Forwarding and optical indices of a graph

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A R T I C L E   I N F O

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Motivated by wavelength-assignment problems for all-to-all traffic in optical networks, we study graph parameters related to sets of paths connecting all pairs of vertices. We consider sets of both undirected and directed paths, under minimisation criteria known as edge congestion and wavelength count; this gives rise to four parameters of a graph G: its edge forwarding index $\pi(G)$, arc forwarding index $\vec{\pi}(G)$, undirected optical index $w(G)$, and directed optical index $\vec{w}(G)$.

In the paper we address two long-standing open problems: whether the equality $\vec{\pi}(G) = \vec{w}(G)$ holds for all graphs, and whether indices $\pi(G)$ and $w(G)$ are hard to compute. For the first problem, we give an example of a family of planar graphs $\{G_k\}$ such that $\vec{\pi}(G_k) \neq \vec{w}(G_k)$. For the second problem, we show that determining either $\pi(G)$ or $w(G)$ is *NP*-hard.

1. Introduction

Many aspects of wavelength routing in optical networks can be modelled in terms of combinatorial graph problems. In this paper we concentrate on a study of properties of *all-to-all routings*, i.e. sets of paths connecting all pairs of vertices of the graph. All-to-all routings serve as a model for gossiping traffic in optical networks; a more detailed discussion of the technical properties of such networks is provided in [14,19].

An undirected graph $G = (V, E)$ of order $n = |V|$ and size $|E|$ can be treated as a model for an optical network, where V represents the set of nodes, while E corresponds to the set of direct physical links between them, with each edge representing a bidirectional communication channel. A sequence of edges $P = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_l, v_{l+1}\})$ such that $\{v_i, v_{i+1}\} \in E$ and $v_i \neq v_j$ for $i \neq j$ is called a *path* of length l in G, with endpoints $\{v_1, v_{l+1}\}$. A pair of paths $\{P_1, P_2\}$ is called *conflicting* if there exists an edge $e \in E$ such that $e \in P_1$ and $e \in P_2$. An *undirected routing* R in graph G is any set of (not necessarily distinct) paths in G. For a given routing R, we define the *conflict graph* $Q(R)$ as the undirected graph with vertex set R and edges corresponding to pairs of conflicting paths of R.

Two of the quality characteristics of a routing most often considered in the literature are defined as follows [3,7].

- *Edge congestion* $\pi(R)$ is given by the formula $\pi(R) = \max_{\{u,v\} \in E} |\{P \in R : \{u, v\} \in P\}|$ and describes the maximum number of paths passing through an edge of the graph.
- *Wavelength count* $w(R)$ is equal to the chromatic number of the conflict graph, $w(R) = \chi(Q(R))$, representing the minimum number of colors which can be assigned to paths of R in such a way that conflicting paths receive different colors. It is easy to see that $w(R) \geq \pi(R)$ for any routing [3].

Analogous parameters can be introduced when considering directed paths. A *directed path* (*dipath*) with endpoints (v_1, v_{l+1}) is a sequence of arcs $\vec{P} = ((v_1, v_2), (v_2, v_3), \ldots, (v_l, v_{l+1}))$ such that $\{v_i, v_{i+1}\} \in E$ and $v_i \neq v_j$ for $i \neq j$. Two dipaths

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conflict if there exists an arc $\vec{e} = (u, v)$, $\{u, v\} \in E$, such that $\vec{e} \in \vec{P}_1$ and $\vec{e} \in \vec{P}_2$. For a directed routing $\vec{R}$, its wavelength count is defined as in the undirected case, $\vec{w}(\vec{R}) = \chi(Q(\vec{R}))$, while its edge congestion is given as the maximum number of paths passing through an arc, $\vec{\pi}(\vec{R}) = \max_{(u,v):\{u,v\} \in E} |\{\vec{P} \in \vec{R} : (u, v) \in \vec{P}\}|$.

An *undirected all-to-all routing* R_A in graph G is defined as any set of $n(n-1)$ paths in graph G such that R_A contains exactly two paths with endpoints $\{u, v\}$ for all $u, v \in V$. Likewise, a *directed all-to-all routing* $\vec{R}_A$ is any set of $n(n-1)$ dipaths containing a dipath with endpoints (u, v) for all $u, v \in V$. The set of all undirected all-to-all routings in G is denoted by $\mathcal{R}_A(G)$, and the set of all directed all-to-all routings by $\vec{\mathcal{R}}_A(G)$. The minimum possible values of edge congestion and wavelength count for an all-to-all routing define four parameters of graph G known as its *indices* [6,8,15]:

- *edge forwarding index* $\pi(G) = \min_{R_A \in \mathcal{R}_A(G)} \pi(R_A)$,
- *arc forwarding index* $\vec{\pi}(G) = \min_{\vec{R}_A \in \vec{\mathcal{R}}_A(G)} \vec{\pi}(\vec{R}_A)$,
- *undirected optical index* $w(G) = \min_{R_A \in \mathcal{R}_A(G)} w(R_A)$,
- *directed optical index* $\vec{w}(G) = \min_{\vec{R}_A \in \vec{\mathcal{R}}_A(G)} \vec{w}(\vec{R}_A)$.

Investigations into the considered indices of graphs have been very intensive in recent years, starting with a paper of Heydemann et al. [15] in 1989. A detailed study of the edge and arc forwarding indices, as well as their generalisations for other types of network traffic, can be found in [8,16,22], while results concerning optical indices and their relation to forwarding indices have been collected in survey papers [3,7,13].

It is well known that for any graph there hold the inequalities $\pi(G) \leq w(G)$ and $\vec{\pi}(G) \leq \vec{w}(G)$ [6]. Motivated by positive results for a related problem concerning so called *one-to-many* traffic [4], there has been an ongoing effort to establish whether there exist tighter bounds on the graph indices. It has been shown that the equality $\pi(G) = w(G)$ is not always true even for a class of trees known as subdivided stars [3]. However, the same question for the directed indices is a long-standing open problem.

Problem 1 (*[2,3,7,13,21]*). Does $\vec{\pi}(G) = \vec{w}(G)$ hold for all graphs?

Partial answers to this problem have been obtained by studying specific graph classes. In particular, the equality is known to hold for all trees [12], cycles [6] and trees of cycles [5], hypercubes [6], some families of recursive circulant graphs [1], Cartesian sums of complete graphs [2,23], tori (Cartesian products of cycles of the same length) of even [2] and odd [21] order, as well as grids (Cartesian products of paths of the same length) of even order [2].

For all the above stated graph classes, there exist polynomial time algorithms for determining exact values of $\vec{\pi}(G)$ and $\vec{w}(G)$. Values of $\pi(G)$ and $w(G)$ can be efficiently determined for hypercubes and cycles [6]. For trees, the value of $\pi(G)$ can be trivially established, but there is as yet no known polynomial time algorithm for finding $w(G)$ [3]. In the general case, the computational complexity of determining all the considered indices has been posed as open.

Problem 2 (*[3,7,15]*). Does there exist an efficient algorithm for computing $\pi(G)$, $w(G)$, $\vec{\pi}(G)$ and $\vec{w}(G)$ for general graphs?

The only known negative evidence concerns a somewhat related parameter called the *vertex forwarding index* of a graph [9], which was shown to be *NP*-hard to determine in the general case [20].

Outline of the paper

In the following sections we address Problems 1 and 2. In Section 2 we show that the inequality $\vec{\pi}(G) = \vec{w}(G)$ is untrue for an infinite family of planar graphs, thus completely solving Problem 1. In Section 3 we give a partial negative answer to Problem 2, showing that indices $\pi(G)$ and $w(G)$ are *NP*-hard to compute in general. Final conclusions are stated in Section 4.

2. An example of a graph family with $\vec{\pi}(G_k) \neq \vec{w}(G_k)$

Let us begin by considering the graph G shown in Fig. 1(a). It is the 3-regular 10-vertex planar graph constructed by connecting corresponding vertices of two disjoint copies of the cycle C_5, the *inner cycle* with vertices $\{i_1, \ldots i_5\}$ and the *outer cycle* with vertices $\{o_1, \ldots o_5\}$, by five *radial edges*.

In all further considerations we assume that $a \in \mathbb{N}$ is an arbitrary, fixed positive integer. We will use the term *outer cycle routing* $\vec{R}$ to refer to a routing consisting of $80a$ dipaths in the considered graph G, such that exactly $4a$ dipaths have the pair of endpoints (o_i, o_j), for all $1 \leq i, j \leq 5$, $i \neq j$. Given an outer cycle routing $\vec{R}$, we denote by R the corresponding set of $80a$ undirected paths in G formed by removing the orientation of dipaths of $\vec{R}$. Thus, exactly $8a$ paths from R have the pair of endpoints $\{o_i, o_j\}$, for all $1 \leq i < j \leq 5$. For a path $P \in R$, we define its *cycle distance* $\text{cd}(P) \in \{1, 2\}$ as the distance between its endpoints in graph G, and its *outer cycle length* $\text{ocl}(P)$ as the number of edges from the outer cycle which belong to P. The *inner cycle length* of path P is defined as $\text{icl}(P) = \max\{0, \text{cd}(P) - \text{ocl}(P)\}$, and is obviously a lower bound on the number of edges from the inner cycle which belong to P. Observe that the following inequality is true for any path:

$$\text{icl}(P) + \text{ocl}(P) \geq \text{cd}(P). \tag{1}$$

We now proceed to state two lemmas which characterise the properties of any outer cycle routing $\vec{R}$ and its undirected counterpart R.
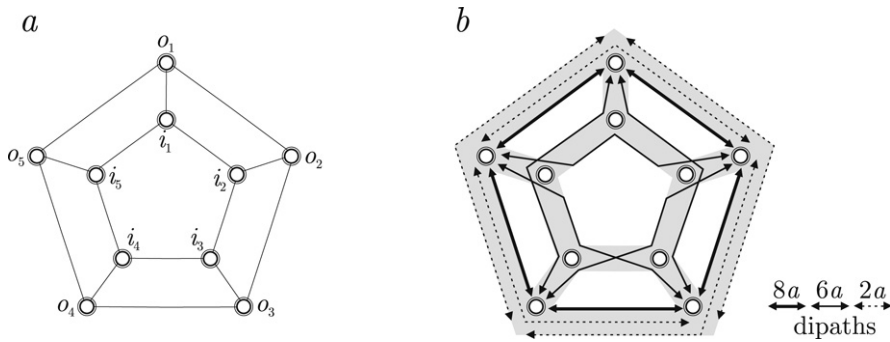
Fig. 1. (a) The graph G, (b) An outer cycle routing $\vec{R}$ with edge congestion $\vec{\pi}(\vec{R}) = 6a$ (lines of different style denote different numbers of dipaths, half of which are directed towards one endpoint, and half towards the other endpoint).

Lemma 3. *Let us consider the disjoint partition of R into three subsets, $R = R_0 \cup R_1 \cup R_2$, given by the formula $R_i = \{P \in R : \text{icl}(P) = i\}$. If $\pi(R) \leq 12a + \varepsilon$, for some $\varepsilon \geq 0$, then $|R_2| \geq 30a - 7.5\varepsilon$.*

Proof. The sum of cycle distances, taken over all paths of outer cycle routing R, is independent of R, since exactly $40a$ paths must have a cycle distance of 1, and $40a$ paths must have a cycle distance of 2. Consequently, we may write $\sum_{P \in R} \text{cd}(P) = 1 \cdot 40a + 2 \cdot 40a = 120a$. By applying inequality (1) for each path $P \in R$, we obtain the following relation:

$$\sum_{P \in R} \text{icl}(P) + \sum_{P \in R} \text{ocl}(P) \geq \sum_{P \in R} \text{cd}(P) = 120a. \tag{2}$$

By definition of the partition of R, each path from R_i, $0 \leq i \leq 2$, contributes exactly i to the total inner cycle length of all paths; thus we have

$$\sum_{P \in R} \text{icl}(P) = |R_1| + 2|R_2|. \tag{3}$$

On the other hand, the total outer cycle length of all paths of R is bounded by the total congestion of the five edges of the outer cycle

$$\sum_{P \in R} \text{ocl}(P) \leq 5\pi(R) \leq 5(12a + \varepsilon) = 60a + 5\varepsilon. \tag{4}$$

Introducing relations (3) and (4) into relation (2) gives the following inequality:

$$|R_1| + 2|R_2| \geq 120a - (60a + 5\varepsilon) \geq 60a - 5\varepsilon. \tag{5}$$

Next, observe that since each path of $R_1 \cup R_2$ uses at least one edge from the inner cycle, it must also contribute to the congestion of at least two of the five radial edges of the graph; thus we may write

$$|R_1| + |R_2| \leq \frac{5\pi(R)}{2} \leq \frac{5}{2}(12a + \varepsilon) = 30a + 2.5\varepsilon. \tag{6}$$

By combining relations (5) and (6) we obtain the sought inequality $|R_2| \geq (60a - 5\varepsilon) - (30a + 2.5\varepsilon) = 30a - 7.5\varepsilon$. □

Lemma 4. *If $\vec{\pi}(\vec{R}) \leq 6a + \varepsilon$, for some $\varepsilon \geq 0$, then $\vec{w}(\vec{R}) \geq 7.5a - 3.75\varepsilon$.*

Proof. Observe that $\pi(R) \leq 2\vec{\pi}(\vec{R}) \leq 12a + 2\varepsilon$, since the congestion of any edge of G at most doubles when converting a directed routing into an undirected routing. Let $\vec{R}_2$ denote the set of dipaths of $\vec{R}$ corresponding to undirected paths from set R_2; by definition, each of the paths from $\vec{R}_2$ contains arcs of at least two edges of the inner cycle. Now, consider any correct coloring of the conflict graph $Q(\vec{R}_2)$. It is easy to see that any color can be assigned to at most four dipaths belonging to $\vec{R}_2$, since any set of five dipaths of $\vec{R}_2$ always contains a pair of dipaths sharing an arc on the inner cycle of G. Consequently, $\vec{w}(\vec{R}_2) = \chi(Q(\vec{R}_2)) \geq \frac{1}{4}|\vec{R}_2|$. Moreover, by Lemma 3 we have $|\vec{R}_2| = |R_2| \geq 30a - 15\varepsilon$. Thus we may write: $\vec{w}(\vec{R}) \geq \vec{w}(\vec{R}_2) \geq \frac{1}{4}|\vec{R}_2| \geq \frac{30a-15\varepsilon}{4}$, which completes the proof. □

Before proceeding to the main theorem of this section, let us briefly observe that the condition stated in Lemma 4 is indeed feasible even for $\varepsilon = 0$.

Lemma 5. *There exists an outer cycle routing $\vec{R}$ such that $\vec{\pi}(\vec{R}) = 6a$.*

Proof. An example of the arrangement of dipaths in such a routing is shown in Fig. 1(b). □
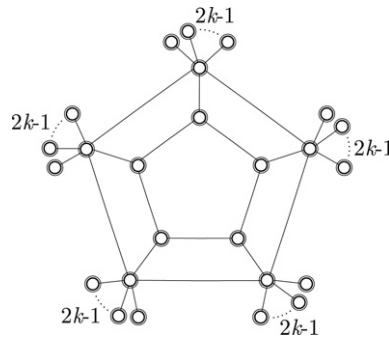
Fig. 2. The graph G_k.

Lemmas 4 and 5 imply that there exists a discrepancy between the minimum possible edge congestion and minimum possible wavelength count of an outer cycle routing in graph G. We will show that the same is true for all-to-all routing in a slightly modified family of graphs.

Theorem 6. *There exists an infinite family of planar graphs $\{G_k\}_{k \in \mathbb{N}}$ such that $\vec{w}(G_k) \neq \vec{\pi}(G_k)$ for all sufficiently large values of k.*

Proof. For any $k \in \mathbb{N}$, let $G_k \supseteq G$ be the $(10k + 5)$-vertex graph obtained from graph G by connecting exactly $2k - 1$ vertices of degree 1 to each of the vertices of the outer cycle (Fig. 2).

We will first show for all G_k that if $\vec{\pi}(G_k) < 6\frac{6}{19}k^2$, then $\vec{w}(G_k) \neq \vec{\pi}(G_k)$. The proof proceeds by contradiction. Supposing that $\vec{w}(G_k) = \vec{\pi}(G_k)$, then there must exist a directed all-to-all routing $\vec{R}_A$ in G_k such that

$$\vec{\pi}(R_A) = \vec{w}(R_A) < 6\frac{6}{19}k^2. \tag{7}$$

Let $\vec{R}_{AG}$ be the set of all dipaths in $G \subseteq G_k$, obtained by restricting each of the dipaths from $\vec{R}_A$ to the set of edges of G (this can be achieved by discarding the first or final arc of a dipath from $\vec{R}_A$, if its respective endpoint is a vertex of degree 1), and leaving out resultant dipaths of length 0. It is easy to see that the subset $\vec{R}$ of routing $\vec{R}_{AG}$ consisting of dipaths with both endpoints in the outer cycle contains exactly $4k^2$ dipaths with the pair of endpoints (o_i, o_j), for all $1 \leq i, j \leq 5, i \neq j$, thus forming a valid outer cycle routing with parameter $a = k^2$ in graph G. Taking into account relation (7), we can write

$$\vec{\pi}(\vec{R}) \leq \vec{\pi}(\vec{R}_{AG}) \leq \vec{\pi}(\vec{R}_A) < 6\frac{6}{19}k^2.$$

However, applying Lemma 4 with $a = k^2$ and $\varepsilon = \frac{6}{19}k^2$ to routing $\vec{R}$ gives $\vec{w}(\vec{R}) \geq 6\frac{6}{19}k^2$; hence

$$\vec{w}(\vec{R}_A) \geq \vec{w}(\vec{R}_{AG}) \geq \vec{w}(\vec{R}) \geq 6\frac{6}{19}k^2,$$

a contradiction with (7).

To prove the theorem, it now suffices to show that $\vec{\pi}(G_k) < 6\frac{6}{19}k^2$ for sufficiently large values of k. Although the exact value of $\vec{\pi}(G_k)$ can be determined by careful analysis of cases, we confine ourselves to a shorter proof that $\vec{\pi}(G_k) \leq 6(k + 1)^2$. Indeed, consider the all-to-all routing $\vec{R}_A$ in graph G_k obtained by extending the outer cycle routing $\vec{R}$ from the proof of Lemma 5 (with parameter $a = k^2$) by means of the following procedure.

(1) Arcs of edges from $G_k \setminus G$ are appended to some dipaths of $\vec{R}$ in such a way that for any pair of vertices u, v of G_k, neither of which belongs to the inner cycle, there is a dipath with endpoints (u, v).
(2) For each vertex i_j of the inner cycle, $1 \leq j \leq 5$, two oppositely directed dipaths are added between i_j and all other vertices of the graph (a dipath of minimum possible length should always be chosen; ties are broken by choosing the dipath with smaller outer cycle length).

For any fixed arc of an edge from G, the congestion does not exceed $6k^2$ at the start of the procedure (since $\vec{\pi}(\vec{R}) = 6k^2$), does not change throughout step 1 of the procedure, and increases by at most $12k + 6$ throughout step 2 (the proof of this fact is straightforward and left to the reader). Moreover, for any directed all-to-all routing in G_k, the congestion of each arc of an edge from $G_k \setminus G$ is equal to $10k + 4$. Consequently, we have $\vec{\pi}(G_k) \leq \vec{\pi}(\vec{R}_A) = \max\{6k^2 + 12k + 6, 10k + 4\} = 6(k + 1)^2$. It is easy to verify that $\vec{\pi}(G_k) < 6\frac{6}{19}k^2$, and therefore also $\vec{w}(G_k) \neq \vec{\pi}(G_k)$, for all values of $k \geq 39$. $\square$

The proof of Theorem 6 implies that the smallest graph known to have a directed optical index different from its arc forwarding index is the 395-vertex graph G_{39}. It would be interesting to find a counterexample of smaller order.

As a side note, we observe that the technique applied in the proof of the theorem can be slightly modified to obtain the following result.

Corollary 7. *For the considered family of graphs* $\{G_k\}_{k \in \mathbb{N}}$, *the following limit inequality is fulfilled:* $\lim_{k \to \infty} \frac{\vec{w}(G_k)}{\pi(G_k)} \geq \frac{20}{19}$.

3. Hardness of computing $\pi(G)$ and $\vec{w}(G)$

For the purposes of the proof, we will introduce an auxiliary concept known as a *k-routing*. A *k-routing* in a graph $G = (V, E)$ is defined as any undirected routing such that each vertex $v \in V$ is the endpoint of exactly k paths, and not more than two paths of the routing have the same pair of endpoints. Consequently, a routing R in G is an all-to-all routing iff it is an $2(n-1)$-routing.

In this section we will first prove that deciding whether a given 4-regular graph G admits a 10-routing R, such that $\pi(R) \leq 3$, is *NP*-complete. Next, we will show by reduction from the previous problem that deciding whether a given graph G admits an all-to-all routing R_A, such that $\pi(R_A) \leq 3$, is *NP*-complete. This directly implies the *NP*-hardness of determining the edge forwarding index $\pi(G)$, and may also be used to show that computing the undirected optical index $\vec{w}(G)$ is *NP*-hard.

Some of the results of the section were briefly announced in [18].

3.1. Reduction from three-dimensional matching to 3-loaded 10-routing

Let us consider the following auxiliary problem, which we will prove to be *NP*-complete.

3-Loaded 10-Routing Problem ($\overline{3L10R}$)	
Instance:	A 4-regular graph G.
Question:	Does G admit a 10-routing R such that $\pi(R) \leq 3$?

The proof proceeds by reduction from the problem of three-dimensional matching, restricted to subcubic instances. This problem, referred to as $\overline{3DM}$, can be formulated in the way described below and is well known to be *NP*-complete [11].

Three-Dimensional Matching Problem ($\overline{3DM}$)	
Instance:	A subcubic bipartite graph $G = (V \cup M, E)$ without vertices of degree 1, where $V = X \cup Y \cup Z$, $\lvert X \rvert = \lvert Y \rvert = \lvert Z \rvert = q$, such that for every vertex $m \in M$ we have $\deg(m) = 3$ and m is adjacent to exactly one vertex from each of the sets X, Y, Z.
Question:	Does there exist a subset $M_S \subseteq M$ of cardinality q covering all the vertices in V?

Theorem 8. *The* $\overline{3L10R}$ *problem is NP-complete.*

Proof. Let $G = (V \cup M, E)$ be any correctly defined instance of the $\overline{3DM}$ problem. We write $V = V_2 \cup V_3$, where V_3 denotes the set of vertices of degree 3 in V, and V_2 denotes the set of vertices of degree 2. Let $G' = (V' \cup M', E')$ be an isomorphic copy of G. The graph G^* is formed by connecting vertices of G with corresponding vertices in G' by edges, formally: $G^* = (V \cup M \cup V' \cup M', E \cup E' \cup \{\{v, v'\} : v \in V \cup M\})$.

We now define the class of graphs (gadgets) H_i, for $i \geq 2$, as follows. The gadget H_2 is presented in Fig. 3(a), together with relevant labels a and b at endpoints of outgoing edges. For $i \geq 2$, the gadget H_i is obtained by connecting a copy of graph H_{i-1} with a copy of graph H_2 using exactly two edges, whose endpoints are correctly labelled (as shown in Fig. 3(c)).

Finally, we convert graph G^* to the form of the 4-regular graph $G^{**} = (V^{**}, E^{**})$. This transformation is presented in Fig. 4. First, for each $m \in M$ we replace the subgraph of G^* induced by the pair of vertices $\{m, m'\}$ by the graph H_3, as shown in Fig. 4(a). Next, all edges connecting the vertex set V_3 with the vertex set V_3' are removed, and the graph $H_{\lvert V_3 \rvert}$ is inserted in their place, Fig. 4(b). Finally, for each $v \in V_2$ we replace the subgraph of G^* induced by the pair of vertices $\{v, v'\}$ by the graph H_2, preserving the labels a and b in accordance with Fig. 4(c). $\quad \square$

We complete the proof by showing that the constructed reduction is legitimate.

Claim. There exists a solution to the $\overline{3DM}$ problem in graph $G = (V \cup M, E)$ iff there exists a solution to the $\overline{3L10R}$ problem in graph $G^{**} = (V^{**}, E^{**})$.

Proof ($\Leftarrow$). Suppose that there exists a 10-routing R with edge congestion 3 in graph G^{**}. Since G^{**} is 4-regular, by an analysis of the total congestion of all edges it is easy to show that routing R fulfills the following conditions.

(1) Routing R consists of paths of lengths 1 and 2 only.
(2) The congestion of every edge from E^{**} with respect to R is equal to exactly 3.
(3) Each vertex from V^{**} is the endpoint of exactly eight paths of length 1 and exactly two paths of length 2 belonging to R.
(4) Each vertex from V^{**} is the mid-point of exactly one path of length 2 belonging to R.
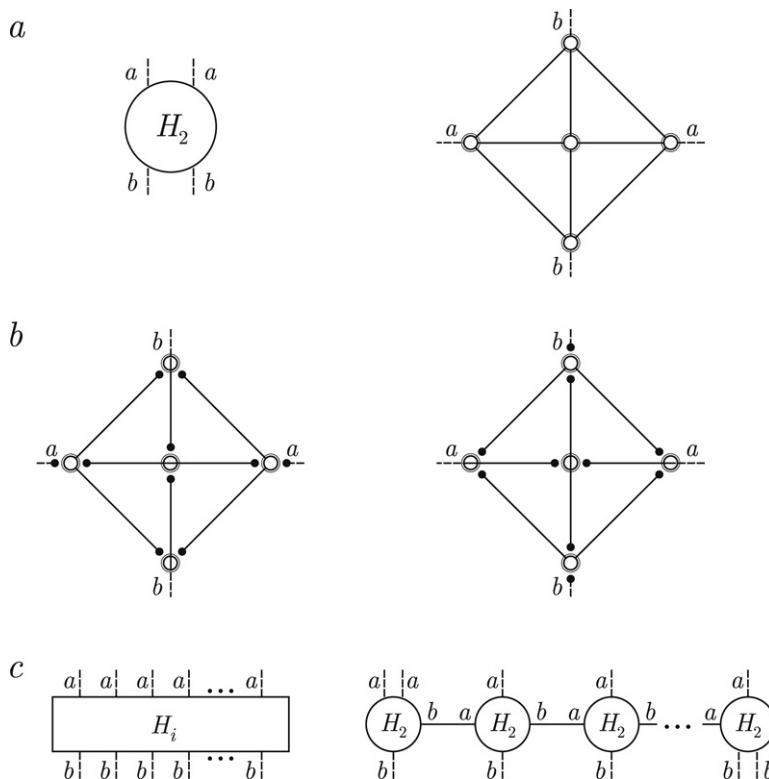
Fig. 3. Construction of the class of gadgets H_i: (a) the gadget H_2 and its labelled points, (b) the only possible arrangements of paths from R_2 in a structure containing H_2, (c) construction of H_i, $i > 2$, from $i - 1$ instances of H_2.

In view of the above properties, routing R can be written in the form of the disjoint union of two sets of paths, $R = R_1 \cup R_2$, where routing R_1 consists of paths of length 1, routing R_2 consists of paths of length 2, and each edge from E^{**} belongs to exactly two paths from R_1 and exactly one path from R_2. Since the set of paths R_1 is uniquely determined by the set of edges E^{**}, we will concentrate on the properties of set R_2 only (taking into account that the endpoints of a path from R_2 cannot be adjacent vertices in G^{**}, since such a pair of vertices is connected by two paths from R_1).

Let us now consider the properties of set of paths R_2 restricted to an arbitrarily chosen subgraph isomorphic to $H_2 \subseteq G^{**}$. Notice that there exist exactly two correct arrangements of paths from R_2 in H_2, Fig. 3(b), because any other arrangement either results in congestion different from 1 for some edge, or in more than two paths from R connecting some pair of vertices. For both the correct arrangements, all endpoints of edges of H_2 which are labelled a are either simultaneously the endpoints, or the mid-points of paths from R_2; endpoints of edges labelled b always serve the opposite role. The construction of H_i for $i > 2$ is such that the interpretation of labels a and b (Fig. 3(c)) remains the same. However, this is not a global property of graph G^{**}, since in some copies of the gadgets H_i the label a may mean a mid-point, while in others it may mean an endpoint of a path from R_2. Intuitively, the solution to the instance G of the $\overline{\text{3DM}}$ problem is derived solely from the interpretation of labels a and b for different copies of H_i in G^{**}.

Without loss of generality, we may assume that in the considered routing, label a beside the gadget $H_{|V_3|}$ in Fig. 4(b) denotes an endpoint of a path from R_2 (if not, all labels a and b in graph G^{**} can be swapped). Therefore, for an arbitrarily chosen vertex $v \in V_3$, the label b beside it denotes the mid-point of some path P_v from R_2. One endpoint of path P_v is obviously in the vertex set of $H_{|V_3|}$, the other, in set M. Path P_v may be thought of as the selector for v in the solution of the $\overline{\text{3DM}}$ problem, i.e. the endpoint of path P_v in set M indicates which element of M_S vertex v is covered by. Vertices from V_2 use a similar selection principle. On the other hand, notice that each graph H_3 corresponding to some $m \in M$ may either have all three labels a denoting endpoints of paths of R_2 (in which case m is considered selected and belongs to the solution M_S) or mid-points of paths of R_2 (in which case m is not selected). The set M_S obtained in this way is thus a valid solution to the $\overline{\text{3DM}}$ problem for instance G.

($\Rightarrow$) Most of the considerations can be performed by reversing the sequence of steps in the first part of the proof. Given a solution to the $\overline{\text{3DM}}$ problem for input G, we can create an appropriate routing in G^{**} by requiring that all labels a beside the gadget $H_{|V_3|}$ denote endpoints of paths from R_2, and that the label a in gadget H_3 inserted in place of some $m \in M$ denotes the endpoint of a path iff $m \in M_S$. The correctness of the routing implied by these conditions is easy to verify. □
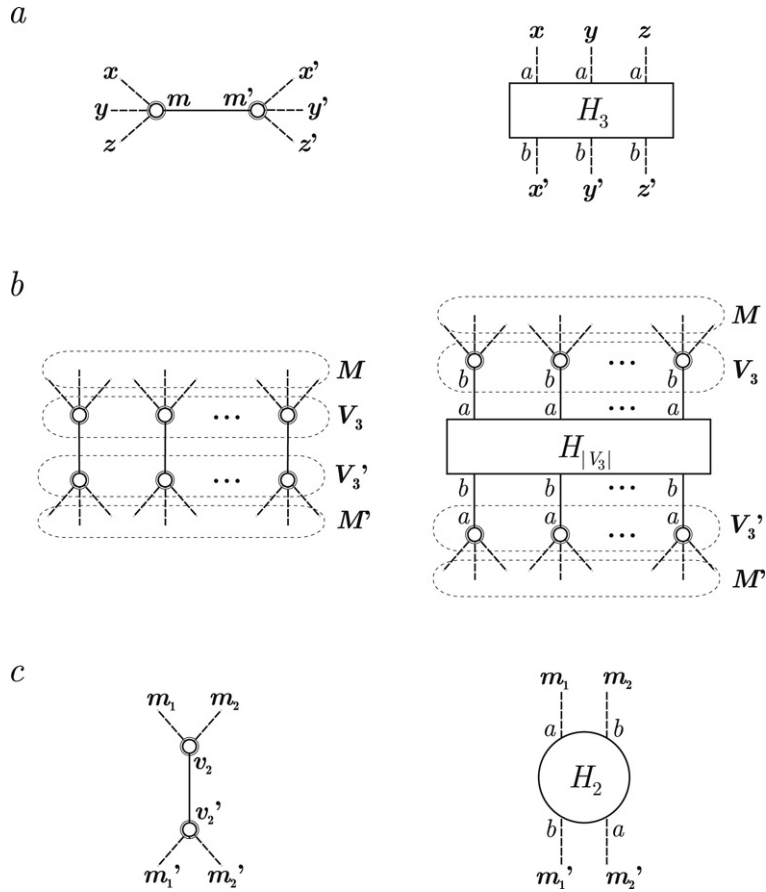
Fig. 4. Reduction from the doubled $\overline{\text{3DM}}$ instance G^* to the $\overline{\text{3L10R}}$ instance G^{**}: (a) replacement for elements of vertex set $M \cup M'$, (b) replacement for elements of vertex set $V_3 \cup V_3'$, (c) replacement for elements of vertex set $V_2 \cup V_2'$.

3.2. Reduction from 3-loaded 10-routing to 3-loaded all-to-all routing

Making use of the results of the previous subsection, we proceed to prove the *NP*-completeness of the all-to-all routing problem.

3-Loaded All-to-all Routing Problem ($\overline{\text{3LAR}}$)	
Instance:	A graph G.
Question:	Does G admit an all-to-all routing R_A such that $\pi(R_A) \leq 3$?

Theorem 9. *The* $\overline{\text{3LAR}}$ *problem is NP-complete.*

Proof. The proof proceeds by reduction from the $\overline{\text{3L10R}}$ problem (Theorem 8). Consider an instance of the $\overline{\text{3L10R}}$ problem, a 4-regular graph $G = (V, E)$ of $n \geq 16$ vertices. We construct the instance G^* of $\overline{\text{3LAR}}$ as the $(3n - 12)$-vertex graph formed by connecting a copy of graph G with a copy of the complete graph $K_{2n-12} = (V_K, E_K)$ by all possible edges, i.e. $G^* = (V \cup V_K, E \cup E_K \cup E_S)$, where $E_S = \{\{u, v\} : u \in V, v \in V_K\}$. $\square$

Claim. There exists a solution to the $\overline{\text{3L10R}}$ problem in graph G iff there exists a solution to the $\overline{\text{3LAR}}$ problem in graph G^*.

Proof ($\Leftarrow$). Let R_A denote a routing which is a valid solution to the $\overline{\text{3LAR}}$ problem in graph G^*. Consider the subset $R' \subseteq R_A$ consisting of all paths from R_A with at least one endpoint in V; we have $|R'| = 2\binom{|V|}{2} + 2|V_K| \, |V| = 5(n - 5)n$. Let $R' = R_1' \cup R_2' \cup R_{\geq 3}'$, where $R_1', R_2', R_{\geq 3}'$ denote disjoint subsets of those paths from R' which use exactly one, two, and at least three edges from $E \cup E_S$, respectively. Remembering that in the all-to-all instance a pair of vertices is connected by exactly two paths, the number of paths R_{1_a}' using exactly one edge from E and no edges from E_S is bounded by $|R_{1_a}'| \leq 2|E| = 4n$, while the number of paths R_{1_b}' using exactly one edge from E_S and no edges from E is bounded by $|R_{1_b}'| \leq 2|V_K| \, |V| = 2(2n-12)n$; hence, $|R_1'| = |R_{1_a}'| + |R_{1_b}'| \leq 4(n - 5)n$. Taking into account that edge congestion is at most 3, we may write $|R_1'| + 2|R_2'| + 3|R_{\geq 3}'| \leq$

$3|E \cup E_S| = 6(n-5)n$. Using the known bound on $|R_1'|$ and the value of $|R'| = |R_1'| + |R_2'| + |R_{\geq 3}'| = 5(n-5)n$, we eventually obtain that all considered inequalities have to be tight, $|R_{1a}'| = 4n$, $|R_{1b}'| = 2(2n-12)n$, $|R_2'| = (n-5)n$, $R_3' = \emptyset$, and that the congestion of each edge from $E \cup E_S$ is equal to 3.

Next, observe that no path of routing R_A can contain both an edge from set E_S and an edge from set E. If such a path existed, it would belong to set R_2' and have one endpoint in set V and the other in V_K. However, all $2|V_K| \, |V| = 2(2n-12)n$ requests with one endpoint in V and the other in V_K are necessarily handled by paths from R_{1b}', a contradiction.

Now, consider an arbitrary vertex $v \in V$, and denote by $E(v)$ and $E_S(v)$ the set of all edges from E and E_S, respectively, which are incident to v; we have $|E(v)| = 4$ and $|E_S(v)| = 2n-12$. By the above observation, no path of R_A which connects vertex v with a vertex from K_V can contain an edge from E. Thus, all the $2|V_K|$ paths connecting v with vertices from K_V use edges from $E_S(v)$. Vertex v is also connected by $2(|V|-1)$ paths to all the remaining vertices from V. The edges from $E_S(v)$ can be used to handle at most $3|E_S(v)|$ paths originating at v, and so at least $2(|V|-1) - (3|E_S(v)| - 2|V_K|) = 10$ paths connecting v with other vertices from V have to use edges from $E(v)$. Observing once more that a path of R_A containing an edge from set E cannot contain edges from outside E, we immediately obtain that routing R_A restricted to graph G is a 3-loaded 10-routing in G.

($\Rightarrow$) Let us now assume that there exists a solution to the $\overline{\text{3L10R}}$ problem in G, induced by 10-routing R. The sought all-to-all routing R_A for G^* may be constructed as follows. First, we assume $R_A = R$, and add to R_A paths of length 1, using each edge from $E_K \cup E_S$ for exactly two such paths. At this point of the construction, the congestion of each edge from E in routing R_A is equal to 3, and the congestion of each edge from $E_K \cup E_S$ is equal to 2. In order to complete the construction, it is only necessary to add to R_A a set of paths connecting those pairs of vertices from V which were not connected in the routing R; we denote the set of all such pairs by E_H. Due to the imposed conditions on edge congestion, this has to be achieved without using edges from E and using all other edges at most once. In order to solve this final subproblem, we act as follows. First, all vertices from V_K are labelled as $(u_1, u_2, \ldots, u_{2n-12})$, respectively. Next, we find an edge coloring of the $(2n-12)$-regular n-vertex multigraph $H = (V, E_H)$ using at most $(2n-12) + 2 = 2n-10$ colors; such a coloring can always be determined in polynomial time by Vizing's theorem since the multigraph has at most two edges connecting each pair of vertices [10]. Let $e = \{v, w\}$ be an arbitrarily chosen edge from E_H and let c_e be the color assigned to it. If $c_e \leq 2n-12$, then the sought path added to R_A for the pair of vertices $\{v, w\}$ lies within E_S and is defined as follows: $(\{v, u_{c_e}\}, \{u_{c_e}, w\})$. Now, suppose that $c_e \in \{2n-11, 2n-10\}$; let c_v and c_w be integers from the range $\{1, \ldots, 2n-12\}$ which do not appear as color values of edges adjacent to v and w, respectively. The path with endpoints $\{v, w\}$ added to R_A then takes the form: $(\{v, u_{c_v}\}, \ldots, \{u_{c_u}, w\})$, where the fragment of the path between u_{c_v} and u_{c_u} consists of edges from E_K, only. Since the number of edges e in the whole graph having $c_e \in \{2n-11, 2n-10\}$ is bounded from above by $n \leq 2n-13$, all the missing paths fragments can be laid out in E_K without excessively increasing the edge congestion (this is a property of routings in complete graphs; see e.g. [17] Prop. 3). The resulting set of paths R_A is now an all-to-all routing in G^* with $\pi(R_A) \leq 3$, which completes the proof. $\square$

The proofs of Theorems 8 and 9 can be slightly modified in order to show that the problem of deciding whether a graph admits an all-to-all routing R_A, such that $\mathtt{w}(R_A) \leq 3$, is also *NP*-complete. It suffices to notice that all the considered routings have edge congestion bounded by 3 iff their conflict graph is 3-colorable (paths of length 1 can receive colors 1 and 2, and all longer paths can receive color 3). Consequently, we may write the following theorem.

Theorem 10. *The problem of deciding whether $\pi(G) \leq 3$ ($\mathtt{w}(G) \leq 3$) is NP-complete.*

4. Final remarks

The negative results presented in this paper lead to the conclusion that all-to-all routings in general graphs do not display 'elegant' mathematical or computational properties, of the sort proven for one-to-many routings [4]. Nevertheless, such properties become apparent when considering the forwarding and optical indices for numerous special graph classes which were mentioned in the introduction. For general graphs, it still remains to be shown whether indices $\vec{\pi}(G)$ and $\vec{\mathtt{w}}(G)$ can be efficiently computed.

Little is also known about the extremal values of ratios $\frac{\mathtt{w}(G)}{\pi(G)}$ and $\frac{\vec{\mathtt{w}}(G)}{\vec{\pi}(G)}$. In order to enable the construction of approximation algorithms for the optical indices, it would be useful to prove that there does not exist a graph family $\{G_k\}$ such that $\lim_{k \to \infty} \frac{\mathtt{w}(G_k)}{\pi(G_k)} = +\infty$, or $\lim_{k \to \infty} \frac{\vec{\mathtt{w}}(G_k)}{\vec{\pi}(G_k)} = +\infty$. Currently, the largest known values of these limits are $\frac{3}{2}$ [3] and $\frac{20}{19}$ (Corollary 7), respectively.

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