

Lower bound on the weakly connected domination number of a tree

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Abstract

We prove that the weakly connected domination number of every tree T on $n \geq 3$ vertices and with n_1 end vertices satisfies the inequality $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ and we characterize the extremal trees.

1 Introduction

Let $G = (V, E)$ be a connected undirected graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v . For a set $X \subseteq V(G)$, the *open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* $N_G[X] = N_G(X) \cup X$. A subset D of $V(G)$ is *dominating* in G if every vertex of $V(G) - D$ has at least one neighbour in D . Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G . The degree of a vertex v is $d_G(v) = |N_G(v)|$. Further, $D \subset V(G)$ is a *connected dominating set* in G if D is dominating and the subgraph $G[D]$ induced by D in G , is connected.

A dominating set $D \subset V(G)$ is a *weakly connected dominating set* in G if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by D , is connected, where E_w is the set of all edges having at least one vertex in D . Dunbar et al. [1] define the *weakly connected domination number* $\gamma_w(G)$ of a graph G to be the minimum cardinality among all weakly connected dominating sets in G .

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $(u - v)$ -path in G . For unexplained terms and symbols see [3].

Here we consider trees on at least three vertices. If T is a tree, let $n = n(T)$ be the order of T and let $n_1 = n_1(T)$ denote the number of end-vertices of T . The set of all end-vertices in T is denoted by $\Omega(T)$. A vertex v is called a *support* if it is adjacent to an end-vertex.

Let D be a weakly connected dominating set of a tree T . We say that D has the property \mathcal{F} if D contains no end-vertex of T . It is obvious that in every tree on at least three vertices exists a minimum weakly connected dominating set having property \mathcal{F} .

We say that vertices $x, y \in D$ are *adjacent in D* if there exists an edge $e = xy$ or if there exists an $(x - y)$ -path whose vertices different from x and y do not belong to D .

It is known [4] that $\gamma(T) \geq (n(T) + 2 - n_1(T))/3$ for a tree T on $n \geq 3$ vertices. Here we prove a similar inequality for the weakly connected domination number, i.e. we show, that $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ and we characterize the family of all trees T for which $\gamma_w(T) = (n(T) + 1 - n_1(T))/2$.

2 Results

We begin with the following result.

Theorem 1 *If T is a tree of order at least three, then $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$.*

Proof. We use induction on n , the order of a tree. The result is trivial for each tree of order 3. Let T be a tree of order $n > 3$ and assume that $\gamma_w(T') \geq (n(T') + 1 - n_1(T'))/2$ for each tree T' with $3 \leq n(T') \leq n - 1$. Let D be a minimum weakly connected dominating set having property \mathcal{F} in T , let $P = (v_0, v_1, \dots, v_l)$ be a longest path in T and let $T' = T - \{v_0\}$ be the subtree of T . Without loss of generality we may assume, that P is chosen in such a way that $d_T(v_1)$ is as large as possible. We consider two cases: $d_T(v_1) > 2$ and $d_T(v_1) = 2$.

Case 1. If $d_T(v_1) > 2$, then D is a minimum weakly connected dominating set of $T' = T - \{v_0\}$ and thus $\gamma_w(T') = \gamma_w(T)$. By induction, $\gamma_w(T') \geq (n(T') + 1 - n_1(T'))/2$ and therefore $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ as $n_1(T') = n_1(T) - 1$ and $n(T') = n(T) - 1$.

Case 2. If $d_T(v_1) = 2$, then we consider two subcases: $\gamma_w(T') < \gamma_w(T)$ and $\gamma_w(T') = \gamma_w(T)$.

Subcase 2.1. If $\gamma_w(T') < \gamma_w(T)$, then it is easy to observe that $\gamma_w(T') = \gamma_w(T) - 1$. By induction, $\gamma_w(T') \geq (n(T') + 1 - n_1(T'))/2$ and consequently, $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ as $n_1(T') = n_1(T)$ and $n(T') = n(T) - 1$.

Subcase 2.2. If $\gamma_w(T') = \gamma_w(T)$, then $v_2 \notin D$ (otherwise $D - \{v_1\}$ would be a weakly connected dominating set of T' and equality $\gamma_w(T') = \gamma_w(T)$ would not hold) and $l \geq 4$. It is easy to observe that $D'' = D - \{v_1\}$ is a minimum weakly connected dominating set of $T'' = T - \{v_0, v_1\}$. We consider separately $d_T(v_2) > 2$ and $d_T(v_2) = 2$.

If $d_T(v_2) > 2$, then $n_1(T'') = n_1(T) - 1$, $n(T'') = n(T) - 2$ and $\gamma_w(T'') = \gamma_w(T) - 1$. Then, by induction, $\gamma_w(T'') \geq (n(T'') + 1 - n_1(T''))/2$ and therefore $\gamma_w(T) \geq (n(T) - n_1(T) + 1)/2$.

If $d_T(v_2) = 2$, then, by induction, $\gamma_w(T'') \geq (n(T'') + 1 - n_1(T''))/2$ and thus $\gamma_w(T) \geq (n(T) - n_1(T) + 1)/2$ as $n_1(T'') = n_1(T)$, $n(T'') = n(T) - 2$ and $\gamma_w(T'') = \gamma_w(T) - 1$. ■

Now we consider a family \mathcal{R} which will provide the extremal trees in Theorem 1.

If T_1 and T_2 are two trees, then let $K(T_1, T_2)$ denote a tree obtained from T_1 and T_2 by identifying one end-vertex of T_1 with one end-vertex of T_2 .



We say that a tree T belongs to the family \mathcal{R} if and only if there exists a sequence of stars S_1, \dots, S_p such that every star has at least three vertices and $T = K(K(\dots K(K(S_1, S_2), S_3) \dots), S_p)$ (see Fig. 1).

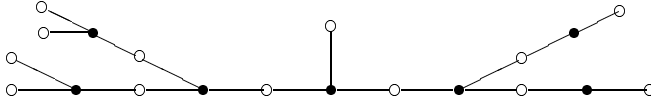


Figure 1: An example of a tree from the family \mathcal{R}

The following two observations are obvious from definition of the family \mathcal{R} .

Observation 1 *If T is a tree belonging to the family \mathcal{R} , then the distance between any two end-vertices of T is an even number.*

Observation 2 *If T belonging to the family \mathcal{R} is a tree obtained from the stars S_1, \dots, S_p and if vertices c_1, \dots, c_p are centers of the stars, then the set $D = \{c_1, \dots, c_p\}$ is a unique minimum weakly connected dominating set of T . Additionally, every non-end vertex belonging to $V(T) - D$, has degree two.*

Proof. Reader may use induction on p to show that $D = \{c_1, \dots, c_p\}$ is a unique minimum weakly connected dominating set of T belonging to the family \mathcal{R} . ■

Now we prove the following theorem.

Theorem 2 *For every tree T on $n \geq 3$ vertices is $\gamma_w(T) = (n(T) - n_1(T) + 1)/2$ if and only if T belongs to the family \mathcal{R} .*

Proof. Let T be a tree belonging to the family \mathcal{R} and let D be a minimum weakly connected dominating set having property \mathcal{F} in T . From Observation 1 and Observation 2 it follows that for every two vertices $u, v \in D$, the distance $d_T(u, v)$ is an even number and the distance between any two adjacent in D vertices is exactly two. Any two vertices adjacent in D are separated by exactly one vertex of degree two belonging to $V(T) - D$. Thus we have $n(T) = |D| + |V(T) - D| = |D| + n_1(T) + (|D| - 1)$, and hence $n(T) = n_1(T) + 2\gamma_w(T) - 1$, which finally gives $\gamma_w(T) = (n(T) - n_1(T) + 1)/2$.

Now we show that if $\gamma_w(T) = (n(T) - n_1(T) + 1)/2$, then T belongs to the family \mathcal{R} . It suffices to show that $\gamma_w(T) > (n(T) - n_1(T) + 1)/2$ if T does not belong to \mathcal{R} .

Let T be a tree not belonging to the family \mathcal{R} . It is obvious, that T is not a star, because every star belongs to \mathcal{R} . Thus $n \geq 4$ and for $n = 4$ T is a path on four vertices and it is easy to check that $\gamma_w(T) > (n(T) - n_1(T) + 1)/2$. Assume that T has at least five vertices and assume that $\gamma_w(T') > (n(T') - n_1(T') + 1)/2$ for each tree T' not belonging to the family \mathcal{R} , where $n(T') \leq n(T) - 1$. Let $P = (v_0, v_1, v_2, \dots, v_l)$ be a longest path in T and let D be a minimum weakly connected dominating set satisfying property \mathcal{F} in T . Without loss of generality we may assume, that P is chosen in such a way that $d_T(v_1)$ is as large as possible. We consider two cases:



$d_T(v_1) > 2$ and $d_T(v_1) = 2$.

Case 1. If $d_T(v_1) > 2$, then $T' = T - \{v_0\}$ does not belong to \mathcal{R} and D is a minimum weakly connected dominating set of T' . Thus we have $\gamma_w(T') = \gamma_w(T)$. By induction, $\gamma_w(T') > (n(T') + 1 - n_1(T'))/2$ and therefore $\gamma_w(T) > (n(T) + 1 - n_1(T))/2$ as $n_1(T') = n_1(T) - 1, n(T') = n(T) - 1$.

Case 2. If $d_T(v_1) = 2$, then we consider $T'' = T - \{v_0, v_1\}$. It is easy to observe that $D - \{v_1\}$ is a minimum weakly connected dominating set of T'' . Thus we have $\gamma_w(T') = \gamma_w(T) - 1$.

If $d_T(v_2) = 2$, then $T'' \notin \mathcal{R}$ (in the other case T would belong to the family \mathcal{R}). By induction $\gamma_w(T'') > (n(T'') - n_1(T'') + 1)/2$ and hence $\gamma_w(T) > (n(T) - n_1(T) + 1)/2$ as $n_1(T'') = n_1(T)$ and $n(T'') = n(T) - 2$.

If $d_T(v_2) > 2$, then we have $n_1(T'') = n_1(T) - 1, n(T'') = n(T) - 2$ and $\gamma_w(T') = \gamma_w(T) - 1$. If $T \notin \mathcal{R}$, then $\gamma_w(T) - 1 > (n(T) - 2 - n_1(T) + 1 + 1)/2$, so $\gamma_w(T) > (n(T) - n_1(T) + 2)/2 > (n(T) - n_1(T) + 1)/2$.

If $T'' \in \mathcal{R}$, then $\gamma_w(T'') = (n(T'') - n_1(T'') + 1)/2$. Since $T'' \in \mathcal{R}$, we have $d_T(v_2) > 2$ (in the other case T would belong to \mathcal{R}). Thus $\gamma_w(T) - 1 = (n(T) - 2 - n_1(T) + 1 + 1)/2$, so $\gamma_w(T) = (n(T) - n_1(T) + 2)/2 > (n(T) - n_1(T) + 1)/2$. ■

Since every connected dominating set is a weakly connected dominating set and every weakly connected dominating set is a dominating set, we have the following inequality chain for every tree T : $\gamma(T) \leq \gamma_w(T) \leq \gamma_c(T)$. It is known [5] that $\gamma_c(T) = n(T) - n_1(T)$, so we also have an upper bound for $\gamma_w(T)$ in terms of $n(T)$ and $n_1(T)$. We can characterize all trees T for which the equality $\gamma_w(T) = n(T) - n_1(T)$ holds.

Theorem 3 *For a tree T on at least three vertices $\gamma_w(T) = n(T) - n_1(T)$ if and only if every non-end vertex of T is a support.*

Proof. Let T be a tree of order $n \geq 3$ and let $\gamma_w(T) = n(T) - n_1(T)$. Let D be a minimum weakly connected dominating set with property \mathcal{F} in T . Since D has property \mathcal{F} and $\gamma_w(T) = n(T) - n_1(T)$, every non-end vertex of T belongs to D . Suppose that in T there exists a non-end vertex u which is not a support. Then $D - \{u\}$ is a weakly connected dominating set of T , a contradiction.

Now assume that T is a tree in which every non-end-vertex is a support and let D be a minimum weakly connected dominating set with property \mathcal{F} in T . Every support belongs to D and no end vertex belongs to D , so $|D| = n(T) - n_1(T) = \gamma_w(T)$. ■

The bound $n - n_1$ is also an upper bound for the domination number γ . Favaron [2] proved that $\gamma(T) \leq (n(T) + n_1(T))/3$ for every tree T on at least 3 vertices. For trees with a great number of end vertices, i.e. for every tree T with $n_1(T) > (n(T))/2$ we have $(n(T) + n_1(T))/3 > n(T) - n_1(T)$, so the bound $n(T) - n_1(T)$ is better than $(n(T) + n_1(T))/3$.



3 Concluding remarks

From [5] and the above results it follows that $(n(T) + 1 - n_1(T))/2 \leq \gamma_w(T) \leq n(T) - n_1(T)$ for every tree T on at least three vertices. The example of the caterpillar given in Fig. 2 shows that the difference between $\gamma_w(T)$ and $(n(T) + 1 - n_1(T))/2$ can be arbitrarily large. It is no problem to observe that $\gamma_w(T_l) - (n(T_l) + 1 - n_1(T_l))/2 = l/6$ for any integer $l \equiv 0 \pmod{3}$.

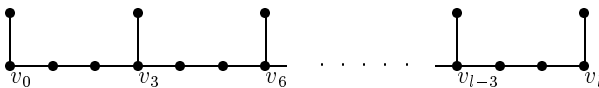


Figure 2: Caterpillar T_l

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