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WEAKLY CONNECTED DOMINATION CRITICAL GRAPHS

Abstract. A dominating set $D \subset V(G)$ is a *weakly connected dominating set* in G if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by D is connected, where E_w is the set of all edges with at least one vertex in D. The *weakly connected domination number* $\gamma_w(G)$ of a graph G is the minimum cardinality among all weakly connected dominating sets in G. The graph is said to be *weakly connected domination critical* (γ_w*-critical*) if for each $u, v \in V(G)$ with v not adjacent to u, $\gamma_w(G + vu) < \gamma_w(G)$. Further, G is k-γ_w*-critical* if $\gamma_w(G) = k$ and for each edge $e \notin E(G)$, $\gamma_w(G + e) < k$. In this paper we consider weakly connected domination critical graphs and give some properties of 3-γ_w-critical graphs.

Keywords: weakly connected domination number, tree, critical graphs.

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1. INTRODUCTION

Let $G = (V, E)$ be a connected simple graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v. For a set $X \subseteq V(G)$, the *open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* is $N_G[X] = N_G(X) \cup X$. We say that a vertex v is *a universal vertex* of G if it is a neighbour of every other vertex of a graph.

A subset D of $V(G)$ is *dominating* in G if every vertex of $V(G) - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. The degree of a vertex v is $d_G(v) = |N_G(v)|$. Further, $D \subseteq V(G)$ is a *connected dominating set* in G if D is dominating and the subgraph $G[D]$ induced by D in G is connected. The minimum cardinality among all connected dominating sets in G is called *connected domination number* of G and is denoted $\gamma_c(G)$.

A dominating set $D \subseteq V(G)$ is a *weakly connected dominating set* in G if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by D is connected, where E_w is the set of all edges with at least one vertex in D. Dunbar et al. [1] defined the *weakly connected domination number* $\gamma_w(G)$ of a graph G to be the minimum cardinality among all weakly connected dominating sets in G.

We say that a set $D \subseteq V(G)$ has the property $\mathcal{F}$ in G if D contains no end-vertex of G.

We say that two vertices $a, b \in D$ are *adjacent in* D in a graph G if $ab \in E(G)$ or there is an $(a$-$b)$-path P in G such that no vertex $v \in P - \{a, b\}$ belongs to D. We denote by $d_G(a, b)$ the distance between two vertices $a, b \in V(G)$.

Here we consider connected graphs only. If G is a graph, let $n = n(G)$ be the order of G and let $n_1 = n_1(G)$ denote the number of end-vertices of G. The set of all end-vertices in G is denoted by $\Omega(G)$. A vertex v is called a *support* if it is adjacent to an end-vertex.

A graph G is said to be γ-*domination critical*, or just γ-critical if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for every edge e in the complement $\overline{G}$ of G. In [2] X.-G. Chen et al. defined the connected domination critical graphs. The graph is said to be *connected domination critical* in the following sense: for each $u, v \in V(G)$ with v not adjacent to u, $\gamma_c(G + vu) < \gamma_c(G)$. Further, G is k-γ_c-*critical* if $\gamma_c(G) = k$ and for each edge $e \notin E(G)$, $\gamma_c(G + e) < k$.

In this paper we study the weakly connected domination critical graphs. The graph is said to be *weakly connected domination critical* (γ_w-*critical*) if for each $u, v \in V(G)$ with v not adjacent to u, $\gamma_w(G + vu) < \gamma_w(G)$. Thus, G is k-γ_w-*critical* if $\gamma_w(G) = k$ and for each edge $e \notin E(G)$, $\gamma_w(G + e) < k$.

2. RESULTS

In [4] the following theorem has been proved.

Theorem 1. *If G is a connected graph, then for any edge $e \in E(\overline{G})$, $\gamma_w(G) - 1 \leq \gamma_w(G + e) \leq \gamma_w(G)$.*

Observation 1. *If G is a connected graph with at most one cycle and D is a weakly connected dominating set in G, then there are at most two vertices a, b adjacent in D such that $d_G(a, b) > 2$ and then $d_G(a, b) = 3$. Additionally, there exists the unique $(a$-$b)$-path P in G whose inner vertices do not belong to D.*

The following result is included in [1].

Theorem 2. *If T is a tree of order n, then $\gamma_w(T) = n - \beta_0(T)$, where β_0 is the cardinality of maximum independent set of T.*

The next observation is the immediate consequence of Theorem 2.

Observation 2. *For a path P_n on n vertices, $\gamma_w(P_n) = \lfloor \frac{n}{2} \rfloor$.*

Theorem 3. *For a cycle C_n, $\gamma_w(C_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $G = C_n$. We may consider a cycle C_n as a path P_n with an added edge $v_1 v_n$, where v_1, v_n are end-vertices of P_n. By Theorem 1 and Observation 2, there is $\gamma_w(C_n) = \gamma_w(P_n + v_1 v_n) \leq \gamma_w(P_n) = \lfloor \frac{n}{2} \rfloor$. Let D be a minimum weakly connected dominating set with property $\mathcal{F}$ in G. From Observation 1, at least $\lfloor \frac{n}{2} \rfloor$ vertices must be in D and thus $\gamma_w(G) \geq \lfloor \frac{n}{2} \rfloor$. Hence $\gamma_w(G) = \lfloor \frac{n}{2} \rfloor$. □

Since $C_n = P_n + v_1 v_n$, where v_1, v_n are end-vertices of P_n, we obtain the following corollary:

Corollary 4. *The path P_n is not γ_w-critical.*

Theorem 5. *The cycle C_n is γ_w-critical if and only if n is even.*

Proof. Let $G = C_n + e$, where e is an edge belonging to $\overline{C_n}$. Since it is easy to observe that the result is true for $n = 3$, we assume $n \geq 4$. We consider two cases.
Case 1. If n is odd, then let $(c_1, c_2, \ldots, c_n)$ be the consecutive vertices of C_n, $e = c_1c_3$ and let D be a minimum weakly connected dominating set of G. Let us denote $P = (c_4, c_5, \ldots, c_n)$ and note that P is a path on $n-3$ vertices.

If both c_1, c_3 belong to D, then D is also a weakly connected dominating set of C_n. Hence $\gamma_w(C_n) \leq |D| = \gamma_w(G)$ and C_n is not γ_w-critical.

If neither c_1 nor c_3 belongs to D, then, since D is dominating, $c_2 \in D$. By Theorem 2, at least $\frac{n-3}{2}$ vertices are needed to dominate P. Thus $\gamma_w(G) \geq \frac{n-3}{2} + 1 = \frac{n-1}{2}$. Since $\gamma_w(C_n) = \lfloor \frac{n}{2} \rfloor$, we have $\gamma_w(G) \geq \gamma_w(C_n)$.

Assume now that (without loss of generality) $c_1 \in D, c_3 \notin D$. By Theorem 2, at least $\frac{n-3}{2}$ vertices are needed to dominate P and thus $\gamma_w(G) = |D| \geq \frac{n-3}{2} + 1 = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor = \gamma_w(C_n)$. Hence C_n is not γ_w-critical.
Case 2. If n is even, then notice that e is a chord of C_n and e belongs to two chordless cycles of G, denote these cycles C_p and $C_m; p, m \geq 3$ and denote $e = c_1c_2$. Let $(c_1, c_2, \ldots, c_p)$ be the consecutive vertices of C_p and $(c_1, c_2, v_3, \ldots, v_m)$ be the consecutive vertices of C_m. Thus $n = p + m - 2$ and $\gamma_w(C_n) = \lfloor \frac{p+m-2}{2} \rfloor$. Since n is even, both m, p are even or both are odd. Thus $\gamma_w(C_n) = \lfloor \frac{p+m-2}{2} \rfloor = \frac{p+m}{2} - 1$.

If both m, p are even, then $D^{'} = \{c_1, c_2, c_4, \ldots, c_{p-2}, v_4, \ldots, v_{m-2}\}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D^{'}| = 2 + \frac{p-4}{2} + \frac{m-4}{2} = \frac{p+m}{2} - 2$. Hence $\gamma_w(G) < \gamma_w(C_n)$ and C_n is γ_w-critical.

If m, p are odd, then $D^{''} = \{c_1, c_3, \ldots, c_{p-1}, v_4, \ldots, v_{m-1}\}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D^{'}| = 1 + \frac{p-3}{2} + \frac{m-3}{2} = \frac{p+m}{2} - 2$. Hence $\gamma_w(G) < \gamma_w(C_n)$ and C_n is γ_w-critical. □

Lemma 6. *If G is γ_w-critical, then there is no support vertex in G which would be adjacent to two or more end-vertices of G.*

Proof. Suppose v is a support vertex which is adjacent to at least two end-vertices, say x, y, of a graph G and let $G^{'} = G + xy$. Let $D^{'}$ be a minimum weakly connected dominating set of $G^{'}$.

If neither x nor y belongs to $D^{'}$, then $D^{''} = D^{'} - \{x, y\} \cup \{v\}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D^{''}| < |D^{'}| = \gamma_w(G^{'})$, which gives a contradiction.

If both x, y do not belong to $D^{'}$, then $v \in D^{'}$ and $D^{'}$ is a weakly connected dominating set of G, again a contradiction.

Suppose (without loss of generality) $x \in D^{'}, y \notin D^{'}$. Then $D^{''} = (D^{'} - \{x\}) \cup \{v\}$ is a weakly connected dominating set of G, a contradiction. □

Lemma 7. *If G is γ_w-critical, then no two support vertices are adjacent.*

Proof. Suppose that u and v are adjacent support vertices of $u^{'}$ and $v^{'}$, respectively, in a connected γ_w-critical graph G. Consider $G^{'} = G + u^{'}v^{'}$ and let $D^{'}$ be a minimum weakly connected dominating set in $G^{'}$. We consider three cases.
Case 1. If both $u^{'}$ and $v^{'}$ belong to $D^{'}$, then $D = (D^{'} - \{u^{'}, v^{'}\}) \cup \{u, v\}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D|$, a contradiction, since $|D| = |D^{'}|$ and G is γ_w-critical.
Case 2. If $u^{'}, v^{'} \notin D^{'}$, then $u, v \in D^{'}$. It is immediate that $D^{'}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D^{'}|$, a contradiction.
Case 3. Without loss of generality, suppose $u^{'} \in D^{'}, v^{'} \notin D^{'}$. Then, since $D^{'}$ is weakly connected, there is $u \in D^{'}$ or $v \in D^{'}$. If both u, v belong to $D^{'}$ or $u \notin D^{'}, v \in D^{'}$, then $D^{'}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D^{'}|$, a contradiction. If $u \in D^{'}, v \notin D^{'}$, then $D = (D^{'} - \{u^{'}\}) \cup \{v\}$ is a weakly connected dominating set of G and $\gamma_w(G) \leq |D| = |D^{'}|$, which gives a contradiction. □

Lemma 8. *If G is γ_w-critical, then for every two supports u, v, there is $d_G(u, v) \geq 3$.*

Proof. By Lemma 7, there is $d_G(u, v) > 1$ for every two supports u, v. Suppose that u and v are support vertices in a connected γ_w-critical graph G and $d_G(u, v) = 2$. Consider $G^{'} = G + uv$ and let $D^{'}$ be a minimum weakly connected dominating set with property $\mathcal{F}$ in $G^{'}$. Since $D^{'}$ is a weakly connected dominating set of G, then $\gamma_w(G) \leq |D^{'}| = \gamma_w(G + uv)$, which gives a contradiction. □

Theorem 9. *No tree is γ_w-critical.*

Proof. Suppose T is γ_w-critical and let $(v_0, \ldots, v_l)$ be a longest path in T. By Lemma 8, $l \geq 5$ and $d_T(v_1) = d_T(v_2) = d_T(v_{l-2}) = d_T(v_{l-1}) = 2$. Let $D^{'}$ be a minimum weakly connected dominating set of $G^{'} = T + v_0v_3$.

If $v_0, v_3 \in D^{'}$, then $D = (D^{'} - \{v_0\}) \cup \{v_1\}$ is a weakly connected dominating set of T and $\gamma_w(T) \leq |D| = |D^{'}| = \gamma_w(G^{'})$, which gives a contradiction.

If $v_0, v_3 \notin D^{'}$, then, since $D^{'}$ is dominating, $v_1, v_2 \in D^{'}$ and $D^{'}$ is also a weakly connected dominating set in T. Thus $\gamma_w(T) \leq |D^{'}| = \gamma_w(G^{'})$, a contradiction.

If $v_0 \in D^{'}$, $v_3 \notin D^{'}$, then if $v_2 \in D^{'}$, $D^{'}$ is a weakly connected dominating set in T, again a contradiction. If $v_2 \notin D^{'}$, then $v_1 \in D^{'}$ and then $D = (D^{'} - \{v_0\}) \cup \{v_3\}$ is a weakly connected dominating set in T, a contradiction.

If $v_0 \notin D^{'}, v_3 \in D^{'}$ then if $v_1 \in D^{'}$, $D^{'}$ is a weakly connected dominating set in T, again a contradiction. If $v_1 \notin D^{'}$, then (by Observation 1) $v_2 \in D^{'}$ and then $D = (D^{'} - \{v_2\}) \cup \{v_1\}$ is a weakly connected dominating set in T, a contradiction. Thus T is not γ_w-critical. □

Since it is easy to observe ([2]) that a connected graph is 2-γ_c-critical if and only if it is 2-γ-critical, we also conclude that G is 2-γ_w-critical if and only if it is 2-γ-critical. 2-γ-critical and 2-γ_c-critical graphs are characterized in [3] and [2], respectively; thus, we also obtain a characterization of 2-γ_w-critical graphs. The situation of k-γ_w-critical graphs with $k \geq 3$ is more complicated. For $k = 3$ there exist graphs which are 3-γ_w-critical, not 3-γ-critical and not 3-γ_c-critical. For example, graph C_6 is not 3-γ_c-critical, since $\gamma_c(C_6) = 4$ and not 3-γ-critical, since $\gamma(C_6) = 2$.

But it is 3-γ_w-critical, since $\gamma_w(C_6) = 3$ and $\gamma_w(C_6 + uv) = 2$, where u and v are any two vertices for which $d_{C_6}(u, v) = 2$ or $d_{C_6}(u, v) = 3$.

We will now characterize 3-γ_w-critical graphs. By Theorem 1, if G is 3-γ_w-critical, then $\gamma_w(G + e) = 2$ for any edge $e \in E(\overline{G})$.

Lemma 10. *If G is 3-γ_w-critical, then diam$(G) \leq 4$.*

Proof. Let G be a connected 3-γ_w-critical graph and suppose G has diameter at least 5. Let $P = (v_1, \ldots, v_l)$ be a diametrical path in G with the length equal to the diameter of G. Obviously $l \geq 6$. Let $D^{'}$ be a minimum weakly connected dominating set of $G + v_1v_l$. Since G is a connected 3-γ_w-critical graph, then $\gamma_w(G + v_1v_l) = 2$ and $|D^{'}| = 2$. If neither v_1 nor v_l belongs to $D^{'}$, then not all vertices $v_2, \ldots, v_{l-2}$ are dominated; if both v_1, v_l do not belong to $D^{'}$, then, since $D^{'}$ is dominating, $v_2, v_{l-1} \in D^{'}$. But, since $l \geq 6$, $D^{'}$ is not weakly connected, a contradiction.

Thus exactly one of v_1, v_l belongs to $D^{'}$. Without loss of generality, let $v_1 \in D^{'}$, $v_l \notin D^{'}$. If $v_2 \in D^{'}$ or $v_3 \in D^{'}$ then, since $l \geq 6$, v_{l-1} is not dominated; hence $v_2, v_3 \notin D^{'}$. Since $D^{'}$ is dominating, $v_4 \in D^{'}$. Then $D^{'}$ is not weakly connected, a contradiction. Thus diam$(G) \leq 4$. □

Fig. 1. A 3-γ_w-critical graph with diameter equal to 4

The result is best possible. Figure 1 shows an example of a 3-γ_w-critical graph with diameter 4.

Theorem 11. *For any $n \geq 6$ there exists a 3-γ_w-critical graph G with n vertices.*

Proof. For $n \geq 6$, we construct G in a following way: we start with a graph K_{n-3} and then obtain a graph H by adding a new vertex v and $n - 5$ edges joining v with any $n - 5$ vertices of K_{n-3}. Finally, to obtain graph G, we add two vertices u, w and edges ua and wb to H, where a and b are vertices of degree $n - 4$ in H.

It is easy to observe that $\{a, b, c\}$, where c is a neighbour of a vertex v, is a minimum weakly connected dominating set of G. We can also find a minimum weakly connected dominating set D of cardinality 2 in $G+e$ for any $e \in \overline{G}$ (for $G+uw$, there is $D = \{c, w\}$; for $G+ub$ and $G+uc$ there is $D = \{b, c\}$, for $G+va$ there is $D = \{a, b\}$ and for $G + uv$ there is $D = \{v, b\}$. The other graphs $G + e$ are isomorphic to the given above). □

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