1. *Homotopy invariance:*  
If  $h \in \mathcal{HNA}_T(\Omega)$  then  $\deg_T(h_t, \Omega) = \deg_T(h_0, \Omega)$  for every  $t \in (0, 1]$ .
2. *Excision:*  
Assume that  $\Omega_0 \subset \Omega$  is  $T$ -invariant and  $f^{-1}(\{0\}) \cap \Omega \subset \Omega_0$ . Then

$$\deg_T(f, \Omega) = \deg_T(f|_{\Omega_0}, \Omega_0).$$

3. *Additivity:*  
Assume that  $\Omega_1, \Omega_2$  are disjoint open  $T$ -invariant subsets of  $\Omega$  such that  $f^{-1}(\{0\}) \cap \Omega \subset \Omega_1 \cup \Omega_2$ . Then

$$\deg_T(f, \Omega) = \deg_T(f|_{\Omega_1}, \Omega_1) + \deg_T(f|_{\Omega_2}, \Omega_2).$$

4. *Existence:*  
If  $\deg_T(f, \Omega) \neq 0$  then there exists a point  $x \in \Omega$  such that  $f(x) = 0$ .

We call  $\deg_T : \mathcal{N} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  the  $T$ -equivariant degree of normal maps. Its properties follow directly from the definition. It is worth pointing out that if  $f \in \mathcal{NA}_T(\Omega)$  then there is the following dependence between  $\deg(f, \Omega)$  and  $\deg_T(f, \Omega)$ :

$$\deg(f, \Omega) = d_0 + 2d_1, \quad d_0 = \deg(f, \Omega(\varepsilon)).$$

Let  $\mathcal{E}$  denote the family of all pairs  $(f, \Omega)$  such that  $f \in \mathcal{A}_T(\Omega)$  and  $\Omega$  is  $T$ -admissible. Applying Theorem 4.1 one can extend the  $T$ -equivariant degree over  $\mathcal{E}$ . Consider  $f \in \mathcal{A}_T(\Omega)$ . There exists  $g \in \mathcal{NA}_T(\Omega)$  such that  $[g] = [f]$ . Set

$$\text{Deg}_T(f, \Omega) = \deg_T(g, \Omega).$$



From Theorem 4.1 it follows that the above formula does not depend on the choice of  $g$ .

DEFINITION 5.1. The map  $\text{Deg}_T : \mathcal{E} \rightarrow \mathbb{Z} \oplus \mathbb{Z}, (f, \Omega) \mapsto \text{Deg}_T(f, \Omega)$ , is called the *T-equivariant degree*.

The next theorem is a natural consequence of Definition 5.1 and Theorem 5.1.

THEOREM 5.2. *The T-equivariant degree possesses the following properties:*

1. If  $h \in \mathcal{HA}_T(\Omega)$  then  $\text{Deg}_T(h_t, \Omega) = \text{Deg}_T(h_0, \Omega)$  for every  $t \in (0, 1]$ .
2. Assume that  $\Omega_0 \subset \Omega$  is  $T$ -invariant and  $f^{-1}(\{0\}) \cap \Omega \subset \Omega_0$ . Then

$$\text{Deg}_T(f, \Omega) = \text{Deg}_T(f|_{\Omega_0}, \Omega_0).$$

3. Assume that  $\Omega_1$  and  $\Omega_2$  are disjoint open  $T$ -invariant subsets of  $\Omega$  such that  $f^{-1}(\{0\}) \cap \Omega \subset \Omega_1 \cup \Omega_2$ . Then

$$\text{Deg}_T(f, \Omega) = \text{Deg}_T(f|_{\Omega_1}, \Omega_1) + \text{Deg}_T(f|_{\Omega_2}, \Omega_2).$$

4. If  $\text{Deg}_T(f, \Omega) \neq 0$  then there exists a point  $x \in \Omega$  such that  $f(x) = 0$ .

**6. T-homotopies versus gradient T-homotopies.** In this section we prove the Parusiński theorem in  $T$ -invariant case.

From now on, we assume that  $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$  and  $p \geq 2$ . Let  $B^n$  denote the open unit ball,  $S^{n-1}$  the unit sphere, and  $D^n$  the unit disc in  $\mathbb{R}^n$  centered at 0. We have  $D^n = B^n \cup S^{n-1}$ . It is trivial that these sets are  $T$ -invariant. Set  $D^p = D^n \cap \mathbb{R}^p$ ,  $B^p = B^n \cap \mathbb{R}^p$  and  $S^{p-1} = S^{n-1} \cap \mathbb{R}^p$ .

Among many generalizations of the Brouwer degree there is the stable equivariant degree. It was considered by several authors (see [2, 4, 17] and the references given there). The stable equivariant degree of the  $T$ -equivariant continuous map  $f : S^{n-1} \rightarrow S^{n-1}$  is the element  $d_T(f) \in \mathbb{Z} \oplus \mathbb{Z}$  given by

$$d_T(f) = (\text{deg}(f, S^{p-1}), \text{deg}(f, S^{n-1})).$$

Let  $[S^{n-1}, S^{n-1}]_T$  denote the set of all  $T$ -equivariant homotopy classes of  $T$ -equivariant continuous self-maps of  $S^{n-1}$ . Let  $[f]_T$  stands for the  $T$ -equivariant homotopy class of  $f : S^{n-1} \rightarrow S^{n-1}$ . D. Ferrario proved that the stable equivariant degree  $d_T$  classifies  $T$ -equivariant continuous self-maps of  $S^{n-1}$  (see Theorem 7.1 in [4]). This means that the map

$$[S^{n-1}, S^{n-1}]_T \ni [f]_T \longmapsto d_T(f) \in \mathbb{Z} \oplus \mathbb{Z}$$

is an injection.

PROPOSITION 6.1. *Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a T-equivariant continuous map such that  $f(S^{n-1}) \subset \mathbb{R}^n \setminus \{0\}$ . Then there exist a T-equivariant continuous map  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a T-equivariant homotopy  $h : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$  such that*

- $h_0 = f, h_1 = \hat{f}$ ,
- $\hat{f}(S^{n-1}) \subset S^{n-1}$ ,
- $\hat{f}(D^n) \subset D^n$ ,
- $h(S^{n-1} \times [0, 1]) \subset \mathbb{R}^n \setminus \{0\}$ .



*Proof.* We check at once that  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by

$$\hat{f}(x) = \begin{cases} |x| \left| f\left(\frac{x}{|x|}\right) \right|^{-1} f\left(\frac{x}{|x|}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

and  $h(x, t) = t\hat{f}(x) + (1 - t)f(x)$  satisfy all the claims. ■

Fix  $f \in \mathcal{A}_T(B^n)$ . Clearly,  $f$  can be extended to a  $T$ -equivariant continuous map over  $\mathbb{R}^n$ . Moreover, two different extensions of  $f$  are linear homotopic and the linear homotopy joining these extensions has no zeroes on  $S^{n-1} \times [0, 1]$ . Therefore we identify  $f$  with its extension. Let  $\hat{f}$  be a map as in Proposition 6.1. Then  $\text{Deg}_T(f, B^n) = \text{Deg}_T(\hat{f}, B^n)$ . Remark that there is one-to-one correspondence between  $\text{Deg}_T(f, B^n) = (d_0, d_1)$  and  $d_T(\hat{f}) = (\text{deg}(\hat{f}, S^{p-1}), \text{deg}(\hat{f}, S^{n-1}))$ . Namely,

$$d_0 = \text{deg}(\hat{f}, S^{p-1}), \quad d_1 = \frac{1}{2}(\text{deg}(\hat{f}, S^{n-1}) - \text{deg}(\hat{f}, S^{p-1})).$$

Therefore the map

$$\mathcal{A}_T[B^n] \ni [f] \longmapsto d_T(\hat{f}) \in \mathbb{Z} \oplus \mathbb{Z}$$

is an injection.

**CONCLUSION 6.2.** *The  $T$ -equivariant degree  $\text{Deg}_T(f, B^n)$  classifies  $T$ -admissible maps from  $D^n$  into  $\mathbb{R}^n$ .*

Another generalization of the Brouwer degree is the  $T$ -equivariant degree for gradient  $T$ -equivariant maps from  $\mathbb{R}^n$  into  $\mathbb{R}^n$ . This degree was considered in [7, 5, 1].

Assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $T$ -equivariant continuous map. We say that  $f$  is  $\nabla_T$ -admissible if  $f(S^{n-1}) \subset \mathbb{R}^n \setminus \{0\}$  and there exists a  $T$ -equivariant  $C^1$  function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f = \nabla\varphi$ . We will denote by  $\nabla\mathcal{A}_T(B^n)$  the set of all  $\nabla_T$ -admissible maps. In the same spirit we introduce the notion of  $\nabla_T$ -admissible homotopy. Let  $h : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$  be a  $T$ -equivariant homotopy. We say that  $h$  is  $\nabla_T$ -admissible if  $h(S^{n-1} \times [0, 1]) \subset \mathbb{R}^n \setminus \{0\}$  and there exists a  $T$ -equivariant  $C^1$  function  $\chi : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}$  such that  $h(x, t) = \nabla_x \chi(x, t)$  for all  $x \in \mathbb{R}^n$  and  $t \in [0, 1]$ .

$f$  is homotopic to  $g$  in  $\nabla\mathcal{A}_T(B^n)$ , if there is a  $\nabla_T$ -admissible homotopy  $h$  joining  $f$  to  $g$ . The  $\nabla_T$ -admissible homotopy class of  $f \in \nabla\mathcal{A}_T(B^n)$  will be denoted by  $[f]_\nabla$ . The set of all  $\nabla_T$ -admissible homotopy classes in  $\nabla\mathcal{A}_T(B^n)$  will be denoted by  $\nabla\mathcal{A}_T[B^n]$ .

The  $T$ -equivariant degree of  $f \in \nabla\mathcal{A}_T(B^n)$  is the element  $\nabla_T \text{deg}(f, B^n) \in \mathbb{Z} \oplus \mathbb{Z}$ . From the construction made by Gęba in [5] (see formula 3.5, Theorems 3.2 and 3.3), it follows that

$$\nabla_T \text{deg}(f, B^n) = \text{Deg}_T(f, B^n).$$

Dancer, Gęba and Rybicki proved that this degree classifies  $\nabla_T$ -admissible maps. More precisely, the map

$$\nabla\mathcal{A}_T[B^n] \ni [f]_\nabla \longmapsto \nabla_T \text{deg}(f, B^n) \in \mathbb{Z} \oplus \mathbb{Z}$$

is a bijection (see Corollary 4.1 and Remark 4.1 in [1]).

**CONCLUSION 6.3.** *The  $T$ -equivariant degree  $\text{Deg}_T(f, B^n)$  classifies  $\nabla_T$ -admissible maps.*



From Conclusions 6.2 and 6.3 we get a nice theorem.

**THEOREM 6.4.** *Assume that  $f, g \in \nabla \mathcal{A}_T(B^n)$ . If  $f$  is homotopic to  $g$  in  $\mathcal{A}_T(B^n)$  then  $f$  is homotopic to  $g$  in  $\nabla \mathcal{A}_T(B^n)$ .*

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